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PERIODIC SOLUTIONS OF A WEAKLY NONLINEAR WAVE EQUATION IN ONE DIMENSION

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In applications of the theory of partial differential equations the following problem may rise up: The existence and uniqueness of classical (twice continuously differentiable) solutions are to be investigated for the wave equation

$$\tilde{u}_{\tau\tau}(\tau, x) - \tilde{u}_{xx}(\tau, x) = g_1(\tau, x) + \varepsilon F_1(\tilde{u}, \varepsilon)(\tau, x)$$

considered in the domain $\{(\tau, x) \mid \tau \in (-\infty, +\infty), x \in \langle 0, \pi \rangle\}$ under the periodicity condition $\tilde{u}(\tau + 2\pi\omega, x) - \tilde{u}(\tau, x) = 0$ and boundary conditions of various types at the points $x = 0$ and $x = \pi$, where g_1 and F_1 are $2\pi\omega$-periodic in the variable τ. The problem with $\omega = 1$ is solved in [1], the special case of boundary conditions of Dirichlet type for an arbitrary ω in [2]. While the both papers utilize the Poincaré method, in this paper a different method is used – solutions are sought in the form of Fourier series with respect to τ, which guarantees the periodicity of the solutions.

Performing the transformation $\tau = \omega t$ and putting $u(t, x) = \tilde{u}(\tau, x)$, the above equation assumes the form

$$u_{tt}(t, x) - \omega^2 u_{xx}(t, x) = g(t, x) + \varepsilon F(u, \varepsilon)(t, x), \tag{0.1}$$

where g and F are 2π-periodic in the variable t, and the periodicity condition mentioned above reads

$$u(t + 2\pi, x) - u(t, x) = 0, \quad t \in (-\infty, +\infty), x \in \langle 0, \pi \rangle. \tag{0.2}$$

As for the boundary conditions the paper deals with the following three types:

$$\begin{aligned} u(t, 0) &= {}^0h(t) + \varepsilon\, {}^0X(u, \varepsilon)(t), \\ u(t, \pi) &= {}^1h(t) + \varepsilon\, {}^1X(u, \varepsilon)(t); \end{aligned} \tag{0.3}$$

$$\begin{aligned} u_x(t, 0) + \alpha_0 u(t, 0) &= {}^0h(t) + \varepsilon\, {}^0X(u, \varepsilon)(t), \\ u_x(t, \pi) + \alpha_1 u(t, \pi) &= {}^1h(t) + \varepsilon\, {}^1X(u, \varepsilon)(t); \end{aligned} \tag{0.4}$$

$$
\begin{aligned}
(0.5) \qquad u(t, 0) &= {}^0h(t) + \varepsilon\, {}^0X(u, \varepsilon)(t)\,, \\
u_x(t, \pi) + \alpha u(t, \pi) &= {}^1h(t) + \varepsilon\, {}^1X(u, \varepsilon)(t)\,.
\end{aligned}
$$

The paper is divided into three paragraphs. The first, preparatory paragraph contains definitions and lemmas, the second one deals with the linear problem, i.e. with the case where $\varepsilon = 0$. The results obtained are used in the final paragraph to solve the weakly nonlinear problem.

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1. SOME DEFINITIONS AND AUXILIARY LEMMAS

$\mathscr{R}$, $\mathscr{N}$ and $\mathscr{M}$ are the symbols for the sets of reals, positive integers and integers, respectively. $\mathscr{C}^{(k)}[a, b]$ (or $\tilde{\mathscr{C}}^{(k)}[a, b]$, if need be) denotes the space of real-valued (or complex-valued) functions the k-th derivative of which is continuous on $\langle a, b\rangle$, the letter $\mathscr{I}$ stands for the closed interval $\langle 0, \pi\rangle$. Finally, let us write $e_k(t) = e^{ikt}$, $t \in \mathscr{R}$, $k \in \mathscr{M}$.

