Časopis pro pěstování matematiky, roč. 98 (1973), Praha

ON BOUNDEDNESS OF THE WEAK SOLUTION FOR SOME CLASS OF QUASILINEAR PARTIAL DIFFERENTIAL EQUATIONS

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(Received January 28, 1971)

Introduction. This paper is connected with my paper [1]. The main aim of this paper is to find a bounded weak solution of the Dirichlet boundary value problem for the equation of the form

$$(1) \qquad -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x}\right) + a_0(x, u) = f(x)$$

where the growth of $a_i(x, p)$ in p ($p \equiv (p_1, \ldots, p_N)$) and $a_0(x, u)$ in u satisfies conditions (3), (4) given below.

Let us consider a bounded domain $\Omega \subset \mathsf{E}^N$ (N-dimensional Euclidean space) with the Lipschitzian boundary $\partial\Omega$. We shall suppose

$$(2) \qquad f(x) \in \mathsf{L}_\infty(\Omega)\,.$$

Let us consider real functions $g(u) \in \mathsf{C}^1(-\infty, \infty)$ for which there exists a positive number u_1 so that

I. $u\, g(u)$ is even and convex for $|u| \geqq u_1$ and

$$\lim_{u \to \infty} (u\, g'(u) + g(u)) = \infty\,.$$

II. For each $l > 1$ there exists a constant $c(l)$ such that

$$g(lu) \leqq c(l)\, g(u) \quad \text{for each} \quad u \geqq u_1\,.$$

III. There exists $l > 1$ such that

$$g(u) \leqq \tfrac{1}{2} g(lu) \quad \text{for each} \quad u \geqq u_1\,.$$

Now, we shall denote by $\boldsymbol{M}_1$; $\boldsymbol{M}_2$; $\boldsymbol{M}_3$ the classes of the functions $g(u)$ satisfying I; I and II; I, II and III respectively.

The functions $a_i(x, p)$ for $i = 0, 1, \ldots, N$ are real and defined for $x \in \Omega$ and $|p| < \infty$. They are continuous in p for almost all $x \in \Omega$ and measurable in x at fixed p. (If $i = 0$, then $p \in E^1$).

Let us have $g_i(u) \in \boldsymbol{M}_3$ for $i = 1, 2, \ldots, N$ and suppose $g_i(u) \geqq g_j(u)$ (or $g_i(u) \leqq g_j(u)$) for all $i, j = 1, 2, \ldots, N$ and $u \geqq u_1$. Then, the conditions for the growth of $a_i(x, p)$ in p are of the form

$$(3) \qquad |a_i(x, p)| \leqq c\left(1 + \sum_{j=1}^{N} \min\left(|g_i(p_j)|, |g_j(p_j)|\right)\right) \quad \text{for} \quad i = 1, 2, \ldots, N\,.$$

Now, let $g_0(u) \in \boldsymbol{M}_1$ such that

$$(4) \qquad |a_0(x, u)| \leqq c(1 + g_0(u))\,.$$

In paper [1], the existence of a weak solution of equation (1) is proved – under the assumption of monotonicity and coerciveness – only if $g_0(u) \in \boldsymbol{M}_3$ in the condition (4). In this case a weak solution is found in the space $\mathbf{W}_{1,G}(\Omega)$. In this paper we shall also work in the space $\mathbf{W}_{1,G}(\Omega)$ and therefore we sketch its construction – for details, see [1].

First we construct Orlicz spaces $L^*_{G_i}(\Omega)$ by means of functions $G_i(u) = u\, g_i(u)$, where $g_i(u)$ for $i = 0, 1, \ldots, N$ are those from conditions (3) and (4). More exactly,

$$G_i(u) = \begin{cases} u\, g_i(u)\,, & \text{for} \quad |u| \geqq u_i \\ c_i|u|^{p_i}\,, & \text{for} \quad |u| \leqq u_i \end{cases}$$

where u_i; c_i; $p_i > 1$ are suitable constants. For the construction of Orlicz spaces, see [3]. Then, we construct the space $\mathbf{W}_{1,G}(\Omega)$ of Sobolev type, see [1], as follows: $\mathbf{W}_{1,G}(\Omega) \equiv \mathbf{W}_{1,G} \equiv \{u \in L^*_{G_0}(\Omega)$, for which the distributive derivatives $\partial u/\partial x_i \in L^*_{G_i}(\Omega)$ for $i = 1, 2, \ldots, N\}$. The norm in this space is defined by

$$\|u\|_{\mathbf{W}_{1,G}} = \sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{G_i} + |u|_{G_0}\,,$$

where $\|\,.\,\|_{G_i}$ is the norm in the Orlicz space $L^*_{G_i}(\Omega)$. Let us denote by ${}^\circ\mathbf{W}_{1,G}$ the subspace of all functions $u \in \mathbf{W}_{1,G}$ satisfying

$$u|_{\partial\Omega} = 0$$

in the sence of traces.

