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ON SOME GRAPH-THEORETICAL PROBLEMS OF V. G. VIZING

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In [2] V. G. VIZING suggests a number of unsolved graph-theoretical problems. Here we shall solve partially two of them.

I.

The first problem we shall investigate is the following one:

Which is the maximal number of edges that a graph with n vertices and with a given Hadwiger number can have?

Here this problem is solved for Hadwiger number 3.

We say that a graph G can be contracted onto a graph H if and only if the graph H can be obtained from G by a finite number of the following operations:

- (a) deleting an edge;
- (b) deleting an isolated vertex;
- (c) identifying two neighbouring vertices, i.e. replacing of two neighbouring vertices x and y by a new vertex neighbouring to exactly all vertices which were neighbouring to at least one of the vertices x and y .

We consider only finite undirected graphs without loops and multiple edges.

The Hadwiger number $\eta(G)$ of a graph G is the maximal number of vertices of a complete graph onto which G can be contracted.

By $\lambda_k(n)$ for any positive integer n we shall denote the maximal number of edges of a graph of Hadwiger number k with n vertices.

The graphs with Hadwiger number 3 are graphs which can be contracted onto a triangle, but not onto a complete graph with four vertices.

If G is a graph, C a circuit in G , then a diagonal arc of C in G is an arc joining two vertices of C whose internal vertices do not belong to C . Two vertex-disjoint diagonal arcs P_1 and P_2 of C will be called topologically crossing if and only if the circuit C and the arcs P_1 and P_2 cannot be drawn in the plane so that the arcs P_1 and P_2 might

be drawn in the interior of the drawing of C without crossing each other. (This term is defined only for the use of this paper.)

The following lemma is evident.

Lemma 1. *A graph G has Hadwiger number not exceeding 3 if and only if no circuit C in G has two vertex-disjoint topologically crossing diagonal arcs.*

We shall prove another lemma.

Lemma 2. *For every positive integer $n \geq 2$ any graph of Hadwiger number 3 with n vertices and maximal possible number of edges is connected and without articulations.*

Proof. Assume that such a graph G is disconnected. Then we join two vertices of different connected components of G by an edge e ; the graph thus obtained will be denoted by G' . The edge e is a bridge in G' , therefore it belongs neither to a circuit, nor to a diagonal arc of some circuit in G' . This means that all circuits and their diagonal arcs in G' are those of G and G' has also Hadwiger number 3, which is a contradiction with the maximality of G .

Now assume that G is connected and contains an articulation a . Let L_1, L_2 be two lobes whose common vertex is a . Let u_1 and u_2 be vertices of L_1 and L_2 respectively joined by an edge with a . Let G'' be a graph obtained from G by adjoining an edge h joining u_1 and u_2 . Let C be a circuit in L_1 , let P be a diagonal arc of C in G'' not contained in G and joining the vertices v_1 and v_2 of C . Then P consists of an arc P_1 from v_1 (or v_2) to a , an arc P_2 from a to u_2 , the edge h and an arc P_3 from u_1 to v_2 (or v_1). Then there exists an arc P' in G joining v_1 and v_2 and consisting of the path P_1 , the edge au_1 and the path P_3 . This is either a diagonal arc of C , or an edge of C (if the vertices v_1, v_2 are identical with u_1, a). If some other diagonal arc P'' of C vertex-disjoint with P forms together with P a pair of topologically crossing diagonal arcs of C in G'' , then P'' is in L_1 , because it contains neither a nor h . This means that P'' forms a pair of topologically crossing diagonal arcs of C also with P' and this pair is also in G , which is a contradiction. Analogously we can consider any circuit in L_2 . A circuit in G which is neither in L_1 nor in L_2 evidently cannot have a diagonal arc in G'' not contained in G . We have proved that by adjoining the edge h no pair of topologically crossing diagonal arcs of any circuit of G is obtained. Now consider a circuit C' in G'' not contained in G . Evidently it consists of an arc R_1 from a to u_1 in L_1 , the edge h and an arc R_2 from u_2 to a in L_2 . Any diagonal arc of C' joins either two vertices of R_1 , or two vertices of R_2 . As L_1 is a lobe, there exists an arc R'_1 joining a and u_1 in L_1 and having no vertex in common with R_1 except for a and u_1 . The arcs R_1, R'_1 form a circuit C_1 in L_1 ; any diagonal arc of C joining two vertices of R_1 is also a diagonal arc of C_1 and any two such arcs which would be topologically crossing in G'' would be topologically crossing also in G . Analogously for R_2 . Finally, a diagonal arc of C joining two vertices of R_1 and a diagonal arc of C joining two

vertices of R_2 cannot evidently be topologically crossing. Therefore G'' has also Hadwiger number 3, which is a contradiction with the maximality of G .