With respect to the method used it is convenient to introduce some functional spaces, derived from Sobolev spaces $\mathscr{W}_2^k(0, 2\pi)$.

Definition 1.1. Let us denote by $\mathscr{H}^r_{2\pi}$ $(r \in \mathscr{N})$ the subspace of real-valued functions $f \in \mathscr{W}_2^r(0, 2\pi)$ fulfilling the relation

$$f^{(k)}_{\text{abs}}(2\pi) = f^{(k)}_{\text{abs}}(0)\,, \quad 0 \leqq k < r\,, \quad k \in \mathscr{M}\,,$$

where $f_{\text{abs}} \in \mathscr{C}^{(r-1)}[0, 2\pi]$ is the absolute continuous representant of f (i.e. $f_{\text{abs}} = f$ almost everywhere in $(0, 2\pi)$). The norm in $\mathscr{H}^r_{2\pi}$ is defined by

$$\|f\|_{\mathscr{H}^r_{2\pi}} = \left[\sum_{k=0}^{r} \int_0^{2\pi} |f^{(k)}(t)|^2 \,dt\right]^{1/2}.$$

Remark 1.1. A function $f \in \mathscr{H}^r_{2\pi}$ will usually be identified with the corresponding $f_{\text{abs}} \in \mathscr{C}^{(r-1)}[0, 2\pi]$.

The following lemma may be easily verified:

Lemma 1.1. *$\mathscr{H}^r_{2\pi}$ $(r \in \mathscr{N})$ is a Banach space.*

Definition 1.2. Let us denote by $\mathscr{C}^{(k)}(\mathscr{I}; \mathscr{H}^r_{2\pi})$ $(r \in \mathscr{N}, k \in \mathscr{M}, k \geqq 0)$ the space of mappings $u : \mathscr{I} \to \mathscr{H}^r_{2\pi}$ which are, together with their derivatives $u^{(n)}$, $0 \leqq n \leqq k$, continuous transformations on $\mathscr{I}$ into $\mathscr{H}^r_{2\pi}$. (We shall often write $\mathscr{C}(\mathscr{I}; \mathscr{H}^r_{2\pi})$ instead of $\mathscr{C}^{(0)}(\mathscr{I}; \mathscr{H}^r_{2\pi})$.) The norm in this space is defined by

$$\|u\|_{\mathscr{C}^{(k)}(\mathscr{I};\mathscr{H}^r_{2\pi})} = \max_{x \in \mathscr{I}} \Big[\sum_{n=0}^{k} \|u^{(n)}(x)\|^2_{\mathscr{H}^r_{2\pi}}\Big]^{1/2}.$$

Also the next lemma can be proved easily:

Lemma 1.2. $\mathscr{C}^{(k)}(\mathscr{I}; \mathscr{H}^r_{2\pi})$ $(r \in \mathscr{N}, k \in \mathscr{M}, k \geqq 0)$ *is a Banach space.*

Remark 1.2. Analogously to Remark 1.1, the mapping $u \in \mathscr{C}^{(k)}(\mathscr{I}; \mathscr{H}^r_{2\pi})$ may be identified with the function $u(x, t)$ (of two real variables) defined on the rectangle $\mathscr{I} \times \langle 0, 2\pi \rangle$, whose derivatives $\partial^{n+s}u/\partial x^n \partial t^s$, $n = 0, 1, \ldots, k$, $s = 0, 1, \ldots, r - 1$, are continuous and fulfil the condition

$$\frac{\partial^{n+s}u}{\partial x^n \partial t^s}(x, 0) = \frac{\partial^{n+s}u}{\partial x^n \partial t^s}(x, 2\pi), \quad x \in \mathscr{I}.$$

