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GENERAL BOUNDARY VALUE PROBLEM FOR AN INTEGRODIFFERENTIAL SYSTEM AND ITS ADJOINT

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(Continuation)**)

4. WEAKLY NONLINEAR BOUNDARY VALUE PROBLEM

Notation. Given a B-space $\mathscr{B}$ with the norm $\|.\|_{\mathscr{B}}$, $u_0 \in \mathscr{B}$ and $\varrho > 0$, the set $\{u \in \mathscr{B} : \|u - u_0\|_{\mathscr{B}} \leqq \varrho\}$ is denoted by $\mathscr{U}(u_0, \varrho; \mathscr{B})$.

Definition 4,1. Let $\mathscr{B}_1, \mathscr{B}_2$ be B-spaces and let $\varepsilon_0 > 0$. An operator $F : u \in \mathscr{B}_1$, $\varepsilon \in [0, \varepsilon_0] \to F(\varepsilon)(u) \in \mathscr{B}_2$ is said to be locally lipschitzian in u near $\varepsilon = 0$ if, given an arbitrary $u_0 \in \mathscr{B}_1$, there exist $\alpha(u_0) > 0$, $\varrho(u_0) > 0$ and $\varepsilon(u_0) > 0$ such that

$$\|F(\varepsilon)(u_2) - F(\varepsilon)(u_1)\|_{\mathscr{B}_2} \leqq \alpha(u_0)\,\|u_2 - u_1\|_{\mathscr{B}_1}$$

for all $u_1, u_2 \in \mathscr{U}(u_0, \varrho(u_0); \mathscr{B}_1)$ and $\varepsilon \in [0, \varepsilon(u_0)]$.

Hereafter we suppose

($\mathscr{A}$) $\qquad A \in \mathscr{L}^1_{n,n}\,, \quad G \in \mathscr{L}^2[\mathscr{BV}]\,, \quad L \in \mathscr{BV}_{n,n} \quad (m = n)\,.$

The mappings

$$\Phi : x \in \mathscr{AC}\,, \quad \varepsilon \in [0, \varepsilon_0] \to \Phi(\varepsilon)(x) \in \mathscr{L}^1\,,$$
$$\Lambda : x \in \mathscr{AC}\,, \quad \varepsilon \in [0, \varepsilon_0] \to \Lambda(\varepsilon)(x) \in \mathscr{R}_n$$

are locally lipschitzian in x near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \varepsilon_0]$ for any $x \in \mathscr{AC}$ fixed, $\varepsilon_0 > 0$.

*) The last paragraph (§ 5) was added.

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Let us consider the weakly nonlinear boundary value problem $(\mathscr{P}_\varepsilon)$

$$\dot{x} = A(t)\,x + \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) + \varepsilon\, \Phi(\varepsilon)\,(x)\,(t)\,, \tag{4,1}$$

$$\int_a^b [\mathrm{d}L(s)]\, x(s) + \varepsilon\, \Lambda(\varepsilon)\,(x) = 0\,, \tag{4,2}$$

where $\varepsilon \geqq 0$ is a small parameter.

We proceed formally as in § 3 and write the problem $(\mathscr{P}_\varepsilon)$ in the equivalent form as the system of equations for $x \in \mathscr{AC}$, $h \in \mathscr{L}^2$ and $c \in \mathscr{R}_n$

$$\begin{aligned} -x(t) + X(t)\,c \; + \int_a^t X(t)\,X^{-1}(s)\,h(s)\,\mathrm{d}s + \varepsilon P_0(\varepsilon)\,(x)\,(t) &= 0\,, \\ -h(t) + H_1(t)\,c + \int_a^b K(t, s)\,h(s)\,\mathrm{d}s \qquad + \varepsilon P_1(\varepsilon)\,(x)\,(t) &= 0\,, \\ Cc + \int_a^b H_2(s)\,h(s)\,\mathrm{d}s \qquad + \varepsilon P_2(\varepsilon)\,(x) \quad &= 0\,, \end{aligned} \tag{4,3}$$

where $X(t)$ has the same meaning as before ((3,3)) and

$$H_1(t) = \int_a^b [\mathrm{d}_s G(t, s)]\, X(s)\,, \quad H_2(t) = \left(\int_t^b [\mathrm{d}L(s)]\, X(s)\right) X^{-1}(t)\,, \tag{4,4}$$

$$K(t, s) = \left(\int_s^b [\mathrm{d}_\sigma G(t, \sigma)]\, X(\sigma)\right) X^{-1}(s)\,, \quad C = \int_a^b [\mathrm{d}L(s)]\, X(s)\,,$$

$$P_0(\varepsilon)\,(x)\,(t) = X(t) \int_a^t X^{-1}(s)\, \Phi(\varepsilon)\,(x)\,(s)\,\mathrm{d}s\,,$$

$$P_1(\varepsilon)\,(x)\,(t) = \int_a^b [\mathrm{d}_s G(t, s)] \left(X(s) \int_a^s X^{-1}(\sigma)\, \Phi(\varepsilon)\,(x)\,(\sigma)\,\mathrm{d}\sigma\right) =$$

$$= \int_a^b \left(\int_s^b [\mathrm{d}_\sigma G(t, \sigma)]\, X(\sigma)\right) X^{-1}(s)\, \Phi(\varepsilon)\,(x)\,(s)\,\mathrm{d}s = \int_a^b K(t, s)\, \Phi(\varepsilon)\,(x)\,(s)\,\mathrm{d}s\,,$$

$$P_2(\varepsilon)\,(x) = \Lambda(\varepsilon)\,(x) + \int_a^b [\mathrm{d}L(s)] \left(X(s) \int_a^s X^{-1}(\sigma)\, \Phi(\varepsilon)\,(x)\,(\sigma)\,\mathrm{d}\sigma\right) =$$

$$= \Lambda(\varepsilon)\,(x) + \int_a^b \left(\int_s^b [\mathrm{d}L(\sigma)]\, X(\sigma)\right) X^{-1}(s)\, \Phi(\varepsilon)\,(x)\,(s)\,\mathrm{d}s =$$

$$= \Lambda(\varepsilon)\,(x) + \int_a^b H_2(s)\, \Phi(\varepsilon)\,(x)\,(s)\,\mathrm{d}s\,.$$

By assumptions of this paragraph $K \in \mathscr{L}_2$, H_1 and $H_2 \in \mathscr{L}^2_{n,n}$ and P_0, P_1 and P_2 are mappings of $\mathscr{AC} \times [0, \varepsilon_0]$ into $\mathscr{AC}$, $\mathscr{L}^2$ and $\mathscr{R}_n$, respectively, locally lipschitzian in x near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \varepsilon_0]$ for any $x \in \mathscr{AC}$ fixed. For example, in the case of P_1 we have for $x_1, x_2 \in \mathscr{AC}$, $t \in J$ and $\varepsilon_1, \varepsilon_2 \in [0, \varepsilon_0]$

$$\|P_1(\varepsilon_2)(x_2)(t) - P_1(\varepsilon_1)(x_1)(t)\| \leqq \beta \operatorname{var}_a^b G(t, \cdot) \|\Phi(\varepsilon_2)(x_2) - \Phi(\varepsilon_1)(x_1)\|_1,$$

where $\beta = \sup_{t,s \in J} \|X(t) X^{-1}(s)\|$. Hence

$$\|P_1(\varepsilon_2)(x_2) - P_1(\varepsilon_1)(x_1)\|_2 \leqq \alpha \|\Phi(\varepsilon_2)(x_2) - \Phi(\varepsilon_1)(x_1)\|_1,$$