Lemma 3. *Let G be a graph of Hadwiger number 3 with n vertices. Let u be its vertex and G_0 the graph obtained from G by deleting u . Let G_0 be a connected graph with p lobes. Then u is joined in G at most with $p + 1$ vertices.*

Proof. First assume that three vertices of a lobe L of G_0 are joined with u in G . Then there exists a circuit in this lobe containing all of them; it can be contracted onto a triangle, whose vertices are these three vertices. This triangle together with u and the edges joining u with its vertices form a complete graph with four vertices and $\eta(G_0) \geq 4$, which is a contradiction. Therefore u can be joined at most with two vertices of the same lobe. Now let u be joined with two vertices v_1, v_2 of a lobe L , none of which is a cut-vertex. If there exists at least one vertex v_3 in G_0 which is joined with u in G and different from v_1 and v_2 , then let a be the cut-vertex belonging to L and separating v_3 from v_1 and v_2 . In L there exists a circuit containing v_1, v_2 and a . The subgraph of G consisting of this circuit, of the edges uv_1, uv_2, uv_3 and of an arc joining a and v_3 in G_0 (none of whose edges is in L) can be contracted onto a complete graph with four vertices. Therefore if G_0 has cut-vertices, at most three vertices of G_0 are joined with u in G and two of them belong to one lobe, not being cut-vertices, the graph G has not the assumed property. We shall continue by induction with respect to p . For $p = 1$ the assertion holds, because G_0 consists of one lobe and we have proved that no three vertices of one lobe can be joined with u in G . Let $r \geq 2$, let the assertion hold for $p < r$. If we delete one lobe L except for the cut-vertices in it from G_0 so that the resulting graph G_1 is connected (this is always possible), then G_1 has $r - 1$ lobes and u is joined in G with at most r vertices of G_1 . Now at most one vertex of L which is no cut-vertex may be joined with u . The lobe L contains only one cut-vertex which is in G_1 (because it is a common vertex of L and some other lobe), thus at most $r + 1$ vertices of G_0 can be joined in G with u .

Now we shall prove

Theorem 1. *Let $\lambda_k(n)$ be the maximal number of edges of a graph G of Hadwiger number k with n vertices. Then*

$$\lambda_3(n) = 2n - 3$$

for any positive integer $n \geq 2$.

Proof. We shall prove the assertion by induction. The graphs with two or three vertices evidently cannot be contracted onto a complete graph with four vertices. The maximal number of edges of a graph with $n = 2$ vertices is $2n - 3 = 1$, the maximal number of edges of a graph with $n = 3$ is $2n - 3 = 3$. For $n = 4$ only the complete graph with 4 vertices has Hadwiger number 4, no other can be contracted onto it.

Thus the graph of Hadwiger number 3 with four vertices and the maximal possible number of edges is the graph obtained from the complete graph with four vertices by deleting one edge. Now let $n = r \geq 5$ and let the assertion hold for $2 \leq n < k$. Let G be a graph with k vertices and $\lambda_3(r)$ edges for which $\eta(G) = 3$. Delete one vertex u from G and denote the obtained graph by G_0 . According to Lemma 2 G_0 is connected. According to Lemma 3 the number of vertices of G_0 joined by edges with u in G is at most $p + 1$, where p is the number of lobes of G_0 . Let the lobes of G_0 be $L_1, \dots, L_p$, let l_i be the number of vertices of L_i for $i = 1, \dots, p$. For the number $r - 1$ of vertices of G_0 we have

$$(1) \quad r - 1 = \sum_{i=1}^p l_i - p + 1.$$

Any lobe of G_0 is a graph with Hadwiger number not exceeding 3 (because this property is evidently hereditary). According to the induction assumption the number of edges of L_i does not exceed $2l_i - 3$ for $i = 1, \dots, p$. For the number m_0 of edges of G_0 we have