Remark 1.3. Any function $h \in \mathscr{H}^r_{2\pi}$ $(r \in \mathscr{N})$ can be extended onto $(-\infty, +\infty)$ 2π-periodically preserving its smoothness (the function extended in this way will be denoted by h as well). This fact enables us to define the "translation of the argument" $(. + t_0) : \mathscr{H}^r_{2\pi} \to \mathscr{H}^r_{2\pi}$ for an arbitrary $t_0 \in \mathscr{R}$ by

$$h(.+t_0)(t) = h(t + t_0), \quad t \in \langle 0, 2\pi \rangle, \; h \in \mathscr{H}^r_{2\pi}.$$

Since $\mathscr{H}^r_{2\pi}$ $(r \in \mathscr{N})$ is a subspace of $\mathscr{L}_2(0, 2\pi)$, any function $h \in \mathscr{H}^r_{2\pi}$ can be expressed in the form of a Fourier series $h = \sum_{n \in \mathscr{M}} h_n e_n$, where $h_n = (1/2\pi) \int_0^{2\pi} h(t)\, e_{-n}(t)\, dt$. Analogously, each $u \in \mathscr{C}^{(k)}(\mathscr{I}; \mathscr{H}^r_{2\pi})$ can be written as $u(x) = \sum_{n \in \mathscr{M}} u_n(x)\, e_n$, $x \in \mathscr{I}$. On the other hand, provided that the coefficients h_n $(n \in \mathscr{M})$ or $u_n(x)$ $(n \in \mathscr{M}, x \in \mathscr{I})$ satisfy certain conditions, the corresponding series converges in the space $\mathscr{H}^r_{2\pi}$ or $\mathscr{C}^{(k)}(\mathscr{I}; \mathscr{H}^r_{2\pi})$, respectively.

Definition 1.1'. Let us denote by $\mathfrak{h}^r$ $(r \in \mathscr{N})$ the space of sequences $h = \{h_n\}_{-\infty}^{\infty}$ of complex numbers satisfying the conditions:

(i) $h_{-n} = \bar{h}_n$, $\quad n \in \mathscr{M}$,

(ii) $\sum_{n \in \mathscr{M}} n^{2r}|h_n|^2 < +\infty$.

The norm is defined (putting $0^0 = 1$) by

$$\|h\|_{\mathfrak{h}^r} = \sqrt{(2\pi)}\, [\sum_{m=0}^{r} \sum_{n \in \mathscr{M}} n^{2m}|h_n|^2]^{1/2}.$$

Definition 1.2'. Let us denote by $\mathscr{C}^{(k)}(\mathscr{I}; \mathfrak{h}^r)$ $(k \geqq 0, k \in \mathscr{M}, r \in \mathscr{N})$ the space of sequences $u = \{u_n\}_{-\infty}^{\infty}$ of complex-valued functions defined on $\mathscr{I}$ with the following properties:

(i) $u_{-n}(x) = \overline{u_n(x)}$, $\quad n \in \mathscr{M}, x \in \mathscr{I}$,

(ii) $u_n \in \tilde{\mathscr{C}}^{(k)}(\mathscr{I})$, $\quad n \in \mathscr{M}$,

(iii) the series $\sum_{n \in \mathscr{M}} n^{2r}|u_n^{(s)}(x)|^2$, $\quad s = 0, 1, \ldots, k$, converge uniformly on $\mathscr{I}$.

The norm is defined by

$$\|u\|_{\mathscr{C}^{(k)}(\mathscr{I};\mathfrak{h}^r)} = \sqrt{(2\pi)} \max_{x\in\mathscr{I}} \left[\sum_{s=0}^{k} \sum_{m=0}^{r} \sum_{n\in\mathscr{M}} n^{2m}|u_n^{(s)}(x)|^2\right]^{1/2}.$$

The following important lemma can be proved quite analogously as Lemma 1.3 in [3].

Lemma 1.3. *The spaces* $\mathscr{H}^r_{2\pi}$ *and* $\mathfrak{h}^r$ $(r \in \mathscr{N})$ *are isomorphic and isometric. This relation also holds between* $\mathscr{C}^{(k)}(\mathscr{I}; \mathscr{H}^r_{2\pi})$ *and* $\mathscr{C}^{(k)}(\mathscr{I}; \mathfrak{h}^r)$ $(r \in \mathscr{N},\ k \in \mathscr{M},\ k \geqq 0)$.

