Časopis pro pěstování matematiky, roč. 98 (1973), Praha

INDEFINITE HARMONIC CONTINUATION

JOSEF KRÁL and JAROSLAV LUKEŠ, Praha

(Received May 24, 1971)

The purpose of this note is to characterize harmonic spaces whose harmonic functions admit indefinite harmonic continuation.

In the classical potential theory harmonic functions are defined as continuous solutions of the Laplace differential equation. In the one-dimensional case these functions reduce to locally affine functions and any harmonic (=affine) function on an interval of the real line R^1 can thus be harmonically continued onto the whole of R^1. We are going to describe all topological spaces which have a similar exceptional property (analoguous to that of the real line in the classical case) in the framework of the Brelot axiomatic theory of harmonic functions.

By a Brelot space we mean a locally compact and locally connected Hausdorff topological space X which is equipped with a sheaf $\mathscr{H}$ associating with each open set $U \subset X$ a real vector – space $\mathscr{H}(U)$ of continuous functions, termed harmonic functions on U, such that the sheaf axiom, the basis axiom and the Brelot convergence axiom are satisfied. We shall say that a Brelot space $(X, \mathscr{H})$ has the continuation property CP if and only if each point $x \in X$ is contained in a domain (=open and connected set) $D \subset X$ such that each harmonic function defined on an arbitrary subdomain of D can be harmonically continued onto D. More precisely: Whenever $D_0 \subset D$ is a domain and $h_0 \in \mathscr{H}(D_0)$, then there is an $h \in \mathscr{H}(D)$ such that $h_0 = \operatorname{Rest}_{D_0} h$ (= the restriction of h to D_0). It is known that if X is a 1-dimensional manifold, then every Brelot space $(X, \mathscr{H})$ has CP (cf. [5]), and one may naturally ask whether there are other Brelot spaces possessing CP, besides those defined on 1-dimensional manifolds. We are going to show that such spaces can be completely described and, as shown by the following theorem, cannot topologically deviate much from 1-dimensional manifolds.

Theorem. *A Brelot space $(X, \mathscr{H})$ enjoys CP if and only if for every $x \in X$ there is a finite number $n \geqq 2$ (depending on x) of arcs*[1]) *$C_1, \ldots, C_n$ in X such that $\bigcup_{i=1}^{n} C_i$*

[1]) By an arc in X we mean a subspace $C \subset X$ which is homeomorphic with the segment $\{a;\ a \in R^1,\ 0 \leqq a \leqq 1\}$.

is a neighborhood of x in X and

$$C_i \cap C_j = \{x\} \quad \textit{whenever} \quad 1 \leqq i < j \leqq n\,.$$

We shall see that the sufficiency of the above condition can be proved quite easily. Its necessity, however, requires some preliminary investigations (note that X is a general locally compact and locally connected space which is not assumed to have a countable base).

We shall first assume in sections 1–5 that $(X, \mathscr{H})$ is a Brelot space with a connected X satisfying the following condition:

(C) For every domain $D_0 \subset X$ and every $h_0 \in \mathscr{H}(D_0)$ there is an $h \in \mathscr{H}(X)$ such that $\mathrm{Rest}_{D_0} h = h_0$.

We shall prove several auxiliary results describing properties of such an X. For $M \subset X$ we denote by $\overline{M}$ and M^* the closure and the boundary of M, respectively. $\mathscr{C}(M)$ will stand for the Banach space of all bounded continuous real-valued functions on M with the usual supremum norm. The number (possibly zero or infinite) of all points in M will be denoted by $n(M)$ $(0 \leqq n(M) \leqq \infty)$. Let us recall that an open set $U \subset X$ is termed regular if it is relatively compact, $U^* \neq \emptyset$ and for each $f \in \mathscr{C}(U^*)$ there is a uniquely determined $H_f^U \in \mathscr{C}(\overline{U})$ such that $\mathrm{Rest}_U H_f^U \in \mathscr{H}(U)$, $\mathrm{Rest}_{U^*} H_f^U = f$ and, besides that, $H_f^U \geqq 0$ whenever $f \geqq 0$.