where

$$\alpha = \beta \|\operatorname{var}_a^b G(t, \cdot)\|_2 .$$

Let $K_0 \in \mathscr{L}_2$, $K_1 \in \mathscr{L}^2_{n,n'}$ and $K_2 \in \mathscr{L}^2_{n',n}$ be again such that $K(t, s) = K_0(t, s) + K_1(t) K_2(s)$, $|||K_0||| < 1$. Let Γ be the resolvent kernel of K_0 and let $\tilde{H}_1$ and $\tilde{K}_1$ be again defined by (3,10). ($\Gamma \in \mathscr{L}_2$, $\tilde{H}_1 \in \mathscr{L}^2_{n,n}$ and $\tilde{K}_1 \in \mathscr{L}^2_{n,n'}$, of course.) Then the system (4,3) becomes

$$-x(t) + U(t)\, b + \varepsilon R_0(\varepsilon)(x)(t) = 0, \qquad Bb + \varepsilon R(\varepsilon)(x) = 0, \tag{4,5}$$

where B is given by (4,4), (3,9), (3,10) and (3,12),

$$U(t) = \left(X(t)\left[I + \int_a^t X^{-1}(s)\, \tilde{H}_1(s)\, ds\right],\ X(t) \int_a^t X^{-1}(s)\, \tilde{K}_1(s)\, ds\right), \tag{4,6}$$

$$R_0(\varepsilon)(x)(t) = P_0(\varepsilon)(x)(t) + X(t) \int_a^t X^{-1}(s)\, P_1(\varepsilon)(x)(s)\, ds,$$

$$R(\varepsilon)(x) = \begin{pmatrix} \int_a^b \tilde{K}_2(s)\, P_1(\varepsilon)(x)(s)\, ds \\ P_2(\varepsilon)(x) + \int_a^b \tilde{H}_2(s)\, P_1(\varepsilon)(x)(s)\, ds \end{pmatrix},$$

$$\tilde{H}_2(t) = H_2(t) + \int_a^b H_2(s)\, \Gamma(s, t)\, ds, \quad \tilde{K}_2(t) = K_2(t) + \int_a^b K_2(s)\, \Gamma(s, t)\, ds,$$

$$h(t) = \tilde{H}_1(t)\, c + \tilde{K}_1(t)\, d + \varepsilon\left[P_1(\varepsilon)(x)(t) + \int_a^b \Gamma(t, s)\, P_1(\varepsilon)(x)(s)\, ds\right],$$

$$d = \int_a^b K_2(s)\, h(s)\, ds, \quad b = (c', d')'.$$

Clearly, $U(t)$ is absolutely continuous on J, $\tilde{H}_2 \in \mathscr{L}^2_{n,n}$, $\tilde{K}_2 \in \mathscr{L}^2_{n',n}$, R_0 and R are mappings of $\mathscr{AC} \times [0, \varepsilon_0]$ into $\mathscr{AC}$ and $\mathscr{R}_{n+n'}$, respectively, locally lipschitzian in x near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \varepsilon_0]$ for any $x \in \mathscr{AC}$ fixed.

The further investigation of our problem rather depends on whether $\det B \neq 0$ or $\det B = 0$. In the former simple (so called noncritical) case the following theorem holds.

Theorem 4,1. *Let the boundary value problem $(\mathscr{P}_\varepsilon)$ be given and let the assumptions $(\mathscr{A})$ be fulfilled. Let the limit problem $(\mathscr{P}_0)$ have only the trivial solution. Then there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in [0, \varepsilon^*]$ there exists a unique solution x_ε^* of $(\mathscr{P}_\varepsilon)$, while $\|x_\varepsilon^*\|_{\mathscr{AC}} \to 0$ for $\varepsilon \to 0+$.*

Proof. Let $(\mathscr{P}_0)$ have only the trivial solution. Then by Corollary 1 of Theorem 3,1 $\det B \neq 0$ and (4,5) becomes

$$x(t) = \varepsilon[R_0(\varepsilon)(x)(t) - U(t) B^{-1} R(\varepsilon)(x)] = \varepsilon T(\varepsilon)(x)(t).$$

It follows immediately from the above argument that the operator $T: \mathscr{AC} \times [0, \varepsilon_0] \to \mathscr{AC}$ is locally lipschitzian in x near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \varepsilon_0]$ for any $x \in \mathscr{AC}$ fixed. Hence the fixed point theorem for contractive operators ([8]) can be applied.

Remark 4,1. The given boundary value problem $(\mathscr{P}_\varepsilon)$ is certainly noncritical e.g. if in (4,3)

a) $\det C \neq 0$ and 1 is not an eigenvalue of $K(t, s) - H_1(t) C^{-1} H_2(s)$,

b) 1 is not an eigenvalue of K and

$$\det\left(C + \int_a^b H_2(s)\left[H_1(s) + \int_a^b Q(s, \sigma) H_1(\sigma)\, d\sigma\right] ds\right) \neq 0,$$

where Q is the resolvent kernel of K.

In the critical case ($\det B = 0$) some further notations are needed.

Notation. $\mathscr{N}_0$ denotes the naturally ordered set $\{1, 2, \ldots, n + n'\}$. If $\mathscr{S}$ is a naturally ordered subset of $\mathscr{N}_0$, then $\mathscr{S}^*$ denotes the naturally ordered complement of $\mathscr{S}$ with respect to $\mathscr{N}_0$. The number of elements of a set $\mathscr{S} \subset \mathscr{N}_0$ is denoted by $\gamma(\mathscr{S})$. Let $C = (c_{i,j})_{i,j \in \mathscr{N}_0}$ be an $(n + n') \times (n + n')$-matrix and let $\mathscr{S} \subset \mathscr{N}_0$, $\mathscr{V} \subset \mathscr{N}_0$, then $C_{\mathscr{S},\mathscr{V}}$ denotes the matrix $(c_{i,j})_{i \in \mathscr{S}, j \in \mathscr{V}}$. Similarly if b is an $(n + n')$-vector ($b = (b_j)_{j \in \mathscr{N}_0}$) and $\mathscr{S} \subset \mathscr{N}_0$, then $b_{\mathscr{S}} = (b_j)_{j \in \mathscr{S}}$. (Analogously for matrix or vector functions and operators.) $\mathscr{N}$ denotes the naturally ordered set $\{1, 2, \ldots, n\}$. The sign $\dotplus$ is defined by $b = b_{\mathscr{S}} \dotplus b_{\mathscr{S}^*}$.