$$m_0 \leq \sum_{i=1}^p (2l_i - 3) = 2 \sum_{i=1}^p l_i - 3p.$$

As u is joined with not more than $p + 1$ vertices of G_0 , for the number m of edges of G we have

$$m \leq m_0 + p + 1 \leq 2 \sum_{i=1}^p l_i - 2p + 1.$$

From (1) we have

$$\sum_{i=1}^p l_i = r + p - 2,$$

therefore

$$m \leq 2r - 3.$$

We have proved that $2n - 3$ is the upper bound for the number of edges of a graph with Hadwiger number 3 with n vertices. It remains to prove that for every $n \geq 2$ this bound is attained. For any given $n \geq 2$ we construct the "fan graph" F_n as follows. The vertices of F_n are $v_1, \dots, v_n$ and its edges are $v_i v_{i+1}$ for $i = 1, \dots, n - 1$ and $v_1 v_j$ for $j = 3, \dots, n$. If $n > 2$, a contraction of any edge leads either to F_{n-1} , or to the graph with two lobes isomorphic to F_r with $2 \leq r < n$. If $n = 2$, then F_2 is a graph consisting of two vertices and one edge. Thus by induction one can prove that F_n cannot be contracted onto a complete graph with four vertices, q.e.d.

In the end we shall consider also $\lambda_1(n)$ and $\lambda_2(n)$. Any graph containing at least one edge can be contracted onto a complete graph with two vertices. Thus $\eta(G) = 1$ if and only if G contains no edges and

$$\lambda_1(n) = 0.$$

Any graph containing at least one circuit can be contracted onto a complete graph with three vertices. Thus $\eta(G) = 2$ if and only if G is a forest with at least in edge and

$$\lambda_2(n) = n - 1.$$

Comparing $\lambda_1(n)$, $\lambda_2(n)$, $\lambda_3(n)$ leads us to a conjecture.

Conjecture. For the maximal number $\lambda_k(n)$ of edges of a graph of Hadwiger number k with n vertices we have

$$\lambda_k(n) = (k - 1)n - \binom{k}{2}$$

for any two positive integers $k, n \geq 2$.

II.

The other problem which will be studied here is the following one:

Which is the maximal number of edges of a connected undirected graph with n vertices, none of whose spanning trees has more than k terminal edges?

We shall denote this number by $\tau(n, k)$. We shall give the solution for some special cases, namely for $k = 2, k = 3, k = n - 3, k = n - 2, k = n - 1$. We study graphs without loops and multiple edges.

Evidently we can define neither $\tau(n, 1)$ nor $\tau(n, n)$, because a spanning tree of a graph with n vertices has at least two and at most $n - 1$ terminal edges.

Before investigating concrete values of k , we shall introduce an auxiliary concept.

If G_0 is a connected subgraph of G , then the degree of G_0 in G is the number of vertices of G not belonging to G_0 which are joined with a vertex of G_0 . If G_0 consists only of one vertex, its degree is equal to the degree of this vertex.

Now we shall prove a lemma.

Lemma 4. *Let G be a connected undirected graph. Then the maximal number of terminal edges of a spanning tree of G is equal to the maximal degree of a connected subgraph of G .*

Proof. Let G_0 be a connected subgraph of G with the maximal degree k . Let $u_1, \dots, u_k$ be the vertices not belonging to G_0 and joined by edges with vertices of G_0 . Choose a spanning tree T_0 of G_0 . Then for any $i = 1, \dots, k$ choose an edge e_i joining u_i with a vertex of G_0 . The graph T'_0 consisting of all vertices of G_0 , vertices $u_1, \dots, u_k$, all edges of G_0 and all edges $e_1, \dots, e_k$ is a tree in which $e_1, \dots, e_k$ are terminal edges. This tree T'_0 is a subtree of a spanning tree T of G which has also at least k terminal edges. (Evidently the number of terminal edges of a subtree of a tree T is less than or equal to the number of terminal edges of T .) On the other hand, let l

be the maximal number of terminal edges of a spanning tree of G . Let T_1 be a spanning tree of G with l terminal edges. Let G_1 be the subgraph of G generated by all vertices which are not terminal in T_1 . Then G_1 has the degree l .

Now we shall prove theorems on the numbers $\tau(n, k)$.

Theorem 2. $\tau(n, 2) = n$ for every $n \geq 3$.

This assertion is evident; we leave the proof to the reader.