1. Lemma. *If D_0, D_1 are regular domains such that*

$$D_0 \subset D_1\,, \tag{1}$$

then $n(D_0^*) \leqq n(D_1^*)$. If, moreover,

$$\overline{D}_0 \subset D_1\,, \tag{2}$$

then $n(D_0^*) < \infty$.

Proof. Assuming (1) we define the mapping T of $\mathscr{C}(D_1^*)$ into $\mathscr{C}(D_0^*)$ by

$$\mathrm{T}f = \mathrm{Rest}_{D_0^*} H_f^{D_1}\,, \quad f \in \mathscr{C}(D_1^*)\,.$$

Clearly, T is a continuous linear mapping. Given an arbitrary $g \in \mathscr{C}(D_0^*)$ we may apply to $h_0 = \mathrm{Rest}_{D_0} H_E^{D_0}$ the process of harmonic continuation described in (C) so as to get an $h \in \mathscr{H}(X)$ with $\mathrm{Rest}_{D_0} h = h_0$. Clearly, $g = \mathrm{Rest}_{D_0^*} h = \mathrm{T}f$, where $f = \mathrm{Rest}_{D_1^*} h \in \mathscr{C}(D_1^*)$. We see that T maps $\mathscr{C}(D_1^*)$ onto $\mathscr{C}(D_0^*)$. The assumption $n(D_1^*) < n(D_0^*)$ would mean that D_1^* is finite and the dimension of $\mathscr{C}(D_1^*)$ is less than the dimension of $\mathscr{C}(D_0^*)$ (which is the image of $\mathscr{C}(D_1^*)$ under T) – a contradiction. Now assume (2) and denote by

$$B_1 = \{f : f \in \mathscr{C}(D_1^*),\ |f| < 1\}$$

the unit ball in $\mathscr{C}(D_1^*)$. By the Harnack principle, the image of B_1 under T is a relatively compact set $\mathrm{T}B_1$ in $\mathscr{C}(D_0^*)$. On the other hand, the Banach theorem assures that T is open, because it maps $\mathscr{C}(D_1^*)$ onto $\mathscr{C}(D_0^*)$. We conclude that the unit ball in $\mathscr{C}(D_0^*)$ is relatively compact and this implies $n(D_0^*) < \infty$.

2. Lemma. *If D is a regular domain, then $1 < n(D^*) < \infty$ and $\bar{D}$ is contained in a domain on which there exists a positive potential.*

Proof. Fix a regular domain D, $y \in D$ and another regular domain D_0 such that $y \in D_0$, $\bar{D}_0 \subset D$. Suppose that $n(D^*) = 1$. By preceding lemma also $n(D_0^*) = 1$, say $D_0^* = \{z\}$. Choose $x \in D \setminus \bar{D}_0$ and denote by C_x and C_y that component of $D \setminus \{z\}$ which contains x and y, respectively. The equality $C_x = C_y = C$ would mean that $C \cap D_0 = C \cap \bar{D}_0$ is open and closed in C and $y \in C \cap D_0$, $x \in C \setminus D_0$, which is a contradiction. We have thus

$$C_x \cap C_y = \emptyset\,, \quad z \in \bar{C}_x \cap \bar{C}_y\,.$$

Next choose a regular domain D_z such that $z \in D_z$, $\bar{D}_z \subset D \setminus \{x, y\}$. Then $C_x \cap D_z \neq \emptyset \neq C_y \cap D_z$ and $x \in C_x \setminus D_z$, $y \in C_y \setminus D_z$, so that the boundary of D_z must meet both C_x and C_y. Consequently, $n(D_z^*) \geqq 2 > n(D^*)$, which violates lemma 1. This contradiction proves the inequality $n(D^*) > 1$.

