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ON UNIQUELY COLORABLE GRAPHS WITHOUT SHORT CYCLES

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1. INTRODUCTION

A uniquely colorable graph is defined in [3] as a graph X which possesses exactly one partition into n color classes, where $n = \chi(X)$ is the chromatic number of the graph X.

Let $\Delta(X)$ denote the maximal degree of a point of X. We shall characterize uniquely colorable graphs X which satisfy $\Delta(X) \leqq n = \chi(X)$. This is related to the problem of the existence of an n-chromatic graph with a large chord in the following way:

A theorem established in [0] states that $\chi(X) \leqq \Delta(X)$, with the exception of an odd cycle and K_n (the complete graph with n points). On the other hand, the question if there is an n-chromatic graph without cycles of length $\leqq k$ has been solved constructively only recently, see [4], [6].

B. GRÜNBAUM conjectured that for every n, k there is an n-regular n-chromatic graph without cycles of length $\leqq k$. (A graph is n-regular if all the points of X have the same degree n). This conjecture is proved to be true for couples (4, 3) and (4, 4), see [1, 2], except the trivial cases.

From our result it will follow that there is no uniquely colorable graph satisfying this conjecture (for $k \geqq 3$, i.e. non-trivial), or that every such graph possesses at least two different colorings. For the same reason, the naturally arising question if there is a uniquely n-colorable graph without cycles of length $\leqq k$, seems to be harder than for n-chromatic graphs in general; none of the known constructions of n-colorable graphs without short cycles gives uniquely colorable graphs.

Nevertheless, we conjecture that the answer to this question is also affirmative. To support this we give here a construction of a uniquely k-colorable graph (for every $k \geqq 1$) without triangles. In fact, we prove that there is a countable number of such graphs for every $k \geqq 1$. This generalizes theorems from [3, 4]*).

*) The examples of graphs given in [3] and [4], p 139 do not serve as examples of uniquely 3-colorable graphs.

2. UNIQUELY COLORABLE GRAPHS WITH SMALL DEGREES

Let us denote by UC_n the class of all uniquely n-colorable graphs. Obviously all connected graphs $X \in UC_2$ which satisfy $\Delta(X) \leqq 2$ are exactly even cycles and paths. Thus, let be $n > 2$ from now on. Let $X \in UC_n$, $\Delta(X) \leqq n$ be a fixed graph, $M = \{M_1, \ldots, M_n\}$ the coloring of X. For $x \in V(X)$ denote by $V(x, X)$ the set of all adjacent points to x in X.

We shall need the following:

Lemma. *Let $\hat{X}$ be the subgraph of X induced on the set $\bigcup\{M_i;\ i \geqq 2\}$. Then $\bigcup\{V(x, X);\ x \in M_1\} = V(\hat{X})$ and if $y \in V(x, X)$ for exactly one $x \in M_1$, then either $y \neq y' \in V(x, X)$ implies $y' \in V(x', X)$ for some $x \neq x' \in M_1$ or $V(x, X) = V(\hat{X})$.*

Proof. $\{V(x, X);\ x \in M_1\}$ is a covering of $V(\hat{X})$, for if there is a $y \in V(\hat{X})$ such that $y \notin V(x, X)$ for every $x \in M_1$, then the coloring M' defined by $M'_1 = M_1 \cup \{y\}$, $M'_i = M_i \setminus \{y\}$, $i \geqq 2$ is different from M. The proof of the second part of the statement proceeds similarly.

Theorem 1. *K_n and $K_{n-1} + \overline{K}_2$ are the only UC_n-graphs X for which it holds $\Delta(X) \leqq n$. (Here $\overline{X}$ denotes the complement of the graph X and $X + Y$ denotes the join (Zykov sum) of the graphs X and Y, see [4]).*

Proof. Let $X \in UC_3$, $\Delta(X) \leqq 3$, then (in the above notation) $\hat{X} \in UC_2$ and by Lemma $\Delta(\hat{X}) \leqq 2$. It is easy to prove that $|\hat{X}| \leqq 4$. It can be verified by examining the individual cases that K_3 and $K_2 + \overline{K}_2$ are the only uniquely 3-colorable graphs under consideration.

It is easy to complete the proof of the statement by induction.

Corollary. *Odd cycles are exactly 2-regular elements of UC_2. There are no n-regular elements of UC_n, $n > 2$.*

Remark. Adding two suitable edges to the graph described in [4], p. 139, one obtains a graph X from UC_3 which has no triangles and for which $\Delta(X) = 4$ holds.

3. UNIQUELY COLORABLE GRAPHS WITHOUT TRIANGLES

Let X be a graph, $M \subseteq V(X)$. The set M is called *an independent subset* if $x, y \in M \Rightarrow [x, y] \notin E(X)$.

Let P_n be the path of length n (i.e. $V(P_n) = \{1, \ldots, n + 1\}$, $[i, i + 1] \in E(P_n)$, $i = 1, \ldots, n$). Define by induction the graphs P_n^i, $i > 0$.

Let $\mathcal{M}^1 = \{M_\iota^1;\ \iota \leqq k^1(n)\}$ be the set of all independent sets $M \subseteq V(P_n)$ with $|M| = 3$ such that there are $i \neq j \in M$ with $|i - j|$ odd.