If $g_i(u) \in \boldsymbol{M}_3$ for $i = 0, 1, \ldots, N$, then the corresponding space $\mathbf{W}_{1,G}$ is reflexive (see [1]). In the general case $g_i(u) \in \boldsymbol{M}_1$, for $i = 0, 1, \ldots, N$, $\mathbf{W}_{1,G}$ need not be reflexive (see [5]) and in that case it is impossible to apply the methods known from the reflexive spaces for seeking the weak solution. In this case the functional (potential) is constructed in paper [1] and its minimum is found.

Considering the growth conditions (3) and (4), we shall proceed analogously, even if the conditions are more general, and we shall prove that the minimum of the functional is attained for a bounded function. Then, it will be easy to prove that this minimum is at the same time also the weak solution.

We shall suppose that the Dirichlet's boundary value condition is given by the trace of a function $u_0 \in \mathbf{W}_{1,G}$, where

$$u(s)|_{\partial\Omega} = u_0(s)|_{\partial\Omega} \in \mathsf{L}_\infty(\partial\Omega) \tag{5}$$

in the sense of traces.

$u \in \mathbf{W}_{1,G}$ is called to be a weak solution of the Dirichlet boundary value problem (1), (5), if $u - u_0 \in {}^\circ\mathbf{W}_{1,G}$ and for all $v \in {}^\circ\mathbf{W}_{1,G}$

$$\sum_{i=1}^{N} \int_\Omega \frac{\partial v}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x}\right) \mathrm{d}x + \int_\Omega v a_0(x, u) \, \mathrm{d}x = \int_\Omega v f \, \mathrm{d}x$$

holds.

By means of the class $\boldsymbol{M}_3$ we can describe a growth, which is near to polynomials, e.g. u^3, $u^3 \ln(|u| + 1)$ etc. However, we call the attention to the fact, that the class $\boldsymbol{M}_3$ is essentialy larger then the set of polynomials $|u|^p$. If $g(u) \in \boldsymbol{M}_3$ then there exist $p, q > 1$ and constants c_1, c_2, u_1 such that (see [1], Assertion [1])

$$c_1 |u|^p \leqq u \, g(u) \leqq c_2 |u|^q \quad \text{for all} \quad |u| \geqq u_1 \, .$$

On the contrary, for all $p, q > 1$ with $q > p$ there exists $g_{p,q}(u) \in \boldsymbol{M}_3$ such that previous inequality holds, while this inequality does not take place for any $p', q' > 1$ with $p < p' < q' < q$.

By means of the class $\boldsymbol{M}_1$ we can describe a larger scale of the growths, e.g.

$$\operatorname{sgn} u \, . \ln(|u| + 1) \, , \quad u \exp(u^2) \quad \text{etc.}$$

If $g(u) \in \boldsymbol{M}_1$ and $g(u) \notin \boldsymbol{M}_3$, then the Orlicz space $\mathsf{L}^*_G(\Omega)$ $(G(u) = u \, g(u))$ is not reflexive, which requires a different method to find a weak solution then in the case of reflexive spaces.

Let us denote by $\mathsf{E}_G(\Omega)$ the closure of the set of all bounded functions in the norm of the space $\mathsf{L}^*_G(\Omega)$. If $g(u) \in \boldsymbol{M}_1$ and $g(u) \notin \boldsymbol{M}_2$, then $\mathsf{E}_G(\Omega)$ is a nowhere dense set in $\mathsf{L}^*_G(\Omega)$. If $g(u) \in \boldsymbol{M}_3$, then $\mathsf{E}_G(\Omega) \equiv \mathsf{L}^*_G(\Omega)$. Let us denote by $P(v)$ the conjugate function to $G(u)$ (see [3]) and by $\mathsf{L}^*_P(\Omega)$ the Orlicz space constructed by means of the generating function $P(v)$ $(P(v) = \max_u (uv - G(u)))$. For $u \in \mathsf{L}^*_G(\Omega)$ and $v \in \mathsf{L}^*_P(\Omega)$ the Hölder inequality $|\int_\Omega u(x) \, v(x) \, \mathrm{d}x| \leqq \|u\|_G \, . \, \|v\|_P$ hlods.

The results obtained here can be transferred without essential difficulties to more general boundary value problems.

Let us denote

$$u(x)]^c \equiv u]^c = \begin{cases} u(x) & \text{for } x \text{ such that } |u(x)| \leqq c \\ c \operatorname{sgn} u(x) & \text{for all other } x \, . \end{cases}$$

Lemma 1. *If $u \in \mathbf{W}_{1,G}(\Omega)$, then $u]^c \in \mathbf{W}_{1,G}(\Omega)$ for each constant $c \geqq 0$.*

Proof. As $L^*_{G_i}(\Omega) \subset L_1(\Omega)$ algebraically and topologically for $i = 0, 1, \ldots, N$, it is $u \in \mathbf{W}^1_1(\Omega)$. Thus, $u]^c \in \mathbf{W}^1_1(\Omega)$ – see [2] (Theorem 2.2, Lemma 2.3). Let us denote by $\partial u/\partial x_i$ the derivative in the sense of distribution of the function $u(x)$ and by $[\partial u/\partial x_i]$ the derivative in the ordinary sense. From the results of B. Levi (see [2], Theorem 2.3),

$$\frac{\partial u}{\partial x_i} = \left[\frac{\partial u}{\partial x_i}\right] \quad \text{almost everywhere in } \Omega\,, \quad i = 1, 2, \ldots, N\,.$$

From this fact easily we deduce the lemma.