Theorem 3. $\tau(n, 3) = n + 2$ for every $n \geq 4$.

Proof. Let G be a graph with n vertices ($n \geq 4$) such that none of its spanning trees has more than three vertices. At first assume that G has a Hamiltonian circuit C consisting of the vertices $u_1, \dots, u_n$ and the edges $u_i u_{i+1}$ for $i = 1, \dots, n-1$ and $u_n u_1$. Assume that there exists an edge $u_i u_j$ where $|i - j| \geq 3$ (the difference is taken modulo n). Without any loss of generality let $i = 1$; then $j \neq 2, j \neq 3, j \neq n-1, j \neq n$. Let T_0 be a subgraph of G consisting of the vertices $u_1, u_2, u_{j-1}, u_j, u_{j+1}, u_n$ and of the edges $u_1 u_2, u_1 u_n, u_1 u_j, u_{j-1} u_j, u_j u_{j+1}$; it is a tree in which all edges except $u_1 u_j$ are terminal, therefore with four terminal edges. The tree T_0 is a subtree of some spanning tree T of G which has at least four vertices, which is a contradiction. Therefore any edge not belonging to C is $u_i u_{i+2}$ for some $i, 1 \leq i \leq n$ (the sum $i + 2$ is taken modulo n). Let there exist an edge $u_1 u_3$ (without any loss of generality) and some other edge $u_j u_{j+2}$ (where $j \neq 1$). Evidently $j \neq 3, j \neq n-1$, because otherwise u_j or u_{j+2} would have the degree at least four. Assume $4 \leq j \leq n-2$. Then there exists a subgraph T_1 of G consisting of the vertices $u_1, \dots, u_{j+2}$ and of the edges $u_1 u_3, u_j u_{j+2}$ and $u_i u_{i+1}$ for $i = 2, \dots, j$. It is a tree with four terminal edges $u_1 u_3, u_2 u_3, u_j u_{j+1}, u_j u_{j+2}$ and we obtain a similar contradiction as in the preceding case. Therefore an edge of G not belonging to C and different from $u_1 u_3$ can be only $u_2 u_4$ or $u_n u_2$; they cannot exist both, because u_2 would have the degree at least four. Therefore G has at most $n + 2$ edges. Now assume that G has no Hamiltonian circuit. Let C_0 be the circuit of the maximal length l in G , let its vertices be $v_1, \dots, v_l$ and its edges $v_i v_{i+1}$ for $i = 1, \dots, l-1$ and $v_l v_1$. Let there exist two vertices w_1, w_2 not belonging to C and joined by edges with vertices of C . If the length of C is at least 5, we can choose an edge e of C such that w_1 and w_2 are joined with the vertices v_i, v_j which are consequently not incident with e . The tree whose edges are all edges of C except e and $v_i w_1, v_j w_2$ (we may have $v_i = v_j$) is a subtree of G with four terminal edges. Thus if the length of C is at least 5, there may exist only one vertex w not belonging to C and joined with a vertex of C . For the edges joining two vertices of C and not belonging to C the same holds as in the case of a Hamiltonian circuit. So assume that there are two such edges; let one of them (without any loss of generality) be $v_1 v_3$ and the other $v_l v_2$. There exist two subtrees of G with three terminal edges not containing w , namely T_1 with the edges $v_i v_{i+1}$ for $i = 2, \dots, l-1$ and $v_1 v_3$ and T_2