Let $\chi = \operatorname{rank}(B) < n + n'$, while

$$\det B_{\mathscr{S}^*,\mathscr{V}^*} \neq 0 \quad \text{and} \quad B_{\mathscr{S},\mathscr{N}_0} - W B_{\mathscr{S}^*,\mathscr{N}_0} = 0, \tag{4,7}$$

$\nu(\mathscr{S}^*) = \nu(\mathscr{V}^*) = \chi$ and W is an $(n + n' - \chi) \times \chi$-matrix. Let us put $\nu = n + n' - \chi$, $B_1 = B_{\mathscr{S}^*,\mathscr{V}^*}$, $B_2 = B_{\mathscr{S}^*,\mathscr{V}}$, $\gamma = b_{\mathscr{V}^*}$ and $\delta = b_{\mathscr{V}}$. Then $(4,5)_2$ yields

(4,8) $$\gamma = -B_1^{-1} B_2 \delta - \varepsilon B_1^{-1} R_{\mathscr{S}^*}(\varepsilon)(x)\,.$$

Inserting (4,8) and $b = \gamma + \delta$ into $(4,5)_1$ we obtain that (4,5) is equivalent to the system of equations for $x \in \mathscr{AC}$ and $\delta \in \mathscr{R}_\nu$,

(4,9) $$-x(t) + V(t)\,\delta + \varepsilon S(\varepsilon)(x)(t) = 0\,,$$
$$T(\varepsilon)(x) = 0\,,$$

where

(4,10) $$V(t) = U_{\mathscr{N},\mathscr{V}}(t) - U_{\mathscr{N},\mathscr{V}^*}(t)\, B_1^{-1} B_2\,,$$

$$S : x \in \mathscr{AC}\,,\quad \varepsilon \in [0, \varepsilon_0] \to S(\varepsilon)(x) = R_0(\varepsilon)(x) - U_{\mathscr{N},\mathscr{V}^*}(.)\, B_1^{-1} R_{\mathscr{S}^*}(\varepsilon)(x) \in \mathscr{AC}\,,$$

$$T : x \in \mathscr{AC}\,,\quad \varepsilon \in [0, \varepsilon_0] \to T(\varepsilon)(x) = R'_{\mathscr{S}}(\varepsilon)(x) - W R_{\mathscr{S}^*}(\varepsilon)(x) \in \mathscr{R}_\nu\,.$$

$V(t)$ is absolutely continuous on J and it is easy to verify that the operators S and T have the same smoothness properties as Φ, Λ, P_0, P_1 etc.

Let $\varepsilon > 0$, then $x \in \mathscr{AC}$ is a solution to the boundary value problem $(\mathscr{P}_\varepsilon)$ iff (x, δ), where

$$\delta = b_{\mathscr{V}} \quad \text{and} \quad b = \begin{pmatrix} x(a) \\ \int_a^b K_2(t) \left(\int_a^b [\mathrm{d}_s G(t, s)]\, x(s) \right) \mathrm{d}t \end{pmatrix} =$$
$$= \begin{pmatrix} x(a) \\ \int_a^b \left[\mathrm{d}_t \int_a^b K_2(s)\, G(s, t)\, \mathrm{d}s \right] x(t) \end{pmatrix},$$

is a solution to (4,9). (All solutions x_0 of the limit problem $(\mathscr{P}_0)$ are given by $x_0(t) = V(t)\,\delta$, where δ is an arbitrary ν-vector.) To investigate further the existence of a solution (and its dependence on ε) to $(\mathscr{P}_\varepsilon)$ various principles in accordance with the smoothness of the operators Φ and Λ may be used. Below we state two existence theorems which can serve as models. The first one is obtained by the use of the Newton method for equations in B-spaces.

Proposition 1. *Let $\mathscr{B}_1$ and $\mathscr{B}_2$ be B-spaces and let $\varepsilon_0 > 0$. Let $\mathscr{U} \subset \mathscr{B}_1$ and let F be an operator: $(u, \varepsilon) \in \mathscr{U} \times [0, \varepsilon_0] \to F(\varepsilon)(u) \in \mathscr{B}_2$. Let us assume that*

(i) *the equation $F(0)(u) = 0$ possesses a solution $u_0 \in \mathscr{U}$;*

(ii) *there exists $\varrho_0 > 0$ such that F is continuous in $(u, \varepsilon) \in \mathscr{U}_0 \times [0, \varepsilon_0] = \mathscr{U}(u_0, \varrho_0; \mathscr{B}_1) \times [0, \varepsilon_0]$ and for all $(u, \varepsilon) \in \mathscr{U}_0 \times [0, \varepsilon_0]$ possesses a G-derivative $F'_u(\varepsilon)(u)$ with respect to u which is continuous in $(u, \varepsilon) \in \mathscr{U}_0 \times [0, \varepsilon_0]$;*

(iii) *$F'_u(0)(u_0)$ possesses a bounded inverse $[F'_u(0)(u_0)]^{-1}$.*

Then there exist $\varepsilon^* > 0$ *and* $\varrho^* > 0$ *such that for any* $\varepsilon \in [0, \varepsilon^*]$ *the equation* $F(\varepsilon)(u) = 0$ *possesses one and only one solution* $u^*(\varepsilon)$ *in* $\mathscr{U}(u_0, \varrho^*; \mathscr{B}_1)$. *The mapping* $\varepsilon \in [0, \varepsilon^*] \to u^*(\varepsilon) \in \mathscr{B}_1$ *is continuous and* $u^*(\varepsilon) \to u_0$ *in* $\mathscr{B}_1$ *if* $\varepsilon \to 0+$.

(For the proof see [19], p. 355. Similar theorems are proved also in [8] or [16].)

Remark 4,1. Let us notice that the assertion of Proposition 1 can be equivalently reformulated as follows.

There exists $\varepsilon^* > 0$ *such that for all* $\varepsilon \in [0, \varepsilon^*]$ *there exists a unique solution* $u^* = u^*(\varepsilon) \in \mathscr{U}_0$ *of the equation* $F(\varepsilon)(u) = 0$ *continuous in* $\varepsilon \in [0, \varepsilon^*]$ *and such that* $u^*(0) = u_0$.

To be able to apply Proposition 1 to the boundary value problem $(\mathscr{P}_\varepsilon)$ we have to add some further assumptions concerning the differentiability of Φ and Λ to those used until now. It is easy to verify that if $\mathscr{U} \subset \mathscr{AC}$ and Φ and Λ are continuous in $(x, \varepsilon) \in \mathscr{U} \times [0, \varepsilon_0]$ and for all $(x, \varepsilon) \in \mathscr{U} \times [0, \varepsilon_0]$ possess a G-derivative with respect to x which is continuous in $(x, \varepsilon) \in \mathscr{U} \times [0, \varepsilon_0]$, then the same holds also for the operators S and T.

Theorem 4,2. *Let the boundary value problem* $(\mathscr{P}_\varepsilon)$ *fulfilling the assumptions* $(\mathscr{A})$ *be given. Let the limit problem* $(\mathscr{P}_0)$ *admit a nonzero solution (i.e.* $\det B = 0$*). Let the matrix function* V *and the operators* T *and* T_0 *be defined by* (4,7), (4,10) *and*

$$T_0 : \delta \in \mathscr{R}_\nu \to T_0(\delta) = T(0)(V(.)\,\delta) \in \mathscr{R}_\nu\,. \tag{4,11}$$

Suppose

(I) *the limit problem* $(\mathscr{P}_0)$ *possesses a solution* x_0 *such that* $T_0(\delta_0) = 0$ *for* $\delta_0 = (b_0)_{\mathscr{V}}$, *where*

$$b_0 = \begin{pmatrix} x_0(a) \\ \int_a^b \left[d_t \int_a^b K_2(s)\, G(s, t)\, ds \right] x_0(t) \end{pmatrix};$$

(II) *there exists* $\varrho_0 > 0$ *such that* Φ *and* Λ *are continuous in* $(x, \varepsilon) \in \mathscr{U}_0 \times \times [0, \varepsilon_0] = \mathscr{U}(x_0, \varrho_0; \mathscr{AC}) \times [0, \varepsilon_0]$ *and for all* $(x, \varepsilon) \in \mathscr{U}_0 \times [0, \varepsilon_0]$ *possess a G-derivative with respect to* x *continuous in* $(x, \varepsilon) \in \mathscr{U}_0 \times [0, \varepsilon_0]$;

(III) *the Jacobian*

$$\det\left(\frac{DT_0}{D\delta}(\delta_0)\right)$$

is nonzero.