On the basis of this lemma it is possible to suppose that $u_0(x)$ is bounded and

$$\sup_{x \in \Omega} \operatorname{ess} |u(x)| \leqq \|u_0\|_{L_\infty(\partial\Omega)} \tag{6}$$

in the sense of traces.

For the construction of the functional to equation (1), we suppose the symmetry

$$\frac{\partial a_i(x, p)}{\partial p_j} = \frac{\partial a_j(x, p)}{\partial p_i} \tag{7}$$

in the sense of distribution for $i, j = 1, 2, \ldots, N$.

Supposing (3), (7), we define

$$\Phi_1(u) = \int_0^1 \mathrm{d}t \int_\Omega \sum_{i=1}^N \frac{\partial u}{\partial x_i}\, a_i\left(x, t\,\frac{\partial u}{\partial x}\right) \mathrm{d}x\,.$$

The functional $\Phi_1(u)$ is continuous on the space $\mathbf{W}_{1,G}$ and has a Gatteaux differential at every point equal to

$$\mathrm{D}\Phi_1(u, v) = \int_\Omega \sum_{i=1}^N \frac{\partial v}{\partial x_i}\, a_i\left(x, \frac{\partial u}{\partial x}\right) \mathrm{d}x\,;$$

for the proof of this assertion see [1] (Lemma 1 and 2, § 2). The functional corresponding to equation (1) is of the form (see [4])

$$\Phi(u) = \Phi_1(u) + \int_0^1 \mathrm{d}t \int_\Omega u a_0(x, tu)\, \mathrm{d}x - \int_\Omega u f\, \mathrm{d}x\,. \tag{8}$$

To obtain the convexity of the functional $\Phi_1(u)$ we shall suppose

$$\sum_{i=1}^N (p_i - q_i)\,[a_i(x, p) - a_i(x, q)] \geqq 0\,. \tag{9}$$

The coerciveness of $\Phi(u)$ will be guaranteed by

$$\sum_{i=1}^{N} p_i a_i(x, p) \geqq c_1 \sum_{i=1}^{N} p_i g_i(p_i) - c_2 \tag{10}$$

and

$$u a_0(x, u) \geqq c_1 u\, g_0(u) - c_2\,, \quad u \in \mathsf{E}^1 \tag{11}$$

where c_1, c_2 are constants and $g_i(u)$ for $i = 0, 1, \ldots, N$ are functions from (3) and (4).

We shall consider such equations (1) for which there exist $g_i(u) \in \boldsymbol{M}_3$ for $i = 1, 2, \ldots, N$, and $g_0(u) \in \boldsymbol{M}_1$ satisfying (3), (4), (10) and (11).

In general, for the functional

$$\Phi_2(u) = \int_0^1 \mathrm{d}t \int_\Omega u a_0(x, tu)\, \mathrm{d}x \tag{12}$$

we admit the value $+\infty$ on the space $\mathsf{L}^*_{G_0}(\Omega)$.

We shall look for the minimum of the functional $\Phi(u)$ on the convex and closed set $u_0 + {}^\circ\mathbf{W}_{1,G}$.

Lemma 2. *If* (2), (3), (4), (7), (10) *and* (11) *are satisfied, then*

$$\lim_{\|u\|_{\mathbf{W}_{1,G}} \to \infty} \Phi(u) = \infty\,, \quad \textit{where} \quad u \in u_0 + {}^\circ\mathbf{W}_{1,G}\,.$$

Proof. First we prove that $\Phi_1(u) \to \infty$, if

$$\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{G_i} \to \infty\,. \tag{13}$$

Let us set

$$\lambda(u) = \left[\sum_{i=1}^{N} \left\|\frac{\partial u}{\partial x_i}\right\|_{G_i}\right]^{-1} \cdot \int_\Omega \sum_{i=1}^{N} \frac{\partial u}{\partial x_i}\, a_i\left(x, \frac{\partial u}{\partial x}\right) \mathrm{d}x\,. \tag{14}$$

Using Hölder's inequality in (14) and regarding [1] (Lemma 1, § 2) we find easily that $\lambda(u)$ is a continuous functional on $\mathbf{W}_{1,G}$, bounded from below on bounded sets. We shall show that $\lambda(u) \to \infty$ if (13) holds. For this purpose it suffices to prove that from every sequence $\{u_n\}$ satisfying (13) a subsequence $\{u_{n_k}\}$ can be extracted such that $\lambda(u_{n_k}) \to \infty$ with $k \to \infty$. From (10) we obtain

$$\sum_{i=1}^{N} \int_\Omega \frac{\partial u_n}{\partial x_i}\, a_i\left(x, \frac{\partial u_n}{\partial x}\right) \mathrm{d}x \geqq c_1 \int_\Omega \sum_{i=1}^{N} \frac{\partial u_n}{\partial x_i}\, g_i\left(\frac{\partial u_n}{\partial x_i}\right) - c_2\,.$$