with the edges $v_i v_{i+1}$ for $i = 3, \dots, l-1$, $v_l v_1, v_l v_2$. If w is joined with some v_i , where $4 \leq i \leq l-1$, then by adding the vertex w and the edge $u_i w$ to T_1 or to T_2 we obtain a tree with four terminal edges. If w is joined with v_3 or v_l , then by adding w and $v_3 w$ or $v_l w$ to T_1 or T_2 respectively we obtain also a tree with four terminal edges. If w is joined with v_1 or v_2 , then v_1 or v_2 has the degree at least four. We have proved that if there are two edges joining vertices of C and not belonging to C (for C of the length at least 5), then C must be a Hamiltonian circuit of G . Now assume that there exists one edge joining two vertices of C and not belonging to C ; analogously to the case when C is Hamiltonian this edge is (without any loss of generality) $v_1 v_3$. Then there exist two subtrees of G not containing vertices outside of C with three terminal edges, namely T_1 with the edges $v_i v_{i+1}$ for $i = 4, \dots, l-1$, $v_l v_1, v_l v_2, v_1 v_3$ and T_2 with the edges $v_i v_{i+1}$ for $i = 2, \dots, l-1$, $v_1 v_3$. If w is joined with v_i for $4 \leq i \leq l-1$, then by adding w and $v_i w$ to T_1 or T_2 we obtain a tree with four terminal edges. If w is joined with v_1 or v_3 , then by adding w and $v_1 w$ or $v_3 w$ to T_1 or T_2 respectively we obtain also a tree with four terminal edges. Thus w can be joined only with u_2 . If there are two vertices x_1, x_2 joined with w and not belonging to C , then by adding the edges $v_2 w, w x_1, w x_2$ to T_1 or T_2 we obtain again a tree with four terminal edges. Thus w can be joined only with one vertex w_1 not belonging to C ; analogously w_1 can be joined only with one vertex w_2 not belonging to C and different from w etc.; therefore the subgraph of G generated by v_2 and all vertices not belonging to C is an arc. We have proved that the subgraph generated by the vertices of C has at most $l+2$ edges, if C is Hamiltonian, or at most $l+1$ edges, if there are some vertices not belonging to C . In the former case C is Hamiltonian and $l = n$, thus $l+2 = n+2$. In the latter case the number of vertices not belonging to C is $n-l$ and, as they generate an arc, the number of edges joining vertices not belonging to C is $n-l-1$ and there is one edge joining a vertex not belonging to C , namely w , with a vertex of C , namely v_2 . The total number of edges of G is at most $n+2$. From the proof it follows that this bound can be always attained. It remains to discuss the case when the length of the longest circuit in G is less than 5. If it is 3, then any circuit of G is a lobe of G , therefore any lobe of G is either a triangle, or a bridge. Assume that two lobes L_1, L_2 of G are triangles. If they have a common vertex, it has the degree at least 4, which is impossible. Otherwise we take an arc joining a vertex v_1 of L_1 with a vertex v_2 of L_2 and having no edge in common with L_1 and L_2 . The tree consisting of this arc, of two edges from L_1 incident with v_1 and of two edges of L_2 incident with v_2 has four terminal edges, namely the edges of L_1 and L_2 incident with v_1 or v_2 . Therefore G can have at most one lobe which is a triangle, the others being bridges. The cyclomatic number of G is at most 1, thus G has at most n edges. If the length of the longest circuit in G is 4, then any lobe of G is either a bridge or a triangle, or it consists of a system of at least two edge-disjoint arcs of the lengths 1 or 2 joining two vertices a and b . Analogously to the preceding case we can prove that there is at most one lobe which is not a bridge. According to the assumption it cannot be a triangle, thus it is of the last type. The number of paths joining a and b

can be at most three, otherwise a and b would have the degree greater than three. If they are two or three, the cyclomatic number of G is 1 or 2 respectively, and the number of vertices of G is n or $n + 1$, respectively.

Theorem 4. $\tau(n, n - 3) = \frac{1}{2}n^2 - \frac{3}{2}n + 5$ for every $n \geq 5$.

Proof. Let G be a graph with n vertices ($n \geq 5$) such that none of its spanning trees has more than $n - 3$ terminal edges. Investigate the complement $\bar{G}$ of G . The graph $\bar{G}$ has the following properties:

- (a) the degree of any vertex of $\bar{G}$ is at least two;
- (b) the diameter of $\bar{G}$ is at most two;
- (c) the complement G of $\bar{G}$ is connected.

If $\bar{G}$ had not the property (a), there would exist some vertex u of $\bar{G}$ of the degree 0 or 1. This vertex would have the degree $n - 1$ or $n - 2$ in G , therefore the star with the center u would be a subtree of G with more than $n - 3$ terminal edges. If $\bar{G}$ had not the property (b), then there would exist two vertices u_1, u_2 of $\bar{G}$ with the distance greater than two. There would not exist any vertex joined with both u_1 and u_2 and these two vertices also would not be joined together. This means that in G any vertex would be joined at least with one of the vertices u_1, u_2 and there would exist the edge u_1u_2 . For any vertex of G different from u_1 and u_2 we choose one edge joining it with u_1 or u_2 ; these edges together with u_1u_2 would form a spanning tree of G with $n - 2$ terminal edges. The condition (c) follows from the text of the problem, because only connected graphs have spanning trees.