Let us introduce the space $\mathscr{U} = \mathscr{C}(\mathscr{I}; \mathscr{H}^3_{2\pi}) \cap \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cap \mathscr{C}^{(2)}(\mathscr{I}; \mathscr{H}^1_{2\pi})$ with the norm

$$\|u\|_{\mathscr{U}} = \max\left\{\|u\|_{\mathscr{C}(\mathscr{I};\mathscr{H}^3_{2\pi})},\ \|u\|_{\mathscr{C}^{(1)}(\mathscr{I};\mathscr{H}^2_{2\pi})},\ \|u\|_{\mathscr{C}^{(2)}(\mathscr{I};\mathscr{H}^1_{2\pi})}\right\}.$$

Analogously: $\mathfrak{u} = \mathscr{C}(\mathscr{I}; \mathfrak{h}^3) \cap \mathscr{C}^{(1)}(\mathscr{I}; \mathfrak{h}^2) \cap \mathscr{C}^{(2)}(\mathscr{I}; \mathfrak{h}^1)$,

$$\|u\|_{\mathfrak{u}} = \max\left\{\|u\|_{\mathscr{C}(\mathscr{I};\mathfrak{h}^3)},\ \|u\|_{\mathscr{C}^{(1)}(\mathscr{I};\mathfrak{h}^2)},\ \|u\|_{\mathscr{C}^{(2)}(\mathscr{I};\mathfrak{h}^1)}\right\}.$$

Then the spaces $\mathscr{U}$ and $\mathfrak{u}$ are isomorphic and isometric as well.

Let $\mathscr{S}$ be a subset of $\mathscr{M}$ and $r \in \mathscr{N}$. Let us denote $\mathfrak{h}^r_{\mathscr{S}} = \{h = \{h_n\}_{-\infty}^{\infty} \in \mathfrak{h}^r \mid h_n = 0$ for all $n \in \mathscr{M} \setminus \mathscr{S}\}$ and $[\mathfrak{h}^r_{\mathscr{S}}]^{\perp} = \mathfrak{h}^r_{\mathscr{M}\setminus\mathscr{S}}$. Then the decomposition $\mathfrak{h}^r = \mathfrak{h}^r_{\mathscr{S}} \dotplus [\mathfrak{h}^r_{\mathscr{S}}]^{\perp}$ is valid. Analogously: $\mathscr{H}^r_{2\pi} = [\mathscr{H}^r_{2\pi}]_{\mathscr{S}} \dotplus [\mathscr{H}^r_{2\pi}]^{\perp}_{\mathscr{S}}$, $\mathscr{U} = \mathscr{U}_{\mathscr{S}} \dotplus \mathscr{U}^{\perp}_{\mathscr{S}}$.

The two following lemmas, quite analogous to Lemmas 5.3 and 5.4 in [1], enable us to transfer the results obtained in the linear case to the weakly nonlinear problem. First the notation used: $[\mathscr{P}_1 \to \mathscr{P}_2]$ denotes the space of all linear continuous transformations from $\mathscr{P}_1$ into $\mathscr{P}_2$ ($\mathscr{P}_1$ and $\mathscr{P}_2$ being normed linear spaces), $\mathscr{B}(p_0; \delta; \mathscr{P}) = \{p \in \mathscr{P} \mid \|p - p_0\| < \delta\}$ is an open ball in the normed linear space $\mathscr{P}$.