As $g_i(u) \in \boldsymbol{M}_3$, for $i = 1, 2, \ldots, N$, it follows from [1] (Theorem 1, Assertion 5) that it is possible to choose a subsequence $\{u_{n_k}\}$ from $\{u_n\}$ such that

$$\left[\sum_{i=1}^{N} \left\|\frac{\partial u_{n_k}}{\partial x_i}\right\|_{G_i}\right]^{-1} \cdot \int_\Omega \sum_{i=1}^{N} \frac{\partial u_{n_k}}{\partial x_i}\, g_i\left(\frac{\partial u_{n_k}}{\partial x_i}\right) \to \infty \quad \text{with} \quad k \to \infty\,.$$

Thus, we conclude that $\lambda(u_{n_k}) \to \infty$ for $k \to \infty$. Now, let $\{u_n\}$ be an arbitrary sequence satisfying (13). From the definition of $\lambda(u)$ we have

$$\Phi_1(u_n) = \int_0^1 \lambda(tu_n)\,\mathrm{d}t \,.\, \sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{G_i} .$$

$\lambda(tu_n)$ is a continuous function in t and so the integral is well-defined. There exists a constant c such that $\lambda(u) \geqq -c$ for all $u \in \mathbf{W}_{1,G}$. Let $\mathsf{K} > 0$. With regard to the properties of $\lambda(u)$, there exists $\mathsf{L} > 0$ such that $\lambda(u) > 2\mathsf{K} + c$ if

$$\sum_{i=1}^N \left\|\frac{\partial u}{\partial x_i}\right\|_{G_i} > \mathsf{L} .$$

Let us choose N_0 such that

$$\sum_{i=1}^N \left\|\frac{\partial u_n}{\partial x_i}\right\|_{G_i} > 2\mathsf{L}$$

for $n > N_0$. Then, we conclude

$$\int_0^1 \lambda(tu_n)\,\mathrm{d}t = \int_0^{1/2} \lambda(tu_n)\,\mathrm{d}t + \int_{1/2}^1 \lambda(tu_n)\,\mathrm{d}t > -\tfrac{1}{2}c + \mathsf{K} + \tfrac{1}{2}c = \mathsf{K}$$

for $n > N_0$. Thus, $\Phi_1(u_n) \to \infty$ with $n \to \infty$ and hence $\Phi_1(u) \to \infty$ if (13) is satisfied.

Now, we prove that

$$\lim_{\|u\|_{G_0}\to\infty} \left(\int_0^1 \mathrm{d}t \int_\Omega u a_0(x, tu)\,\mathrm{d}x - \int_\Omega uf\,\mathrm{d}x\right) = \infty .$$

For this purpose we prove that there exists a constant c such that

$$(15) \qquad \int_0^{|u|} a_0(x, s)\,ds - c_3|u| \geqq \tfrac{1}{2}c_1 \frac{u}{2} g_0\left(\frac{u}{2}\right) - c$$

where $c_3 = \|f\|_{L_\infty(\Omega)}$. Let us suppose that $u > 0$. Then we obtain from (11), (4)

$$\int_0^u a_0(x, s)\,ds - c_3 u \geqq c_1 \int_0^u g_0(s)\,ds - c_4 u .$$

Because of $g_0(u) \in \mathbf{M}_1$ there exists $s_0 > u_1$ such that $g_0(u)$ is increasing, odd for $|u| \geqq s_0$ and satisfies $\lim_{u\to\infty} g_0(u) = \infty$. For $u > 2s_0$ we obtain

$$\int_0^u g_0(s)\,\mathrm{d}s - \frac{c_4}{c_1} u \geqq \int_{u/2}^u g_0(s)\,\mathrm{d}s - c_5 u \geqq \frac{u}{2} g_0\left(\frac{u}{2}\right) - c_5 u$$

and hence

$$\int_0^u g_0(s)\,\mathrm{d}s - \frac{c_4}{c_1} u \geqq \frac{1}{2}\frac{u}{2} g_0\left(\frac{u}{2}\right) - c_6$$

so that (15) is proved for $u > 0$.