Then there exists $\varepsilon^* > 0$ *such that for all* $\varepsilon \in [0, \varepsilon^*]$ *there exists a unique solution* $x^*(\varepsilon)$ *to* $(\mathscr{P}_\varepsilon)$ *continuous in* $\varepsilon \in [0, \varepsilon^*]$ *as a mapping* $[0, \varepsilon^*] \to \mathscr{AC}$ *and such that* $x^*(0) = x_0$.

Proof. Let us denote $\mathscr{B} = \mathscr{AC} \times \mathscr{R}_v$ and

$$F : (x, \delta) \in \mathscr{B}\,, \quad \varepsilon \in [0, \varepsilon_0] \to \begin{pmatrix} -x + V(.)\,\delta + \varepsilon S(\varepsilon)\,(x) \\ T(\varepsilon)\,(V(.)\,\delta + \varepsilon S(\varepsilon)\,(x)) \end{pmatrix} \in \mathscr{B}\,.$$

($\mathscr{B}$ is a B-space with the norm $\|(x, \delta)\|_{\mathscr{B}} = \|x\|_{\mathscr{AC}} + \|\delta\|$.)

We shall verify that the operator F fulfils all the assumptions of Proposition 1.

(i) For $(x, \delta) \in \mathscr{B}$ we have

$$F(0)\,(x, \delta) = \begin{pmatrix} -x + V(.)\,\delta \\ T(0)\,(V(.)\,\delta) \end{pmatrix} = \begin{pmatrix} -x + V(.)\,\delta \\ T_0(\delta) \end{pmatrix}.$$

Let x_0 be a solution to $(\mathscr{P}_0)$ such that $T_0(\delta_0) = 0$ for $\delta_0 = (b_0)_{\mathscr{V}}$, where

$$b_0 = \begin{pmatrix} x_0(a) \\ \int_a^b \left[\mathrm{d}_t \int_a^b K_2(s)\, G(s, t)\, \mathrm{d}s \right] x_0(t) \end{pmatrix}.$$

Then $x_0 = V(.)\,\delta_0$ and hence $F(0)\,(x_0, \delta_0) = 0$.

(ii) Since the operators S and T have the same smoothness properties as Φ and Λ, there exist $\varepsilon_1 > 0$ and $\varrho_1 > 0$ such that F fulfils the assumption (ii) of Proposition 1 on $\mathscr{U}_1 \times [0, \varepsilon_1] = \mathscr{U}((x_0, \delta_0), \varrho_1; \mathscr{B}) \times [0, \varepsilon_1]$ while for $(x, \delta, \varepsilon) \in \mathscr{U}_1 \times [0, \varepsilon_1]$ and $(\bar{x}, \bar{\delta}) \in \mathscr{B}$,

$$[F'_{(x,\delta)}(\varepsilon)\,(x, \delta)]\,(\bar{x}, \bar{\delta}) =$$

$$= \begin{pmatrix} -\bar{x} + V(.)\,\bar{\delta} + \varepsilon[S'_x(\varepsilon)\,(x)]\,\bar{x} \\ [T'_x(\varepsilon)\,(V(.)\,\delta + \varepsilon S(\varepsilon)\,(x))]\,(V(.)\,\bar{\delta}) + \varepsilon[T'_x(\varepsilon)\,(V(.)\,\delta + \varepsilon S(\varepsilon)\,(x))]\,[S'_x(\varepsilon)\,(x)]\,\bar{x} \end{pmatrix}.$$

In particular

$$J_0(\bar{x}, \bar{\delta}) = [F'_{(x,\delta)}(0)\,(x_0, \delta_0)]\,(\bar{x}, \bar{\delta}) = \begin{pmatrix} -\bar{x} + V(.)\,\bar{\delta} \\ [T'_x(0)\,(V(.)\,\delta)]\,(V(.)\,\bar{\delta}) \end{pmatrix} = \begin{pmatrix} -\bar{x} + V(.)\,\bar{\delta} \\ \left[\dfrac{\mathrm{D}T_0}{\mathrm{D}\delta}\,(\delta_0)\right] \bar{\delta} \end{pmatrix}.$$

(iii) Given an arbitrary couple $(x, \delta) \in \mathscr{B}$,

$$J_0(\bar{x}, \bar{\delta}) = \begin{pmatrix} x \\ \delta \end{pmatrix}$$

iff

$$\bar{\delta} = \left[\frac{\mathrm{D}T_0}{\mathrm{D}\delta}\,(\delta_0)\right]^{-1} \delta \quad \text{and} \quad \bar{x} = V(.)\,\bar{\delta} + x\,.$$

Thus the operator J_0 possesses an inverse

$$J_0^{-1} : (x, \delta) \in \mathscr{B} \to \begin{pmatrix} x + V(.) \left[\frac{DT_0}{D\delta}(\delta_0)\right]^{-1} \delta \\ \left[\frac{DT_0}{D\delta}(\delta_0)\right]^{-1} \delta \end{pmatrix} \in \mathscr{B},$$

the boundedness of J_0^{-1} being obvious.

Applying Proposition 1 we complete the proof.

The system (4,9) can be simplified by means of the following

Proposition 2. *There exists* $\varepsilon_1 > 0$ *such that for every* $\varepsilon \in [0, \varepsilon_1]$ *and* $\delta \in \mathscr{R}_v$ *there exists a unique solution* $x = \Xi(\varepsilon)(\delta) \in \mathscr{AC}$ *of the equation*

$$(4{,}9)_2 \qquad -x + V(.)\,\delta + \varepsilon S(\varepsilon)(x) = 0\,,$$

the operator $\Xi : \mathscr{R}_v \times [0, \varepsilon_1] \to \mathscr{AC}$ *being continuous in* (δ, ε) *and locally lipschitzian in* δ *near* $\varepsilon = 0$.

Proof. The existence and uniqueness of the desired solution $x = \Xi(\varepsilon)(\delta)$ for all $\delta \in \mathscr{R}_v$ and $\varepsilon \in [0, \varepsilon_2]$ with some $\varepsilon_2 > 0$ and the continuity of Ξ in $(\delta, \varepsilon) \in \mathscr{R}_v \times [0, \varepsilon_2]$ are evident. Given an arbitrary $\delta_0 \in \mathscr{R}_v$, let us denote

$$x_0 = V(.)\,\delta_0 = \Xi(0)(\delta_0)\,.$$

Let $\beta = \beta(\delta_0) > 0$, $\varepsilon_3 = \varepsilon(\delta_0) > 0$ $(\varepsilon_3 \leqq \varepsilon_2)$ and $\varrho = \varrho(\delta_0) > 0$ be such that

$$\|S(\varepsilon)(x_1) - S(\varepsilon)(x_1)\|_{\mathscr{AC}} \leqq \beta \|x_2 - x_1\|_{\mathscr{AC}}$$

for all $x_1, x_2 \in \mathscr{U}(x_0, \varrho; \mathscr{AC})$ and $\varepsilon \in [0, \varepsilon_3]$. In virtue of the continuity of Ξ in (δ, ε) there exist $\sigma = \sigma(\delta_0) > 0$ and $\varepsilon_4 = \varepsilon_4(\delta_0) > 0$ $(\varepsilon_4 \leqq \varepsilon_3)$ such that $\Xi(\varepsilon)(\delta) \in \mathscr{U}(x_0, \varrho; \mathscr{AC})$ for all $\delta \in \mathscr{U}(\delta_0, \sigma; \mathscr{R}_v)$ and $\varepsilon \in [0, \varepsilon_4]$. Hence for $\delta_1, \delta_2 \in \mathscr{U}(\delta_0, \sigma; \mathscr{R}_v)$ and $\varepsilon \in [0, \varepsilon_4]$

$$\|\Xi(\varepsilon)(\delta_2) - \Xi(\varepsilon)(\delta_1)]_{\mathscr{AC}} \leqq \|V\|_{\mathscr{AC}} \|\delta_2 - \delta_1\| + \varepsilon\beta \|\Xi(\varepsilon)(\delta_2) - \Xi(\varepsilon)(\delta_1)\|_{\mathscr{AC}}\,.$$

Wherefrom, putting $\varepsilon_1 = \varepsilon_1(\delta_0) = \min(\varepsilon_4, (2\beta)^{-1})$ our assertion follows.