We can construct a graph G_0 satisfying the conditions (a), (b), (c) and having $2n - 5$ edges. This is the graph whose vertex set is $u_1, u_2, v_1, \dots, v_{n-4}, w_1, w_2$ and whose edges are u_1v_i and u_2v_i for $i = 1, \dots, n - 4$, and further u_1w_1, w_1w_2, u_2w_2 . This graph G_0 contains n vertices and $2n - 5$ edges. We shall prove that there does not exist any graph with less than $2n - 5$ edges satisfying the conditions (a), (b), (c). Assume that there exist a graph G_1 with n vertices and less than $2n - 5$ edges ($n \geq 5$) satisfying the conditions. At least one of the vertices of G_1 must have the degree less than four; in the opposite case G_1 would contain at least $2n$ edges. Thus also the vertex connectivity degree of G_1 is at most 3. Let R be a cut set of G_1 with the minimal number of vertices. At first assume that $|R| = 1$, thus $R = \{a\}$, where a is some cut vertex. If u, v are two vertices of G_1 separated by a , then they must be both joined with a , because their distance cannot be greater than two and any arc joining them must contain a . As these vertices were chosen arbitrarily, this implies that a is joined with all other vertices of G_1 . Then a is joined with no other vertex in the complement of G_1 and is an isolated vertex; therefore this complement is not connected, which contradicts (c). Assume $|R| = 2$, thus $R = \{a_1, a_2\}$. Let $K_1, \dots, K_t$ be the connected components of the graph obtained from G_1 by deleting the vertex set R and all edges incident to it. Assume that in K_1 (without any loss of generality)

there exists a vertex u_1 joined with a_1 and not with a_2 and a vertex u_2 joined with a_2 and not with a_1 . Let v be a vertex of some K_i for $i \neq 1$. It must have the distance at most 2 from both u_1 and u_2 , therefore it must be joined with both a_1 and a_2 . As v was chosen quite arbitrarily, any vertex of $\bigcup_{i=2}^l K_i$ must be joined with both a_1 and a_2 .

Let m be the total number of vertices of $\bigcup_{i=2}^l K_i$; then the number of edges not incident with vertices of K_1 is at least $2m$. The component K_1 contains $n - m - 2$ vertices. It must be connected, thus it contains at least $n - m - 3$ edges. Each vertex of K_1 must be joined with some vertex of R , therefore there are at least $n - m - 2$ edges joining vertices of K_1 with vertices of R . The graph G_1 has then at least $2m + (n - m - 2) + (n - m - 3) = 2n - 5$ edges. Now assume that in K_1 there is a vertex u_1 joined with a_1 and not with a_2 , but all vertices of K_1 are joined with a_1 . Then in K_i for each $i = 2, \dots, l$ also all vertices are joined with a_1 and there may also exist in it some vertices joined with a_1 and not with a_2 . Let M be the set of vertices of G_1 not belonging to R joined with a_1 and not joined with a_2 . Let M_i for $i = 1, \dots, l$ be the intersection of M with the vertex set of K_i . Consider a connected component of the subgraph of G_1 generated by the set M_i ; let p be its number of vertices. As this component C is connected, it contains at least $p - 1$ edges. As any of its vertices is joined with a_1 , we have further p edges incident with vertices of this component. This component C is a subgraph of some K_i and evidently a proper subgraph; otherwise no vertex of K_i would be joined with a_2 and a_1 would be a cut vertex separating vertices of K_i from other vertices of G_1 . Therefore there exists at least one edge joining a vertex of C with some other vertex of K_i . We have at least $2p$ edges incident with vertices of C and with no other vertices of M . Therefore if $|M| = q$, then there exist $2q$ edges incident with vertices of M (this number was obtained as a sum over all such components C). Any of the vertices not belonging to $M \cup R$ are joined with both a_1 and a_2 . As the number of vertices not belonging to $M \cup R$ is $n - q - 2$, we have $2n - 2q - 4$ edges joining these vertices with the vertices of R . Thus G_1 has at least $2q + (2n - 2q - 4) = 2n - 4$ edges. If all vertices not belonging to R are joined with both a_1 and a_2 , the graph G_1 has evidently also at least $2n - 4$ edges.