If $u < 0$, then conditions (11), (4) imply

$$-a_0(x, -s) \geqq c_1\, g_0(s) - c_7 \quad \text{for all} \quad s > 0\,.$$

Using the above estimates we deduce

$$\int_0^u a_0(x, s)\,\mathrm{d}s - c_3|u| = \int_0^{-u} -a_0(x, -s)\,\mathrm{d}s - c_3|u| \geqq$$

$$\geqq c_1 \int_0^{-u} g_0(s)\,\mathrm{d}s - c_8|u| \geqq \frac{1}{2}\frac{u}{2}\, g_0\left(\frac{u}{2}\right) - c_9\,.$$

Using (15) we deduce

$$\int_0^1 \mathrm{d}t \int_\Omega u a_0(x, tu)\,\mathrm{d}x - \int_\Omega uf\,\mathrm{d}x \geqq \int_\Omega \int_0^{u(x)} a_0(x, s)\,\mathrm{d}s - c_3 \int_\Omega |u|\,\mathrm{d}x =$$

$$= \int_\Omega \left(\int_0^{u(x)} a_0(x, s)\,\mathrm{d}s - c_3|u| \right) \mathrm{d}x \geqq \tfrac{1}{2}c_1 \int_\Omega \frac{u}{2}\, g_0\left(\frac{u}{2}\right) \mathrm{d}x - c\,.$$

Finally,

$$\int_\Omega \frac{u(x)}{2}\, g_0\left(\frac{u(x)}{2}\right) \mathrm{d}x \to \infty\,, \quad \text{if} \quad \|u\|_{G_0} \to \infty$$

holds (see [1]) and the proof is complete.

In the space $\mathbf{W}_{1,G}$ we introduce the *X-convergence as follows (see [1]):

$$u_n \underset{*\mathrm{X}}{\to} u\,, \quad \text{for} \quad u_n, u \in \mathbf{W}_{1,G}\,,$$

if

$$\int_\Omega u_n v^{(0)}\,\mathrm{d}x \to \int_\Omega u v^{(0)}\,\mathrm{d}x \quad \text{and} \quad \int_\Omega \frac{\partial u_n}{\partial x_i}\, v^{(i)}\,\mathrm{d}x \to \int_\Omega \frac{\partial u}{\partial x_i}\, v^{(i)}\,\mathrm{d}x$$

with $n \to \infty$, for all $v^{(i)}(x) \in \mathrm{E}_{P_i}(\Omega)$ and all $i = 0, 1, 2, \ldots, N$. $\mathrm{P}_i(v)$ is the conjugate function to $G_i(u)$.

Lemma 3. *Let us suppose* (3), (7) *and* (9). *Then, the functional $\Phi_1(u)$ is lower semicontinuous with respect to the *X-convergence and it is bounded from below and from above on bounded sets of $u_0 + {}^\circ\mathbf{W}_{1,G}$.*

Proof. Suppose $\{u_n\}$, $u \in u_0 + {}^\circ\mathbf{W}_{1,G}$ and $u_n \underset{*\mathrm{X}}{\to} u$ with $n \to \infty$. From (9) we deduce

$$\Phi_1(u_n) - \Phi_1(u) \geqq \mathrm{D}\Phi_1(u, u_n - u)\,.$$

With regard to the *X-convergence and to $g_i(u) \in \boldsymbol{M}_3$ for $i = 1, 2, \ldots, N$ we conclude that $\partial u_n/\partial x_i \rightharpoonup \partial u/\partial x_i$ with $n \to \infty$ in $\mathrm{L}^*_{G_i}(\Omega)$ (the weak convergence in the space

$L^*_{G_i}(\Omega)$). On the other hand, from [1] (Lemma 1, § 2) and with respect to (3) we conclude

$$a_i\left(x, \frac{\partial u}{\partial x}\right) \in L^*_{P_i} \equiv (L^*_{G_i})'$$

(the dual space to $L^*_{G_i}$), $i = 1, 2, \ldots, N$. Thus,

$$\lim_{n\to\infty} D\Phi_1(u, u_n - u) = 0 \quad \text{and hence} \quad \liminf_{n\to\infty} \Phi_1(u_n) \geqq \Phi_1(u)\,.$$

The space $^\circ\mathbf{W}_{1,G}$ is closed with respect to the *X-convergence (see [1], Lemma 1, § 3) and hence also $u_0 + {}^\circ\mathbf{W}_{1,G}$ is closed with respect to the *X-convergence. Further, from every bounded sequence from $\mathbf{W}_{1,G}$ we can choose a subsequence convergent with respect to the *X-convergence to some element from $\mathbf{W}_{1,G}$ (see [1], Lemma 1, § 3).

Now, let us assume that $\Phi_1(u_n) \to -\infty$ for some bounded sequence $\{u_n\}$ in $u_0 + {}^\circ\mathbf{W}_{1,G}$. Then, there exist $u \in u_0 + {}^\circ\mathbf{W}_{1,G}$ and a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \underset{*X}{\to} u$. Because of the lower semicontinuity, $\Phi_1(u) = -\infty$, holds. On the other hand, $\Phi_1(u)$ is well-defined on $u_0 + {}^\circ\mathbf{W}_{1,G}$ (see [1], Lemma 1 and Lemma 2, § 2) which is a contradiction. Regarding the Hölder inequality we deduce from [1] (Lemma 1, § 2) that $\Phi_1(u)$ is bounded from above on bounded sets.