Let P_n^1 be the graph defined by: $V(P_n^1) = V(P_n) \cup \mathcal{M}^1$, $[x, y] \in E(P_n^1)$ iff either $x, y \in V(P_n)$ and $[x, y] \in E(P_n)$ or $x = M_\iota^1 \in \mathcal{M}^1$ and $y \in M_\iota^1$.

Let P_n^j be defined for all $j \leqq i$, $i \geqq 1$.

Let $\mathcal{M}^{i+1} = \{M_\iota^{i+1};\ \iota \leqq k^{i+1}(n)\}$ be the set of all independent sets $M \subseteq V(P_n^i)$ such that $|M| = i + 3$, $M \cap \mathcal{M}^j \neq \emptyset$ for every $j \leqq i$ and there are $k \neq m \in M \cap \cap V(P_n)$, $|k - m|$ being odd.

Define the graph P_n^{i+1} by: $V(P_n^{i+1}) = V(P_n^i) \cup \mathcal{M}^{i+1}$ $[x, y] \in E(P_n^{i+1})$ iff either $x, y \in V(P_n^i)$ and $[x, y] \in E(P_n^i)$ or $x = M_\iota^{i+1} \in \mathcal{M}^{i+1}$ and $y \in M_\iota^{i+1}$. By the definition, the graph does not contain a triangle for every $i \geqq 1$. Further, it is obvious that $\chi(P_n^i) \leqq i + 2$.

We shall prove

Theorem 2. *Let $k \geqq 1$. Let $n > 16(k + 2)(2k)^{2k+3}$. Then $P_n^k \in UC_{k+2}$.*

Proof. Let $C = \{C_1, \ldots, C_{k+2}\}$ be a coloring of P_n^k. We distinguish two cases.

1) There are three classes, say C_1, C_2, C_3 such that $|C_i \cap V(P_n)| \geqq n/(k + 2)$, $i = 1, 2, 3$. We prove first that there are $(2k)^k$ pairwise disjoint sets M_ι^1 from $\mathcal{M}^1$ such that all of them are colored exactly by 3 different colors (not necessarily 1, 2, 3). Suppose to the contrary that there are no such sets from $\mathcal{M}^1$. Then $|C_1 \cup C_2 \cup C_3| \leqq \leqq \frac{1}{2}n + 3(2k)^k$ (since $|C_i \cap V(P_n)| \geqq n/(k + 2)$, C_1, C_2 and C_3 cannot contain "too many" couples i, j with $|i - j|$ odd, and the same argument shows that there is a set $A \subset C_1 \cup C_2$ such that $|A \cap C_\iota| \geqq (2k)^k$, $i = 1, 2$ and $i \neq j \in A$ implies $|i - j|$ even. Thus there is at least $\frac{1}{2}n - 3(2k)^k)/4 > (k - 1)(2k)^k$ elements $i \in V(P_n)$ such that $|i - j|$ is odd for every $j \in A$. From these facts a contradition easily follows).

Now we shall construct an $M_\iota^k \in \mathcal{M}^k$ such that $M_\iota^k \cap C_i \neq \emptyset$ for every $i = 1, 2, \ldots \ldots, k + 2$. This will contradict the assumption that C is a coloring.

Put $m = (2k)^k$. Without loss of generality, let $M_1^1, \ldots, M_m^1$ be sets from $\mathcal{M}^1$ such that $M_i^1 \cap M_j^1 = \emptyset$ for $i \neq j \leqq m$ and $M_i^1 \cap C_j \neq \emptyset$ for $i = 1, \ldots, m$ and $j = = 1, 2, 3$. Since $m = (2k)(2k)^{k-1}$ there is $2(2k)^{k-1} = 2m_1$ elements of the set $\{M_1^1, \ldots, M_m^1\}$ which are colored by the same color $\geqq 4$, wihout loss of generality let us assume that $M_i^1 \in C_4$ for $i = 1, \ldots, 2m_1$. Define $M_{2i}^2 \in \mathcal{M}^2$, $i = 1, \ldots, m_1$ by $M_{2i}^2 = \{M_{2i-1}^1\} \cup M_{2i}^1$. (It is $M_{2\iota}^2 \in \mathcal{M}^2$ since M_i^1 are pairwise disjoint.) Further $M_{2\iota}^2 \cap C_j \neq \emptyset$ for $i = 1, \ldots, m_1$ and $j = 1, 2, 3, 4$.

Now, without loss of generality, we can find again M_j^2, $j = 1, \ldots, 2m_2 = 2(2k)^{k-2}$ such that $\{M_1^2, \ldots, M_{2m_2}^2\} \subseteq C_i$ for an $i \geqq 5$, say for $i = 5$. We can define $M_{2i}^3 = = \{M_{2\iota-1}^2\} \cup M_{2\iota}^2$, $i = 1, \ldots, m_2$. It is $M_{2\iota}^3 \in \mathcal{M}^3$ and $M_{2i}^3 \cap C_j \neq \emptyset$ $j = 1, \ldots, 5$. This procedure can be continued inductively and finally we get an $M_\iota^k \in \mathcal{M}^k$ for which $M^k \cap C_j \neq \emptyset$, $j = 1, \ldots, k + 2$.

2) There are exactly two classes, say C_1, C_2 such that

$$|C_\iota \cap V(P_n)| \geqq \frac{n}{k + 2}, \quad i = 1, 2.$$