Remark 4,2. It could be shown that if $\delta_0 \in \mathscr{R}_v$, $x_0 = V(.)\,\delta_0$ and S possesses for all $(x, \varepsilon) \in \mathscr{U}(x_0, \varrho_1; \mathscr{AC}) \times [0, \varepsilon_1]$ $(\varrho_1 > 0)$ a G-derivative with respect to x continuous in $(x, \varepsilon) \in \mathscr{U}(x_0, \varrho_1; \mathscr{AC}) \times [0, \varepsilon_1]$, then there exist $\varepsilon_2 > 0$ and $\varrho_2 > 0$

such that for all $(\delta, \varepsilon) \in \mathscr{U}(\delta_0, \varrho_2; \mathscr{R}_\nu) \times [0, \varepsilon_2]$ Ξ possesses a G-derivative with respect to δ continuous in $(\delta, \varepsilon) \in \mathscr{U}(\delta_0, \varrho_2; \mathscr{R}_\nu) \times [0, \varepsilon_2]$. (For $\bar{\delta} \in \mathscr{R}_\nu$

$$[\Xi'_\delta(\varepsilon)(\delta)]\,\bar{\delta} = (i - \varepsilon[S'_x(\varepsilon)(\Xi(\varepsilon)(\delta))])^{-1}(V(.)\,\bar{\delta}),$$

where i denotes the identity operator in $\mathscr{AC}$.)

Inserting $x = \Xi(\varepsilon)(\delta)$ into $(4,9)_2$ we get

$$\Theta(\varepsilon)(\delta) = T(\varepsilon)(\Xi(\varepsilon)(\delta)) = 0. \tag{4,12}$$

The second existence theorem for the critical case is based on the notion of the Brouwer topological degree and does not require any assumptions of the differentiability of Φ and Λ. It follows from the following proposition. (For the definition of the Brouwer topological degree see J. CRONIN [4].)

Proposition 3. *Let $\mathscr{G}$ be a bounded open set in $\mathscr{R}_\nu$ and let f be a continuous mapping of the closure $\bar{\mathscr{G}}$ of $\mathscr{G}$ in $\mathscr{R}_\nu$ into $\mathscr{R}_\nu$. Let $f(\delta) \neq 0$ on the frontier $\partial\mathscr{G}$ of $\mathscr{G}$ in $\mathscr{R}_\nu$ and let the degree $\mathrm{d}(f, \mathscr{G}, 0)$ of f with respect to $0 \in \mathscr{R}_\nu$ and $\mathscr{G}$ be nonzero. Then the equation $f(\delta) = 0$ has at least one solution in $\mathscr{G}$ and there exists $\eta > 0$ such that for every continuous mapping $g : \bar{\mathscr{G}} \to \mathscr{R}_\nu$ with $\sup_{\delta \in \partial\mathscr{G}} \|f(\delta) - g(\delta)\| < \eta$ there exists in $\mathscr{G}$ at least one solution of the equation $g(\delta) = 0$.*

Proof. The mapping

$$h : \delta \in \bar{\mathscr{G}}, \quad t \in [0, 1] \to h(\delta, t) = f(\delta) + (1 - t)(g(\delta) - f(\delta))$$

is a continuous mapping of $\bar{\mathscr{G}} \times [0, 1]$ into $\mathscr{R}_\nu$ with $h(\delta, 0) = g(\delta)$ and $h(\delta, 1) = f(\delta)$. If

$$\|f(\delta)\| \geqq 2\eta > 0 \quad \text{and} \quad \|f(\delta) - g(\delta)\| < \eta \quad \text{on} \quad \partial\mathscr{G},$$

then for all $\delta \in \partial\mathscr{G}$ and $t \in [0, 1]$

$$\|h(\delta, t)\| \geqq \|f(\delta)\| - \|f(\delta) - g(\delta)\| > \eta > 0.$$

Proposition 2 is now an immediate consequence of Existence Theorem ([4]. p. 32) and of Theorem of Invariance under Homotopy ([4], p. 31).

Theorem 4,3. *Let the boundary value problem $(\mathscr{P}_\varepsilon)$ fulfilling the assumptions $(\mathscr{A})$ be given. Let the limit problem $(\mathscr{P}_0)$ admit a nonzero solution (i.e. $\det B = 0$). Let the matrix function V and the operators T and T_0 be given by (4,7), (4,10) and (4,11). Suppose*

(I) *the limit problem $(\mathscr{P}_0)$ possesses a solution x_0 such that $T_0(\delta_0) = 0$ for $\delta_0 = (b_0)_{\mathscr{V}}$, where*

$$b_0 = \begin{pmatrix} x_0(a) \\ \displaystyle\int_a^b \left[\mathrm{d}_t \int_a^b K_2(s)\, G(s, t)\, \mathrm{d}s\right] x_0(t) \end{pmatrix}$$

(II) *there exists a bounded open subset* $\mathcal{G}$ *of* $\mathcal{R}_\nu$ *such that* $T_0(\delta) \neq 0$ *for* $\delta \in \partial\mathcal{G}$ *and* $d(T_0, \mathcal{G}, 0) \neq 0$.

Then there exists $\varepsilon^* > 0$ *such that for every* $\varepsilon \in [0, \varepsilon^*]$ *there exists at least one solution to* $(\mathcal{P}_\varepsilon)$.

Proof. It is easy to verify that the operator $T_0 : R_\nu \times [0, \varepsilon_0] \to \mathcal{R}_\nu$ is locally lipschitzian in $\delta \in \mathcal{R}_\nu$ near $\varepsilon = 0$ and continuous in $\varepsilon \in [0, \eta_1]$ with some $\eta_1 > 0$ small enough for any $\delta \in \mathcal{R}_\nu$ fixed. By Heine-Borel Covering Theorem we may assume that there exists $\eta_2 > 0$ such that Θ is uniformly continuous in $(\delta, \varepsilon) \in \bar{\mathcal{G}} \times \times [0, \eta_2]$. Applying Proposition 3 to the equation (4,12) we complete the proof.