Theorem 1. *Let us suppose* (2), (3), (4), (7), (9), (10) *and* (11). *Then, the functional* $\Phi(u)$ *attains its minimum on the set* $u_0 + {}^\circ\mathbf{W}_{1,G}$.

Proof. Evidently, the functional from (15) is bounded from below on $L^*_{G_0}(\Omega)$ ($G_0(u) = u\, g_0(u)$) and with regard to Lemma 2 also $\Phi(u)$ is bounded from below on the set $u_0 + {}^\circ\mathbf{W}_{1,G}$. Let us consider a minimizing sequence $\{u_n\} \in u_0 + {}^\circ\mathbf{W}_{1,G}$. This sequence is bounded in the norm of the space $\mathbf{W}_{1,G}$, because of Lemma 2. There exist a subsequence $\{u_{n_k}\}$ and $u \in u_0 + {}^\circ\mathbf{W}_{1,G}$ such that $u_{n_k} \underset{*X}{\to} u$, if $k \to \infty$. Since $\mathbf{W}_{1,G} \subset \mathbf{W}^1_1(\Omega)$ (algebraically and topologically), we conclude by means of Theorems on imbeddings that there exists a subsequence $\{z_n\}$ from $\{u_{n_k}\}$ such that $z_n \to u$ in the norm of the space $L_1(\Omega)$ and, moreover, $z_n(x) \to u(x)$ almost everywhere in Ω, with $n \to \infty$. There exists a constant c such that $\Phi(z_n) \leqq c$. $\Phi_1(v)$ and $\int_\Omega vf\, dx$ are bounded from below and from above on bounded subsets of $\mathbf{W}_{1,G}$-see Lemma 2 and [1] (Lemma 1, 2 § 2) so that the functional $\Phi_2(v)$ from (12) is bounded on the sequence $\{z_n\}$. As a consequence of Fatou's lemma we obtain

$$\int_0^1 dt \int_\Omega \liminf_{n\to\infty} z_n a_0(x, tz_n)\, dx \leqq \liminf_{n\to\infty} \int_0^1 dt \int_\Omega z_n a_0(x, tz_n)\, dx\,.$$

Finally, Lemma 2 implies:

$$\Phi(u) \leqq \liminf_{n\to\infty} \Phi_1(z_n) + \liminf_{n\to\infty} \int_0^1 dt \int_\Omega z_n a_0(x, tz_n)\, dx - \lim_{n\to\infty} \int_\Omega z_n f\, dx \leqq \liminf_{n\to\infty} \Phi(z_n)\,.$$

Thus, $\Phi(v)$ attains its minimum on the set $u_0 + {}^\circ\mathbf{W}_{1,G}$ at a point u.

In the following theorem we shall prove that every point of the minimum of $\Phi(u)$ is from $L_\infty(\Omega)$. To that end we shall suppose additionally

$$\sum_{i=1}^{N} p_i a_i(x, p) \geqq 0 \quad \text{for all} \quad p\,. \tag{16}$$

Theorem 2. *Let the assumptions of Theorem* 1 *be fulfilled and in addition to it let us suppose* (6) *and* (16). *Then, every point of the minimum of $\Phi(u)$ is from $L_\infty(\Omega)$.*

Proof. With regard to Theorem 1, we can suppose that $\Phi(v)$ attains its minimum at a point $u \in u_0 + {}^\circ\mathbf{W}_{1,G}$. We prove the theorem by contradiction. Suppose that u is not from $L_\infty(\Omega)$. Let us consider an increasing sequence $\{C_n\}$, $C_1 > \|u_0\|_{L_\infty(\partial\Omega)}$, with $C_n \to \infty$ for $n \to \infty$. Further, let us consider the sequence $u(x)]^{C_n}$. In accordance with Lemma 1, $u]^{C_n} \in u_0 + {}^\circ\mathbf{W}_{1,G}$. Using the notation from Lemma 1, we get

$$\left[\frac{\partial u]^{C_n}}{\partial x_i}\right] = \left[\frac{\partial u}{\partial x_i}\right], \quad \text{or} \quad \left[\frac{\partial u]^{C_n}}{\partial x_i}\right] = 0$$

almost everywhere in Ω, for $i = 1, 2, \ldots, N$.

Let us denote

$$K_{C_n} \equiv \{x \in \Omega;\ |u(x)| > C_n\}\,.$$

From (16) we deduce

$$\Phi_1(u]^{C_n}) = \int_0^1 \mathrm{d}t \int_\Omega \sum_{i=1}^{N} \frac{\partial u]^{C_n}}{\partial x_i}\, a_i\left(x, t\,\frac{\partial u]^{C_n}}{\partial x}\right) \mathrm{d}x =$$

$$= \int_0^1 \mathrm{d}t \int_{\Omega - K_{C_n}} \frac{\partial u}{\partial x_i}\, a_i\left(x, t\,\frac{\partial u}{\partial x}\right) \mathrm{d}x \leqq \Phi_1(u)\,.$$

Now, it suffices to prove that there exists an N_0 such that

$$\Phi_2(u]^{C_n}) - \int_\Omega u]^{C_n} f\,\mathrm{d}x < \Phi_2(u) - \int_\Omega u f\,\mathrm{d}x \tag{17}$$

holds for all $n \geqq N_0$.