Since D^* contains at least two points, we may fix two strictly positive linearly independent functions $f_1, f_2 \in \mathscr{C}(D^*)$ and employ (C) to continue $\mathrm{H}_{f_1}^D$ and $\mathrm{H}_{f_2}^D$ harmonically onto the whole of X obtaining thus h_1 and h_2 in $\mathscr{H}(X)$, respectively. Both h_1 and h_2 being positive on $\bar{D}$ we may fix a domain $D_1 \supset \bar{D}$ such that h_1 and h_2 remain positive on D_1. Since h_1 and h_2 are non-proportional on D_1, we conclude that there is a positive potential on the harmonic space $(D_1, \mathrm{Rest}_{D_1} \mathscr{H})$ (= the restriction of the harmonic space $(X, \mathscr{H})$ to D_1). Applying proposition 7.1 of R. M. Hervé [4] (cf. p. 440) we get a regular domain $D_2 \subset D_1$ such that $\bar{D} \subset D_2$ which, by lemma 1, guarantees $n(D^*) < \infty$.

3. Lemma. *Let $D \neq \emptyset$ be a relatively compact domain, $F \in \mathscr{C}(\bar{D})$, $\mathrm{Rest}_D F \in \mathscr{H}(D)$ and suppose that the constant functions are harmonic on D. If real numbers u, v do not belong to $F(D^*)$ and satisfy the inequalities*

$$\min F(D^*) < u < v < \max F(D^*)\,,$$

then the system S of all components of

$$D_{uv} = \{z : z \in D,\ u < F(z) < v\}$$

is finite.

Proof. Denote by d_u the distance of u from $E_v = \{v\} \cup F(D^*)$. Similarly, let d_v denote the distance of v from $E_u = \{u\} \cup F(D^*)$. With each $x \in \bar{D}_{uv}$ we associate an open neighborhood D_x as follows. If $x \in D_{uv}$ then D_x is the component of D_{uv} containing x. If $x \in D^*_{uv}$ then D_x will be an open set containing x such that the diameter of $F(\bar{D} \cap D_x)$ is less than $\frac{1}{2} \min (d_u, d_v)$. The system

$$\{D_x; x \in \bar{D}_{uv}\} \tag{3}$$

must contain a finite subcover

$$D_{x_1}, \ldots, D_{x_p} \tag{4}$$

of the compact $\bar{D}_{uv}$. Suppose that there is a component C of D_{uv} such that $F(C^*) \cap \cap \{u, v\} = \emptyset$. Then C is closed in D and, consequently, $C = D = D_{uv}$, which is impossible, because the inequalities $\min F(D^*) < u, v < \max F(D^*)$ guarantee that D_{uv} is a proper subset of D.

We have thus

$$F(C^*) \cap \{u, v\} \neq \emptyset$$

for every $C \in S$. Consider now an arbitrary $C \in S$ and suppose, for definiteness, that $v \in F(C^*)$ (the case $u \in F(C^*)$ may be settled by a symmetric argument). Since $F(C) \subset F(D_{uv}) \subset \{a; a \in R^1, a < v\}$, F cannot be constant on C and the minimum principle together with the inclusions $F(C^*) \subset \{u, v\} \cup F(D^*)$ imply $F(C^*) \cap E_u \neq \emptyset$. C being connected we conclude that there is a $z \in C$ with

$$|F(z) - v| = \tfrac{1}{2} d_v \,.$$

If $x \in D^*_{uv} (\subset D^* \cup \{y; y \in D, F(y) = u \text{ or } F(y) = v\})$, then $F(x) \in \{v\} \cup E_u$ and $|F(x) - F(z)| \geqq \frac{1}{2} d_v$, so that $z \notin D_x$. We see that C is the only element of (3) containing z. Thus C must occur in (4) and $S \subset \{D_{x_1}, \ldots, D_{x_p}\}$.