Remark 4,3. The methods of this paragraph can be also applied if $L \in \mathcal{BV}_{m,n}$ and $\Lambda : \mathcal{AC} \to \mathcal{R}_m$, where generally $m \neq n$. Of course, the situation is no more predetermined so largely by the fact whether the limit problem $(\mathcal{P}_0)$ admits a nonzero solution or not. Let the $(m + n') \times (n + n')$-matrix B be defined by (4,4), (3,9), (3,10) and (3,12). Let the $n \times (n + n')$-matrix function U and the operators $R_0 : \mathcal{AC} \times \times [0, \varepsilon_0] \to \mathcal{AC}$ and $R : \mathcal{AC} \times [0, \varepsilon_0] \to \mathcal{R}_{n+n'}$ be given by (4,4) and (4,6). Then again an n-vector function $x \in \mathcal{AC}$ is a solution to the boundary value problem $(\mathcal{P}_\varepsilon)$ iff a couple (x, b), where

$$b = \begin{pmatrix} x(a) \\ \int_a^b \left[d_t \int_a^b K_2(s)\, G(s, t)\, ds \right] x(t) \end{pmatrix},$$

is a solution to the system of operator equations ((4,5))

$$-x + U(.)\, b + \varepsilon R_0(\varepsilon)(x) = 0,$$
$$Bb + \varepsilon R(\varepsilon)(x) = 0.$$

Let $m < n$ and rank $(B) = m + n'$. Let us denote $\mathcal{M} = \{1, 2, \ldots, m + n'\}$ and let $\mathcal{V} \subset \mathcal{N}_0$ be such that $\nu(\mathcal{V}) = n - m$ and det $B_{\mathcal{M},\mathcal{V}^*} \neq 0$. Putting $\gamma = b_{\mathcal{V}^*}$, $\delta = b_{\mathcal{V}}$, $B_1 = B_{\mathcal{M},\mathcal{V}^*}$ and $B_2 = B_{\mathcal{M},\mathcal{V}}$, (4,5) becomes

$$-x + V(.)\, \delta + \varepsilon S(\varepsilon)(x) = 0, \tag{4,13}$$

where the $n \times (n - m)$-matrix function V and the operator S are given by (4,10). Given an arbitrary $\delta_0 \in \mathcal{R}_{n-m}$, the function $x_0 = V(.)\, \delta_0$ is a solution to the limit problem $(\mathcal{P}_0)$ and by Proposition 2 there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in [0, \varepsilon^*]$ there exists a unique solution $x^*(\varepsilon)$ to $(\mathcal{P}_\varepsilon)$ continuous in $\varepsilon \in [0, \varepsilon^*]$ as a mapping $[0, \varepsilon^*] \to \mathcal{AC}$ and such that $x^*(0) = x_0$. The given boundary value problem $(\mathcal{P}_\varepsilon)$ can be treated similarly as the noncritical case for $m = n$, although the limit problem $(\mathcal{P}_0)$ possesses a nonzero solution. On the other hand, if $\varepsilon > 0$, $m > n$ and rank $(B) = = n + n'$, then (4,5) is equivalent to the system

$$-x + \varepsilon S(\varepsilon)(x) = 0, \quad T(\varepsilon)(x) = 0 \tag{4,14}$$

with S and T defined analogously as in (4,10). Now the function x is uniquely determined by $(4,14)_1$ and to be a solution to the given problem $(\mathscr{P}_\varepsilon)$ with $\varepsilon > 0$ it has to satisfy $(4,14)_2$. Hence the boundary value problem $(\mathscr{P}_\varepsilon)$ has generally no solution, though the limit problem $(\mathscr{P}_0)$ has only the trivial solution (cf. Corollary 1 of Theorem 3,1). In the other cases we meet an analogous situation.

5. LINEAR BOUNDARY VALUE PROBLEM — FUNCTIONAL ANALYSIS APPROACH

Let us turn back to the linear boundary value problem $(\mathscr{P})$ given by

$$(5,1) \qquad \dot{x} - A(t)\,x - \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) = f(t)\,,$$

$$(5,2) \qquad \int_a^b [\mathrm{d}L(s)]\, x(s) \quad = l\,,$$

where $A \in \mathscr{L}^1_{n,n}$, $f \in \mathscr{L}^1$, $G \in \mathscr{L}^2[\mathscr{BV}]$, $L \in \mathscr{BV}_{m,n}$ and $l \in \mathscr{R}_m$. Without any loss of generality we may assume that for all $t \in J$ $G(t, .)$ and L are continuous from the right on the open interval (a, b).

In [20] D. Wexler derived the true adjoint (in the sense of functional analysis) to the boundary value problem

$$\dot{x} - A(t)\,x = f(t)\,, \quad Lx = l\,,$$

where $A \in \mathscr{L}^1_{n,n}$, $f \in \mathscr{L}^1$, L is a continuous linear mapping of $\mathscr{AC}$ into some B-space Λ and $l \in \Lambda$. In this paragraph we apply his ideas to the boundary value problem $(\mathscr{P})$. The special form of the operator L and the different choice of a dual space to the space $\mathscr{C}$ of continuous functions on J (measures are replaced by functions of bounded variation) enables us to prove that the problem $(\mathscr{P}^*)$ derived in § 3 ((3,16), (3,17)) is equivalent to the true adjoint of $(\mathscr{P})$.

First, we have to introduce some new notations.

$\mathscr{L}^\infty$ denotes the B-space of all row n-vector functions measurable and essentially bounded on J. It is well-known that $\mathscr{L}^\infty$ is a dual B-space to the B-space $\mathscr{L}^1 = \mathscr{L}^1_{n,1}$ of column n-vector functions L-integrable on J. The value of a functional $y' \in \mathscr{L}^\infty$ on $x \in \mathscr{L}^1$ is given by

$$\langle x, y' \rangle_{\mathscr{L}} = \int_a^b y'(s)\, x(s)\, \mathrm{d}s$$

and the norm of y' is $\|y'\|_\infty = \sup\operatorname{ess}_{t \in J} \|y'(t)\|$. Functions from $\mathscr{L}^\infty$ which coincide a.e. on J are identified with one another.

$\mathscr{BV}^+$ is the B-space of all row n-vector functions of bounded variation on J and continuous from the right on (a, b) ($\mathscr{BV}^+ \subset \mathscr{BV}_{1,n}$). $\mathscr{C}^*$ denotes the dual B-space to the space $\mathscr{C}$ of column n-vector functions continuous on J, i.e. $\mathscr{C}^*$ is formed by all functions from $\mathscr{BV}^+$ which vanish at a. Given an arbitrary functional $y' \in \mathscr{C}^*$, its value on $x \in \mathscr{C}$ is given by

$$\langle x, y' \rangle_{\mathscr{C}} = \int_a^b [\mathrm{d}y'(t)]\, x(t)$$

and $\|y'\|_{\mathscr{C}^*} = \mathrm{var}_a^b\, y'$. The zero element of $\mathscr{C}^*$ is the function vanishing everywhere on J.

$\mathscr{AC}^*$ denotes the dual B-space to the B-space $\mathscr{AC}$ of column n-vector functions absolutely continuous on J. The value of a functional $y' \in \mathscr{AC}^*$ on $x \in \mathscr{AC}$ is denoted by $\langle x, y' \rangle_{\mathscr{AC}}$. Let us notice that we can consider ([20] 2,1) $\mathscr{C}^* \subset \mathscr{AC}^*$ and $\langle x, y' \rangle_{\mathscr{AC}} = \langle x, y' \rangle_{\mathscr{C}}$ for $x \in \mathscr{AC}$ and $y' \in \mathscr{C}^*$. Moreover, since the topology of $\mathscr{AC}$ is stronger than that induced by $\mathscr{C}(\|x\|_{\mathscr{C}} = \sup_J \|x(t)\|)$ and $\mathscr{AC}$ is dense in $\mathscr{C}$, the zero elements of $\mathscr{AC}^*$ and $\mathscr{C}^*$ coincide.