Using the mean value theorem for the integral we deduce

$$\Phi_2(u) - \Phi_2(u]^{C_n}) - \int_\Omega (u - u]^{C_n})\, f\,\mathrm{d}x \geqq \tag{18}$$

$$\geqq \int_\Omega \int_{u]^{C_n}}^{u(x)} a_0(x, s)\,\mathrm{d}s - c\int_\Omega |u - u]^{C_n}|\,\mathrm{d}x =$$

$$= \int_\Omega (u - u]^{C_n})\, a_0(x, u]^{C_n} + \vartheta(x)\,(u - u]^{C_n}))\,\mathrm{d}x - c\int_\Omega |u - u]^{C_n}|\,\mathrm{d}x\,.$$

The function $\vartheta(x)$ satisfying $0 \leqq \vartheta(x) \leqq 1$ can be determined in such a way that it is measurable – see [7] (footnote at the lemma 5,1).

Condition (11) implies that

$$\operatorname{sgn}(u - u]^{C_n}) = \operatorname{sgn} a_0(x, u]^{C_n} + \vartheta(x)(u - u]^{C_n}))$$

for sufficiently large n and

$$|a_0(x, u]^{C_n} + \vartheta(x)(u - u]^{C_n}))| \to \infty \quad \text{with} \quad n \to \infty .$$

From this and from (18) we deduce (17). Thus, for sufficiently large n we obtain

$$\Phi(u]_{C_n}) < \Phi(u)$$

which gives a contradiction with the minimum property of u. Hence the proof is complete.

Theorem 3. *Let the assumptions of Theorem 2 be fulfilled. Then, there exists a bounded weak solution of the boundary value problem* (1), (5)

Proof. Theorem 2 guarantees the existence of the minimum at a point $u \in u_0 + {}^\circ\mathbf{W}_{1,G} \cap \mathsf{L}_\infty(\Omega)$. We shall show that this minimum is the weak solution. Let us take $v \in \mathscr{D}(\Omega)$ ($\mathscr{D}(\Omega)$ is the set of all functions which have all the derivatives in Ω and possess a compact support in Ω.

$\Phi(u + tv)$ is a continuous function in t and has the derivative at the point $t = 0$. As $u + tv \in u_0 + {}^\circ\mathbf{W}_{1,G}$ and u is a point of the minimum,

$$\frac{\mathrm{d}}{\mathrm{d}t} \Phi(u + tv)|_{t=0} = 0$$

must hold. This means that

$$(19) \qquad \int_\Omega \sum_{i=1}^N \frac{\partial v}{\partial x_i} a_i\left(x, \frac{\partial u}{\partial x}\right) \mathrm{d}x + \int_\Omega v a_0(x, u) \,\mathrm{d}x = \int_\Omega v f \,\mathrm{d}x$$

for all $v \in \mathscr{D}(\Omega)$. But $\overline{\mathscr{D}(\Omega)} = {}^\circ\mathbf{W}_{1,G} \cap \mathsf{E}_{G_0}$, where the closure is with respect to the norm of the space $\mathbf{W}_{1,G}$ (see [1], [2]). Since $g_0(u)$ is bounded, $g_0(u) \in E_{P_0} \subset L^*_{P_0}$ ($P_0(v)$ is conjugate to $G_0(u)$). Further,

$$a_i\left(x, \frac{\partial u}{\partial x}\right) \in \mathsf{L}^*_{P_i} \equiv (\mathsf{L}^*_{G_i})'$$

($P_i(v)$ being conjugate to $G_i(v)$).

Thus, we obtain (19) for $v \in {}^\circ\mathbf{W}_{1,G} \cap \mathsf{E}_{G_0}$ by a limiting process (see Hölder's inequality). Now, let us take $v \in {}^\circ\mathbf{W}_{1,G}$. Let us consider the sequence $v]^{C_n}$ (the notation is that from Theorem 2). With regard to Lemma 1, $v]^{C_n} \in {}^\circ\mathbf{W}_{1,G} \cap \mathsf{E}_{G_0}$. It is evident from the definition of the *X-convergence that $v]^{C_n} \underset{*X}{\to} v$. As $a_i(x, \partial u/\partial x) \in E_{p_i}$ and $g_0(u) \in E_{p_0}$, we obtain (19) for $v \in {}^\circ\mathbf{W}_{1,G}$ by a limiting process. It means that u is a weak solution.

In the sequel we replace condition (16) by other conditions. Let us suppose the existence of $\partial a_i(x, p)/\partial x_i$ and

$$\frac{\partial a_i(x, 0)}{\partial x_i} \in L_\infty(\Omega) \quad \text{for all} \quad i = 1, 2, \ldots, N\,. \tag{20}$$

Theorem 4. *Let us suppose* (2), (3), (4), (6), (7), (9), (10), (11) *and* (20). *Then there exists a bounded weak solution of the boundary value problem* (1), (5).