4. Lemma. *Every regular domain (considered as a subspace of X) has a countable basis.*

Proof. Let D be a regular domain. Then there is a (strictly) positive $h_0 \in \mathscr{C}(\bar{D})$ which is harmonic on D. Employing the harmonic continuation (see (C)) we get an $h \in \mathscr{H}(X)$ with $\mathrm{Rest}_{\bar{D}}\, h = h_0$. There is a domain $D_1 \supset \bar{D}$ such that h remains positive on D_1. Passing from the Brelot space $(D_1, \mathrm{Rest}_{D_1} \mathscr{H})$ to the new space whose harmonic functions are obtained by the standard procedure of dividing the original harmonic functions by h, we get a connected Brelot space enjoying (C) on which constant functions are harmonic; besides that, D is again a regular domain in the new space. This consideration shows that we may assume for the proof of our lemma that the constant functions are harmonic on X. We know from lemma 2 that $D^* = \{x_1, \ldots, x_n\}$ is finite.

With each n-tuple of rational numbers $[r_1, \ldots, r_n] = r$ we associate an $F_r \in \mathscr{C}(\bar{D})$ which is harmonic on D and satisfies

$$F_r(x_j) = r_j \,, \quad 1 \leqq j \leqq n \,.$$

If, besides that, the rational numbers u, v satisfy the conditions

(5) $$\min_j r_j < u < v < \max_j r_j ,$$

(6) $$\{u, v\} \cap \{r_1, \ldots, r_n\} = \emptyset ,$$

then we denote by S^r_{uv} the system of all components of $\{z; z \in D, u < F_r(z) < v\}$. In view lemma 3, S^r_{uv} is finite, so that the system

$$S = \bigcup S^r_{uv}$$

(where $r = [r_1, \ldots, r_n]$ runs over all n-tuples of rational numbers and u, v run over all pairs of rational numbers satisfying the corresponding conditions (5), (6)) is countable. We are going to prove that S is a basis of D. Let z be an arbitrary point in D and let U be an arbitrary regular domain such that $z \in U \subset \bar{U} \subset D$. According to lemmas 1 and 2, $U^* = \{y_1, \ldots, y_s\}$, where $2 \leqq s \leqq n$. Define $g \in \mathscr{C}(U^*)$ by

$$g(y_1) = 1 , \quad g(y_k) = 0 \quad \text{for} \quad 2 \leqq k \leqq s .$$

Then

$$0 < H^U_g(z) < 1 ,$$

because constants are harmonic on $D \supset \bar{U}$. Fix $\varepsilon > 0$ small enough to secure

$$2\varepsilon < H^U_g(z) < 1 - 2\varepsilon$$

and apply harmonic continuation to get an $h \in \mathscr{C}(\bar{D})$ with $\mathrm{Rest}_D\, h \in \mathscr{H}(D)$ and $\mathrm{Rest}_U\, h = H^U_g$. Noting that $h = H^D_h$ on $\bar{D}$ and making use of the fact that the values attained by H^D_f at the points $y_1, \ldots, y_s$, z depend continuously on $f \in \mathscr{C}(D^*)$, we choose rational numbers r_j approximating the values $h(x_j)$ $(1 \leqq j \leqq n)$ in such a way that the following inequalities hold for F_r corresponding to $r = [r_1, \ldots, r_n]$:

$$|F_r(z) - H^U_g(z)| < \varepsilon , \quad |F_r(y_k) - g(y_k)| < \varepsilon , \quad 1 \leqq k \leqq s .$$

Then

(7) $$F_r(y_1) > 1 - \varepsilon > F_r(z) > \varepsilon > \max \{F_r(y_k); 2 \leqq k \leqq s\} .$$

Further choose rational numbers u, v satisfying (6) and

(8) $$\varepsilon < u < F_r(z) < v < 1 - \varepsilon ,$$

so that $u, v \in F_r(\bar{D}) = \{a; a \in R^1, \min_j r_j \leqq a \leqq \max_j r_j\}$. Let C be the component of $\{w; w \in D, u < F_r(w) < v\}$ containing z. In view of (7), (8), $F_r(U^*)$ does not meet $F_r(C) \subset \{a; a \in R^1, u < a < v\}$. Consequently, $U^* \cap C = \emptyset$ and $C \subset U$, because $z \in C \cap U$. We have thus found a $C \in S$ with $z \in C \subset U$, which shows that S is a basis.