The operators

$$D : x \in \mathscr{AC} \to \dot{x} \in \mathscr{L}^1 , \qquad A : x \in \mathscr{AC} \to A(t)\, x(t) \in \mathscr{L}^1 ,$$

$$G : x \in \mathscr{AC} \to \int_a^b [\mathrm{d}_s G(t, s)]\, x(s) \in \mathscr{L}^1 , \quad \mathscr{B}_1 : x \in A\mathscr{C} \to Dx - Ax - Gx \in \mathscr{L}^1$$

and

$$\mathscr{B}_2 : x \in \mathscr{AC} \to \int_a^b [\mathrm{d}L(s)]\, x(s) \in \mathscr{R}_m$$

are linear and continuous. Hence the operator

$$\mathscr{B} : x \in \mathscr{AC} \to \begin{pmatrix} \mathscr{B}_1 x \\ \mathscr{B}_2 x \end{pmatrix} \in \mathscr{L}^1 \times \mathscr{R}_m \tag{5,3}$$

is linear and continuous, too. Its adjoint $\mathscr{B}^*$ is a linear continuous operator $\mathscr{L}^\infty \times \mathscr{R}_m^* \to \mathscr{AC}^*$ defined on $(y', \lambda') \in \mathscr{L}^\infty \times \mathscr{R}_m^*$ by

$$\langle \mathscr{B}_1 x, y' \rangle_{\mathscr{L}} + \lambda'(\mathscr{B}_2 x) = \langle x, \mathscr{B}^*(y', \lambda') \rangle_{\mathscr{AC}} \quad \text{for all} \quad x \in \mathscr{AC} .$$

The boundary value problem (P) can be now written in the form

$$\mathscr{B}x = \begin{pmatrix} f \\ l \end{pmatrix} . \tag{5,4}$$

Let us derive an explicit form for $\mathscr{B}^*$. For $x \in \mathscr{AC}$ and $(y', \lambda') \in \mathscr{L}^\infty \times \mathscr{R}_m^*$ we have

$$\langle x, \mathscr{B}^*(y', \lambda') \rangle_{\mathscr{AC}} = \langle \mathscr{B}_1 x, y' \rangle_{\mathscr{L}} + \lambda'(\mathscr{B}_2 x) = \langle Dx, y' \rangle_{\mathscr{L}} - \langle Ax, y' \rangle_{\mathscr{L}} - \\ - \langle Gx, y' \rangle_{\mathscr{L}} + \lambda'(\mathscr{B}_2 x) = \langle x, D^* y' - A^* y' - G^* y' + \mathscr{B}_2^* \lambda' \rangle_{\mathscr{AC}}$$

and

$$\mathscr{B}^*(y', \lambda') = D^* y' - A^* y' - G^* y' + \mathscr{B}_2^* \lambda' ,$$

where D^*, A^*, G^* and $\mathscr{B}_2^*$ are adjoint operators to D, A, G and $\mathscr{B}_2$, respectively. Thus the adjoint equation to (5,4) is

$$D^* y' - A^* y' - G^* y' + \mathscr{B}_2^* \lambda' = 0 \tag{5,5}$$

(where 0 means the zero element of $\mathscr{AC}^*$, of course).

Given an arbitrary $x \in \mathscr{AC}$ and $y' \in \mathscr{L}^\infty$, it holds by Lemma 2,7

$$\int_a^b y'(t) \left(\int_a^b [\mathrm{d}_s G(t, s)]\, x(s) \right) \mathrm{d}t = \int_a^b \left[\mathrm{d}_t \int_a^b y'(s) \left(G(s, t) - G(s, a) \right) \mathrm{d}s \right] x(t) .$$

As a consequence, since $\int_a^b y'(s) \left(G(s, t) - G(s, a) \right) \mathrm{d}s \in \mathscr{C}^*$, we have

$$\langle x, G^* y' \rangle_{\mathscr{AC}} = \langle Gx, y' \rangle_{\mathscr{L}} = \left\langle x, \int_a^b y'(s) \left(G(s, t) - G(s, a) \right) \mathrm{d}s \right\rangle_{\mathscr{C}}$$

and

$$G^* : y' \in \mathscr{L}^\infty \to \int_a^b y'(s) \left(G(s, t) - G(s, a) \right) \mathrm{d}s \in \mathscr{C}^* . \tag{5,6}$$

By a similar argument the operators A^* and $\mathscr{B}_2^*$ are defined by

$$A^* : y' \in \mathscr{L}^\infty \to \int_a^t y'(s)\, A(s)\, \mathrm{d}s \in \mathscr{C}^* \tag{5,7}$$

and

$$B_2^* : \lambda' \in \mathscr{R}_m^* \to \lambda' \left(L(t) - L(a) \right) \in \mathscr{C}^* . \tag{5,8}$$

Furthermore,

$$D^* : y' \in \mathscr{C}^* \to -y'(t) + R(y')(t) \in \mathscr{C}^* , \tag{5,9}$$

where

$$R(y')(t) = \begin{cases} y'(a) & \text{for } t = a , \\ 0 & \text{for } a < t < b , \\ y'(b) & \text{for } t = b . \end{cases} \tag{5,10}$$

The operator $Dx - Ax$ maps $\mathscr{AC}$ onto $\mathscr{L}^1$. Hence $y' \in \mathscr{L}^\infty$ being an arbitrary solution to $D^* y' - A^* y' = 0$, $y'(t) = 0$ a.e. on J. Moreover, given an arbitrary $g' \in \mathscr{C}^*$, the equation

$$D^* y' - A^* y' = g' \tag{5,11}$$

has a solution in $\mathscr{L}^\infty$ iff

$$\int_a^b [\mathrm{d} g'(s)]\, X(s) = 0 , \tag{5,12}$$

where X denotes again the fundamental matrix solution of $Dx - Ax = 0$ (cf. (3,3)). Suppose $g' \in \mathscr{C}^*$ and (5,11) has a solution in $\mathscr{L}^\infty$. Then this solution is unique in $\mathscr{L}^\infty$. Let us put for $t \in J$

$$z'(t) = -\left(\int_a^t [dg'(s)]\, X(s)\right) X^{-1}(t)\,.$$

Since $z' \in \mathscr{C}^*$ and $R(z')(t) \equiv 0$ by (5,10) and (5,12), we have by (5,7), (5,9), Lemma 1,1 and (3,3)

$$D^*z' - A^*z' = -z'(t) + \int_a^t \left(\int_a^s [dg'(\sigma)]\, X(\sigma)\right) X^{-1}(s)\, A(s)\, ds =$$

$$= -z'(t) + \int_a^t [dg'(s)] \left(X(s) \int_s^t X^{-1}(\sigma)\, A(\sigma)\, d\sigma\right) = g'(t)\,.$$

It follows that z' is the unique solution of (5,11) in $\mathscr{L}^\infty$. Applying this to (5,5) and taking into account (5,6)–(5,8), we obtain that to any solution $(y', \lambda') \in \mathscr{L}^\infty \times \mathscr{R}_m^*$ of (5,5) there exists a solution (η', λ') of (5,5) such that $\eta' \in \mathscr{BV}^+$, η' is continuous at a from the right and at b from the left and $y'(t) = \eta'(t)$ a.e. on J ($y' = \eta'$ in $\mathscr{L}^\infty$). Consequently, to find all solutions of (5,5) in $\mathscr{L}^\infty \times \mathscr{R}_m^*$, it is sufficient to consider instead of $\mathscr{B}^*$ its restriction $\mathscr{B}_0^*$ on $\mathscr{V} \times \mathscr{R}_m^*$, where $\mathscr{V}$ is formed by all functions from $\mathscr{BV}^+$ which are continuous at a from the right and at b from the left. By (5,6)–(5,9)

$$\mathscr{B}_0^*(y', \lambda') = -y'(t) + R(y')(t) - \int_a^t y'(s)\, A(s)\, ds + \lambda'(L(t) - L(a)) -$$

$$- \int_a^b y'(s)\,(G(s, t) - G(s, a))\, ds \in \mathscr{C}^*\,.$$

In other words, the equation (5,5) for $(y', \lambda') \in \mathscr{L}^\infty \times \mathscr{R}_m^*$ is equivalent to the equation

