if $x = 8$, *then* $\alpha(x) = 4$, $\beta(x) = 5$; *if* $x = 9$, *then* $\alpha(x) = 5$, $\beta(x) = 6$. *Otherwise*

$$(12) \qquad \alpha(x) < \beta(x) \leqq \frac{x + 1}{2}.$$

Proof. The cases $x \leqq 10$ are easily verifiable; the value of the function α for $x \leqq 9$ have been given by J. Sedláček [3]. From (2) it follows that (12) holds for $x = 11$. The graph $D(2, 2, 2)$ leads to estimate (12) for $x = 12$. There is no graph with cyclomatic number 2 which has 13 spanning trees, and any graph with a greater cyclomatic number has more than 13 spanning trees; hence (11) holds for $x = 13$. There is no graph with cyclomatic number 2 or 3 which has 22 spanning trees, and any graph with a greater cyclomatic number has more than 22 spanning trees; hence (11) holds for $x = 22$. If $x \geqq 106$ it is possible to use exactly one of the estimates (2)–(10) for it; this one estimate then leads to estimate (12).

Now, let us assume that $14 \leqq x < 106$, $x \neq 22$. In so far as it is possible to use for such an x any of estimates (1)–(10), we obtain estimate (12) for it. There remain the cases $x = 19, 31, 34, 46$ and 61; for these x it is possible to obtain estimate (12) by graphs $D(1, 3, 4)$, $D(1, 3, 7)$, $D(1, 4, 6)$, $D(2, 3, 8)$ and $D(3, 4, 7)$ in turn. The proof is complete.

Now we shall turn to other relationship between the functions α and β.

Theorem 2. *Let z be an integer such that $z \geqq 11$ and $z \neq 13, 22$. Then there is no graph having simultaneously $\alpha(z^{z-2})$ vertices, $\beta(z^{z-2})$ edges and z^{z-2} spanning trees.*

Proof. The only graph having $\alpha(z^{z-2})$ vertices and z^{z-2} spanning trees is the complete graph having z vertices; it has $z(z - 1)/2$ edges. From (1) and (12) it follows that $\beta(z^{z-2}) \leqq (z - 2)\,\beta(z) \leqq (z - 2)(z + 1)/2 < z(z - 1)/2$. The proof is complete.

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