Proof. Let us consider $a'_i(x, p) = a_i(x, p) - a_i(x, 0)$, $a'_i(x, p)$ satisfying all the assumptions of Theorem 3. Really, condition (9) implies (16). With respect to (20) and Theorem 3, there exists a bounded weak solution of the equation

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a'_i\left(x, \frac{\partial u}{\partial x}\right) + a_0(x, u) = f + \sum_{i=1}^{N} \frac{\partial a_i(x, 0)}{\partial x_i}\,,$$

i.e.,

$$\sum_{i=1}^{N} \int_\Omega \frac{\partial v}{\partial x_i} a'_i\left(x, \frac{\partial u}{\partial x}\right) \mathrm{d}x + \int_\Omega v a_0(x, u)\, \mathrm{d}x = \tag{21}$$

$$= \int_\Omega \left(f + \sum_{i=1}^{N} \frac{\partial a_i(x, 0)}{\partial x_i}\right) v\, \mathrm{d}x\,, \quad \text{for all} \quad v \in {}^\circ\mathbf{W}_{1,G}\,.$$

Using Green's theorem, we obtain

$$-\int_\Omega \sum_{i=1}^{N} \frac{\partial a_i(x, 0)}{\partial x_i} v\, \mathrm{d}x = \int_\Omega \sum_{i=1}^{N} a_i(x, 0) \frac{\partial v}{\partial x_i}\, \mathrm{d}x$$

and then the identity (21) implies the required result.

We give now conditions for the uniqueness of the weak solution.

Assertion 1. *Let us suppose that the assumptions of Theorem* 3, *or Theorem* 4, *are fufilled. Then, there exists a unique weak solution of the problem* (1), (5), *if* (*i*), *or* (*ii*), *is satisfied*:

(i) $$(s_1 - s_2)\left[a_0(x, s_1) - a_0(x, s_2)\right] > 0 \quad \text{for} \quad s_1 \neq s_2\,.$$

(ii) *In condition* (9), *the equality holds only when* $p \equiv q$; *and further* $(s_1 - s_2) \,.$ $. \left[a_0(x, s_1) - a_0(x, s_2)\right] \geqq 0$.

Proof. If $u_1 \neq u_2$ were two solutions of (1), (5), then

$$\int_{\Omega} \sum_{i=1}^{N} \frac{\partial(u_1 - u_2)}{\partial x_i} \left[a_i\left(x, \frac{\partial u_1}{\partial x}\right) - a_i\left(x, \frac{\partial u_2}{\partial x}\right)\right] dx +$$

$$+ \int_{\Omega} (u_1 - u_2) \left[a_0(x, u_1) - a_0(x, u_2)\right] dx = 0$$

which yields a contradiction in case (*i*), as well as in case (*ii*).

Now, we present some consequences of Theorem 3, or Theorem 4, considering the known results about the regularity of the bounded weak solution – see [6], [2].

We shall suppose more special conditions instead of (3), (10)

$$\sum_{i=1}^{N} p_i a_i(x, p) \geqq c_1 |p|^m - c \tag{22}$$

$$\sum_{i=1}^{N} |a_i(x, p)| \, (1 + |p|) \leqq c_2 (1 + |p|)^m \,, \quad \text{where} \quad m > 1 \,. \tag{23}$$

$C^{0,\alpha}(\bar{\Omega})$ is the space of the Hölder functions with the norm

$$\|u\|_{C^{0,\alpha}(\bar{\Omega})} = \max_{x \in \bar{\Omega}} |u(x)| + \sup_{x,y \in \bar{\Omega}} \frac{|u(x) - u(y)|}{|x - y|^{\alpha}} \,.$$

We shall suppose that the boundary value condition is given by the function

$$u_0(x) \in \mathbf{W}_m^1(\Omega) \cap C^{0,\alpha}(\bar{\Omega}) \quad \text{for some} \quad 0 < \alpha < 1 \,. \tag{24}$$

Assertion 2. *Let the assumptions of Theorem* 3, *or Theorem* 4 *with* (24) *be fulfilled. Let us suppose* (22), (23) *instead of* (3), (10). *Then, there exists* β, $0 < \beta \leqq \alpha$ *such that the weak solution is from* $\mathbf{W}_m^1 \cap C^{0,\beta}(\bar{\Omega})$.

This assertion is a consequence of [6] (Theorem 1.1, Chap. 4) and Theorem 3, or Theorem 4.

Now, let us consider the equation

$$-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial u}{\partial x_j} \right) + a_0(x, u) = f(x) \,. \tag{1'}$$

Let us suppose

$$a_{ij}(x) \in C^{1,\alpha}(\bar{\Omega}) \tag{25}$$

$$f(x) \in C^{0,\alpha}(\bar{\Omega}) \,. \tag{26}$$

$C^{1,\alpha}(\bar{\Omega})$ being the set of all functions the first partial derivatives of which are from $C^{0,\alpha}(\bar{\Omega})$.