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A BAIRE FUNCTION NOT COUNTABLY DECOMPOSABLE INTO CONTINUOUS FUNCTIONS

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In connection with a problem of KARTÁK [1], VRKOČ recently constructed [2] a measurable real function f on $I = [0, 1]$ such that I cannot be partitioned into countably many sets A_n with each restriction $f \mid A_n$ continuous. He asked whether for every Baire function there does exist such a partition of I into Borel sets. Here it will be shown that, on the contrary, there exists a function of Baire class 1 for which there exists no such partition whatever, even into non-Borel sets.

Theorem 1. *If $f: A \to I$ is continuous, where A is a subset of I, then given $\varepsilon > 0$ there exists a closed set $F \subseteq I \times I$ such that $F \cap \mathrm{Gr}(f) = \emptyset$ and $m(F_x) \geqq 1 - \varepsilon$ for all $x \in I$.*

Proof. For each element $u \in A$, let $J_u = \{(u, y) : |y - f(u)| < \frac{1}{2}\varepsilon\}$ and $K_u = (\{u\} \times I) \setminus J_u$. Denote by E the closure of the set $D = \cup\{K_u : u \in A\}$. First, we observe that $E \cap \mathrm{Gr}(f) = \emptyset$. Indeed, given any point $(u, f(u)) \in \mathrm{Gr}(f)$, we can choose $\delta > 0$ so small that

$$v \in A \,\&\, |v - u| < \delta \Rightarrow |f(v) - f(u)| < \varepsilon/4\,;$$

then the open rectangle with centre at $(u, f(u))$, width 2δ, and height $\frac{1}{2}\varepsilon$ contains no point of D.

Next, we prove that *for every $x \in \bar{A}$, the set $I \setminus E_x$ is an interval of length at most ε, open relative to I.* Since E_x is closed, it is enough to show that if $y_1, y_2 \in I \setminus E_x$ and $y_1 < y < y_2$ then (i) $y_2 - y_1 < \varepsilon$ and (ii) $y \in I \setminus E_x$. Consider any $u \in A$ with $|u - x| < \delta$, where δ is the smaller of the distances of (x, y_1) and (x, y_2) from E; then $(u, y_1) \in J_u$ and $(u, y_2) \in J_u$, and (i) follows. Moreover $|y - f(u)| < \frac{1}{2}\varepsilon - \min(y_2 - y, y - y_1)$; hence the open rectangle with centre (u, y), width 2δ, and height $2 \min(y_2 - y, y - y_1)$ contains no point of D, and this establishes (ii).

To construct F we adjoin to E a large part of each strip $S = (c, d) \times I$, where (c, d) is an interval of $I \setminus \bar{A}$; namely, the whole of S except for an open "corridor"