5. Lemma. *If D_1, D_2 are arbitrary domains contained in a regular domain, then*

$$D_1 \subset D_2 \Rightarrow n(D_1^*) \leqq n(D_2^*)\,.$$

Proof. Suppose that $n(D_1^*) > n(D_2^*)$ for a couple of domains $D_1 \subset D_2$ contained in a regular domain D. Let $D_2^* = \{z_1, \ldots, z_s\}$, choose an $(s + 1)$-tuple of points $x_1, \ldots, x_{s+1} \in D_1^*$ and associate with every i a connected neighborhood V_i of x_i such that $V_1, \ldots, V_{s+1}$ are mutually disjoint. Further choose $y_i \in V_i \cap D_1$ $(i = 1, \ldots, s + 1)$ and consider the compact $K = \{y_1, \ldots, y_{s+1}\}$. By lemma 2, $\bar{D}$ is contained in a Brelot space carrying a positive potential. This permits us to apply proposition 7.1 of R. M. Hervé [4] guaranteeing the existence of a regular domain D_0 with $K \subset D_0$, $\bar{D}_0 \subset D_1$. In view of lemma 2, $n(D_0^*) < \infty$. Since every V_i meets both D_0 (note that $y_i \in V_i \cap D_0$) and its complement (note that $x_i \in V_i \setminus D_1$), we conclude that $V_i \cap D^* \neq \emptyset$ so that D_0^* must contain at lest $s + 1$ different points $u_1, \ldots, u_{s+1}$.

Define $f_i \in \mathscr{C}(D_0^*)$ by

$$f_i(u_i) = 1\,, \quad f_i(D_0^* - \{u_i\}) = \{0\}$$

and apply harmonic continuation (see (C)) to $H_{f_i}^{D_0}$ so as to obtain an $h_i \in \mathscr{H}(X)$ with $\mathrm{Rest}_{D_0^*} h_i = f_i$ $(i = 1, \ldots, s + 1)$. Since D_2^* contains only s elements, we may fix real constants $a_1, \ldots, a_{s+1}$, not all zero, such that

$$h = a_1 h_1 + \ldots + a_{s+1} h_{s+1}$$

vanishes identically on D_2^*. By the minimum principle (which is applicable, because $D_2 \subset D$ and D is regular) we conclude that $h = 0$ on D_2. In particular, $0 = h(u_i) = a_i$ $(i = 1, \ldots, s + 1)$, which is a contradiction.

Now we are in position to prove the following

6. Proposition. *If the space X is connected and the Brelot space $(X, \mathscr{H})$ satisfies (C), then every $x \in X$ has a neighborhood of the form $\bigcup_{i=1}^{n} C_i$, where $n \geqq 2$ and $C_1, \ldots, C_n$ are arcs in X (whose number depends on the choice of $x \in X$) such that*

$$C_i \cap C_j = \{x\} \quad \textit{whenever} \quad 1 \leqq i < j \leqq n\,. \tag{9}$$