Finally assume that $|R| = 3$, thus $R = \{a_1, a_2, a_3\}$. We shall prove that in each of the components $K_1, \dots, K_l$, except at most one, either there exists a vertex joined with all vertices of R , or there exist two vertices, each of which is joined with two vertices of R . Assume that K_1 has not this property; i.e. that at most one vertex of K_1 is joined with two vertices of R , any other vertex being joined exactly with one vertex of R . If each vertex of K_1 is joined only with one vertex of R , there must exist three vertices u_1, u_2, u_3 of K_1 so that u_i is joined with a_i for $i = 1, 2, 3$, and with no other vertex of R (otherwise the vertex connectivity degree of G_1 would be less than three). Any vertex of K_i for $i = 2, \dots, l$ must have the distance at most two from all three vertices u_1, u_2, u_3 , therefore it must be joined with all the vertices a_1, a_2, a_3 . If there

exists a vertex v of K_1 joined with two vertices a_1, a_2 (without any loss of generality) of R and not with a_3 and all other vertices are joined only with one vertex of R each, then there exists a vertex u_3 of K_1 joined with a_3 and with no other vertex of R . Any vertex of K_i ($i = 2, \dots, l$) must have the distance from both v and u_3 at most 2, therefore it must be joined with a_3 and one of the vertices a_1, a_2 . If this K_i contains only one vertex, it must be joined with all vertices of R , because we have assumed that the vertex connectivity degree of G_1 is 3 and therefore each vertex has the degree at least 3. If K_i contains two different vertices w_1, w_2 , any of them must be joined with a_3 and one of the vertices a_1, a_2 . Any of the components K_i ($i = 1, \dots, l$) must contain at least $k_i - 1$ edges, where k_i is the number of its vertices, and there are at least k_i edges joining its vertices with vertices of R ; therefore there are at least $2k_i - 1$ edges incident with vertices of K_i . But if for some K_i this number is exactly $2k_i - 1$, this means that any vertex of K_i is joined exactly with one vertex of R ; then any vertex of K_j for $j \neq i$ is joined with all vertices of R . Then the graph G_1 contains at least $3(n - k_i - 3) + 2k_i - 1 = 3n + k_i - 10$ vertices, which is more than $2n - 5$, because $n \geq 5$. If exactly one vertex of K_i is joined with two vertices of R and any other vertex of K_i is joined only with one vertex of R , then there are at least $2k_i$ edges incident with vertices of K_i and any vertex of K_j for $j \neq i$ must be joined at least with two vertices of R ; if such K_j consists only of one vertex, it is joined with all vertices of R , otherwise there exists at least one edge of K_j . Thus there are at least $2k_j + 1$ edges incident with vertices of K_j for $j \neq i$ (k_j is the number of vertices of K_j) and the total number of edges of G_1 is at least $2n - 5$. If in each K_i either there are two vertices joined with two vertices of R , or there is a vertex joined with all vertices of R , then there are $2k_i + 1$ edges incident with vertices of K_i and G_1 has at least $2n - 4$ edges. We have proved that there does not exist any graph satisfying (a), (b), (c) and having less than $2n - 5$ edges. The existence of such a graph with exactly $2n - 5$ edges had been proved before. The graph G with the property that none of its spanning trees has more than $n - 3$ terminal edges and with the maximal possible number of edges is a complement of such a graph. Therefore its number of edges is $\frac{1}{2}n(n - 1) - (2n - 5) = \frac{1}{2}n^2 - \frac{3}{2}n + 5$, q.e.d.

Theorem 5. $\tau(n, n - 2) = \frac{1}{2}n^2 - n$ for n even, $\tau(n, n - 2) = \frac{1}{2}n^2 - n - \frac{1}{2}$ for n odd, $n \geq 4$.

Proof. The only tree with n vertices and $n - 1$ terminal edges is a star. A star can be a spanning tree of a graph G if and only if G contains a vertex u joined with all other vertices, i.e. of the degree $n - 1$. Therefore we look for a graph G with n vertices with the maximal number of edges, in which no vertex has the degree $n - 1$. For n even such a graph is a regular graph of the degree $n - 2$; it contains $\frac{1}{2}n^2 - n$ edges. For n odd such a graph does not exist, but there exists a graph, one of whose vertices has the degree $n - 3$ while all others have the degree $n - 2$. This is evidently the required graph and its number of edges is $\frac{1}{2}n^2 - n - \frac{1}{2}$.