Then one can easily prove that there are two color classes, say C_1, C_2, for which there is a set of pairs $N^1 \subseteq C_1 \times C_2$ such that it holds:

$$(i, j) \in N^1 \Rightarrow 1 < |i - j| \quad \text{is an odd number ;}$$
$$(i, j) \neq (i', j') \in N^1 \Rightarrow i \neq i' \quad \text{and} \quad j \neq j' \; ; \quad |N^1| = (2k)^{2k} .$$

Now we can go on similarly as in the above procedure:

The set of all M_i^1 for which $M_i^1 \supset \{i, j\}$, where $(i, j) \in N^1$, cannot be colored by less than three colors; thus there are again $2m_1 . m$ sets M_i^1 which are colored by the same color and which are pairwise disjoint (this can be easily managed). Define analogously as above $N_{2i}^2 = \{M_{2i-1}^1\} \cup \{a_{2i}, b_{2i}\}$, $i = 1, \ldots, m_1 . m$. (Here $(a_{2i}, b_{2i}) \in N^1$, $a_{2i}, b_{2i} \notin M_{2i-1}^1$, which can be done by a suitable numbering of sets under consideration.) From the sets M_i^2 containing an N_{2i}^2 we can again choose $2m_2 . m$ disjoint sets which are colored by the same color. We can then define N_{2i}^2 and so on. Finally we can define m pairwise disjoint sets N_i^k such that $|N_i^k| = k + 1$ and $N_i^k \cap C_i \neq \emptyset$, $i = 1, \ldots, k + 1$. Now there are two possible cases. Let first $x \in V(P_n^{k-1}) \cap C_{k+2} \neq \emptyset$. Since we have m pairwise disjoint N_i^k with the above properties, there is N_i^k with $N_i^k \cap x = \emptyset$. Thus $N^k \cup \{x\} \in \mathcal{M}^k$, a contradiction. Let $V(P_n^{k-1}) \cap C_{k+2} = \emptyset$. Then it can be easily proved by induction that C is uniquely determined by $C_1 \cup C_2 = V(P_n)$, $C_{i+2} = \mathcal{M}^i$, $i = 1, \ldots, k$.

References

[0] *R. L. Brooks:* On coloring the nodes of a network. Proc. Cambridge Phil. Soc. *37* (1941), 194—197.

[1] *V. Chvátal:* The smallest triangle free 4-chromatic 4-regular graph (To appear).

[2] *B. Grünbaum:* A problem in graph coloring (To appear).

[3] *F. Harary, S. T. Hedetniemi* and *R. W. Robinson:* Uniquely colorable graphs. J. Comb. Th. (1969), 260—270.

[4] *F. Harary:* Graph theory. Addison Wesley, Reading 1969.

[5] *L. Lovász:* On chromatic number of finite set-systems, Acta Math. Acad. Sci. Hungar. *19* (1968), 59—67.

[6] *J. Nešetřil:* *k*-chromatic graphs without cycles of length ≤ 7 (in Russian) Comment. Math. Univ. Carolinae *7* (1966), 373—376.

Author's address: 180 00 Praha 8 - Karlín, Sokolovská 83 (Matematicko-fyzikální fakulta UK).