Proof. Consider an arbitrary point $x \in X$ and fix a regular domain $D_1 \ni x$. It follows easily from lemma 5 that there is a regular domain D with $x \in D \subset \bar{D} \subset D_1$ such that $n(D_0^*) = n(D^*)$ for every domain D_0 satisfying $x \in D_0 \subset D$. In view of lemma 4, $\bar{D}$ is a metrizable continuum. Let y be an arbitrary point in D^* and let D_2 be an arbitrary regular domain containing y. Since $n(D^*) + n(D_2^*) < \infty$, the boundary of $\bar{D} \cap D_2$ in the space $\bar{D}$ is finite. We see that y has in $\bar{D}$ arbitrarily small neighbourhoods with a finite boundary, whence it follows (see [6], p. 209) that $\bar{D}$ is locally connected at y. Thus $\bar{D}$ is a locally connected metrizable continuum such that every sufficiently small neighborhood of x in $\bar{D}$ has at least $n = n(D^*) \geqq 2$ boundary points. Employing the so-called "n-Beinsatz" of K. MENGER (cf. [6], p. 203) we conclude that there are arcs $C_1, \ldots, C_n$ in $\bar{D}$ satisfying (9). Denote by y_i the end-point of C_i different from x and choose a domain $U \subset D$ containing x such that

$$U \cap \{y_1, \ldots, y_n\} = \emptyset .$$

Order C_i naturally from x to y_i and denote by x_i the first point on C_i belonging to $C_i \setminus U$ $(i = 1, \ldots, n)$. Assuming $U \setminus \bigcup_{i=1}^{n} C_i \neq \emptyset$ we fix $x_0 \in U \setminus \bigcup_{i=1}^{n} C_i$ and choose an arc C_0 connecting x and x_0 in U; this is possible, because U is arc-wise connected (see [6], § 45, pp. 182, 184). Let $\hat{U}$ be the component of $U \setminus \{x_0\}$ containing x and denote by $\hat{C}_j$ the component of $C_j \setminus \{x_j\}$ containing x $(0 \leqq j \leqq n)$. Then

$$\bigcup_{j=0}^{n} \hat{C}_j \subset \hat{U}$$

and $\{x_0, x_1, \ldots, x_n\} \subset \hat{U}^*$, which contradicts lemma 5, because the domain $\hat{U} \subset D$ cannot have more than n boundary points. Thus $U \subset \bigcup_{i=1}^{n} C_i$ and $\bigcup_{i=1}^{n} C_i$ is a neighborhood of x.

Now it is easy to present a proof of the theorem. Applying proposition 6 locally one immediately obtains the "only if" part of the theorem. In order to prove the "if" part of the theorem consider an arbitrary point $x \in X$ and fix the arcs $C_1, \ldots, C_n$ satisfying (9) such that $\bigcup_{i=1}^{n} C_i$ is a neighborhood of x. We may clearly suppose that the interior D of $\bigcup_{i=1}^{n} C_i$ is a regular domain; the proof will be complete if we show that every $h_0 \in \mathscr{H}(D_0)$ defined on a subdomain $D_0 \supset D$ can be harmonically continued so as to yield an $h \in \mathscr{H}(D)$. This is celar if $x \in D_0$, because then $\tilde{C}_i = C_i \cap D \setminus \{x\}$ are one-dimensional manifolds and, by [5] (see lemma 1.21), h_0 can be continued harmonically from $C_i \cap D_0 \setminus \{x\}$ onto $\tilde{C}_i$ for $i = 1, \ldots, n$. If $x \notin D_0$, then D_0 can meet only one of the arcs, say C_1, and we may continue h_0 harmonically onto $\tilde{C}_1$. Let $C_i \setminus D = \{x_i\}$ $(i = 1, \ldots, n)$ and define $f_0, f_1 \in \mathscr{C}(D^*)$ by

$$f_1(D^*) = \{1\} = f_0(D^* \setminus \{x_1\}) , \quad f_0(x_1) = 0 .$$

Then $H^D_{f_0}(x) > 0$ and $H^D_{f_0}, H^D_{f_1}$ are easily seen to be linearly independent on $\tilde{C}_1$. Consequently, one may choose real constants a_1, a_2 such that $h_0 = a_1 H^D_{f_0} + a_2 H^D_{f_1}$ on C_1 (see [5], lemma 1.6) and $a_1 H^D_{f_0} + a_2 H^D_{f_1}$ yields the required extension of h_0.

Corollary. *In order that a Brelot space $(X, \mathscr{H})$ possess the following property*