(5,13)
$$-y'(t) + R(y')(t) - \int_a^t y'(s)\, A(s)\, ds + \lambda'(L(t) - L(a)) -$$

$$- \int_a^b y'(s)\,(G(s, t) - G(s, a))\, ds = 0 \quad \text{on} \quad J$$

for $(y', \lambda') \in \mathscr{V} \times \mathscr{R}_m^*$. In particular, (5,13) yields

$$y'(a) - y'(a) = 0 \quad \text{for} \quad t = a\,,$$

(5,14)
$$y'(t) = -\int_a^t y'(s)\, A(s)\, ds + \lambda'(L(t) - L(a)) - \int_a^b y'(s)\,(G(s, t) - G(s, a))\, ds$$

for $t \in (a, b)$,

and

$$(5,15)\quad 0 = -\int_a^b y'(s)\,A(s)\,\mathrm{ds} + \lambda'(L(b) - L(a)) - \int_a^b y'(s)\,(G(s,b) - G(s,a))\,\mathrm{ds}$$

for $t = b$.

Furthermore, from (5,14) we have

$$(5,16)\quad y'(a) = y'(a+) = \lambda'(L(a+) - L(a)) - \int_a^b y'(s)\,(G(s,a+) - G(s,a))\,\mathrm{ds}$$

and consequently (5,14) becomes

$$(5,17)\quad y'(t) = y'(a) - \int_a^t y'(s)\,A(s)\,\mathrm{ds} + \lambda'(L(t) - L(a+)) - \\ - \int_a^b y'(s)\,(G(s,t) - G(s,a+))\,\mathrm{ds} \quad \text{for} \quad t \in (a,b).$$

Making use of (5,15), (5,14) can be modified as follows

$$(5,18)\quad y'(t) = \int_t^b y'(s)\,A(s)\,\mathrm{ds} - \lambda'(L(b) - L(t)) + \\ + \int_a^b y'(s)\,(G(s,b) - G(s,t))\,\mathrm{ds} \quad \text{for} \quad t \in (a,b).$$

Thus

$$(5,19)\quad y'(b) = y'(b-) = -\lambda'(L(b) - L(b-)) + \int_a^b y'(s)\,(G(s,b) - G(s,b-))\,\mathrm{ds}$$

and

$$(5,20)\quad y'(t) = y'(b) + \int_t^b y'(s)\,A(s)\,\mathrm{ds} + \lambda'(L(t) - L(b-)) - \\ - \int_a^b y'(s)\,(G(s,t) - G(s,b-))\,\mathrm{ds} \quad \text{for} \quad t \in (a,b).$$

Let us define

$$G_0(t,s) = \begin{cases} G(t,a+) & \text{for } t \in J \text{ and } s = a, \\ G(t,s) & \text{for } t \in J \text{ and } a < s < b, \\ G(t,b-) & \text{for } t \in J \text{ and } s = b, \end{cases} \qquad L_0(s) = \begin{cases} L(a+) & \text{for } s = a, \\ L(s) & \text{for } a < s < b, \\ L(b-) & \text{for } s = b, \end{cases}$$

$C(t) = G(t, a+) - G(t, a)$ and $D(t) = G(t, b) - G(t, b-)$ for $t \in J$ and

$$M = L(a+) - L(a), \quad N = L(b) - L(b-).$$

Then from (5,16), (5,17), (5,19) and (5,20) we can conclude that the equation (5,13) (and hence also (5,5)) is equivalent to the system of equations for $(y', \gamma') \in \mathscr{L}^\infty \times \mathscr{R}_m^*$ $(\gamma' = -\lambda')$

$$(5,21) \qquad y'(t) = y'(a) - \int_a^t y'(s)\, A(s)\, ds - \gamma'(L_0(t) - L_0(a)) - \\ - \int^b y'(s)\,(G_0(s, t) - G_0(s, a))\, ds \quad \text{on} \quad J,$$

$$(5,22) \quad y'(a) = -\gamma' M - \int_a^b y'(s)\, C(s)\, ds, \quad y'(b) = \gamma' N + \int_a^b y'(s)\, D(s)\, ds.$$

In the introduced notation, the original boundary value problem ($\mathcal{P}$) assumes the form

$$\dot{x} = A(t)\, x + C(t)\, x(a) + D(t)\, x(b) + \int_a^b [d_s G_0(t, s)]\, x(s) + f(t),$$

$$M\, x(a) + N\, x(b) + \int_a^b [dL_0(s)]\, x(s) = l$$

and (5,21), (5,22) is exactly its adjoint ($\mathcal{P}^*$) derived in § 3 ((3,16), (3,17)).

As a consequence we have that *the adjoint* ($\mathcal{P}^*$) *of* ($\mathcal{P}$) *from* § 3 *and the true adjoint* (5,5) *of* ($\mathcal{P}$) *are equivalent.*

From the fundamental "alternative" theorem concerning linear equations in B-spaces ([5] VI, § 6) and from Theorem 3,1 it follows that the operator $\mathcal{B}$ of the boundary value problem ($\mathcal{P}$) defined by (5,3) has a closed range in $\mathcal{L}^1 \times \mathcal{R}_n$.

Remark. The closedness of the range $\mathcal{B}(\mathcal{AC})$ of the operator $\mathcal{B}$ can be also shown directly in a similar way as D. Wexler did in [20] § 3 for the operator

$$x \in \mathcal{AC} \to \begin{pmatrix} \dot{x} - A(t)\, x \\ Lx \end{pmatrix} \in \mathcal{L}^1 \times \mathcal{R}_m,$$

where L is a continuous linear mapping of $\mathcal{AC}$ into some B-space Λ. In fact, let the matrix B and the operator

$$\Psi : \begin{pmatrix} f \\ l \end{pmatrix} \in \mathcal{L}^1 \times \mathcal{R}_m \to \Psi(f, l) = w \in \mathcal{R}_{m+n'}$$

be defined by (4,4), (3,9), (3,10) and (3,12). Let us put

$$\Theta : b \in R_{n+n'} \to Bb \in \mathcal{R}_{m+n'}.$$

Given $f \in \mathcal{L}^1$ and $l \in \mathcal{R}_m$, the corresponding boundary value problem ($\mathcal{P}$) possesses a solution (i.e. $(f', l')' \in \mathcal{B}(\mathcal{AC})$) iff $\Psi(f, l) \in \Theta(\mathcal{R}_{n+n'})$. Hence

$$\mathcal{B}(\mathcal{AC}) = \Psi_{-1}(\Theta(\mathcal{R}_{n+n'})).$$

Since Ψ and Θ are continuous linear operators and $\dim \Theta(\mathcal{R}_{n+n'}) < \infty$, the set $\Psi_{-1}(\Theta(\mathcal{R}_{n+n'}))$ is certainly closed.