Lemma 1.4. *Let the operator* $W = W(u, d)(\varepsilon)$ *map* $\mathscr{P} \times \mathscr{D} \times \langle 0, \varepsilon_0 \rangle$ *into* $\mathscr{P}$ *($\mathscr{P}$, $\mathscr{D}$ being Banach spaces), let it be continuous and have continuous Gateaux's derivatives (further only "G-derivatives")* W'_u, W'_d *on the domain* $\mathscr{B}(0; \varrho; \mathscr{P}) \times \mathscr{B}(\tilde{d}; \delta_0; \mathscr{D}) \times \langle 0, \varepsilon_0 \rangle$. *Let* $V \in [\mathscr{D} \to \mathscr{P}]$. *Then there exist numbers* δ^* *and* ε^* $(0 < \delta^* < \delta_0,\ 0 < \varepsilon^* \leqq \varepsilon_0)$ *such that the equation*

$$u = V(d) + \varepsilon\, W(u, d)(\varepsilon)$$

has a unique solution $u = U(d)(\varepsilon) \in \mathscr{P}$ *for each* $d \in \mathscr{B}(\tilde{d}; \delta^*; \mathscr{D})$ *and* $\varepsilon \in \langle 0, \varepsilon^* \rangle$ *continuous in* ε. *This solution has the G-derivative* U'_d *continuous in both variables* d *and* ε.

Lemma 1.5. *Let the operator* $P = P(d)(\varepsilon)$ *map for every* $\varepsilon \in \langle 0, \varepsilon_0 \rangle$ *an open set* $\mathscr{G} \subset \mathscr{D}_1$ *into* $\mathscr{D}_2$ *($\mathscr{D}_1$, $\mathscr{D}_2$ being Banach spaces), let the following assumptions be fulfilled:*

(i) *The equation* $P(d_0)(0) = 0$ *has a solution* $d_0^* \in \mathscr{G}$.

(ii) *The operator P is continuous and has the G-derivative* $P'_d = P'_d(d)(\varepsilon)$ *continuous on the set* $\mathscr{B}(d_0^*; \varrho; \mathscr{G}) \times \langle 0, \varepsilon_0 \rangle$ ($\varrho > 0$ *being a suitably chosen number*).

(iii) *There exists* $Q = [P'_d(d_0^*)(0)]^{-1} \in [\mathscr{D}_2 \to \mathscr{D}_1]$.

Then there exists $\varepsilon_1 \in (0, \varepsilon_0\rangle$ *such that the equation*

$$P(d)(\varepsilon) = 0$$

has for $\varepsilon \in \langle 0, \varepsilon_1 \rangle$ *a unique solution* $d^* = d^*(\varepsilon) \in \mathscr{G}$ *continuous on* $\langle 0, \varepsilon_1 \rangle$ *and such that* $d^*(0) = d_0^*$.

2. THE LINEAR PROBLEM AND ITS SOLUTIONS

2.1. The formulation of the problem. Before solving the weakly nonlinear problem it is convenient to solve the linear one. In accordance with the previous paragraph we shall formulate this problem as follows:

Let functions $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})$, ${}^i h \in \mathscr{H}^2_{2\pi}$ $(i = 0, 1)$ and real numbers $\omega > 0$, α_0, α_1, α be given. Every function $u \in \mathscr{U}$ that satisfies the equation

$$-\omega^2 u''(x) + \Delta_t u(x) = g(x), \quad x \in \mathscr{I} \tag{2.1.1}$$

in sense of $\mathscr{C}(\mathscr{I}; \mathscr{H}^1_{2\pi})$ (where $\Delta_t = \mathrm{d}^2/\mathrm{d}t^2$ means the Laplace operator) and the boundary conditions

$$\begin{aligned} u(0) &= {}^0h, \\ u(\pi) &= {}^1h \end{aligned} \tag{2.1.2}$$

or

$$\begin{aligned} u'(0) + \alpha_0 u(0) &= {}^0h, \\ u'(\pi) + \alpha_1 u(\pi) &= {}^1h \end{aligned} \tag{2.1.3}$$

or

$$\begin{aligned} u(0) \qquad\qquad &= {}^0h, \\ u'(\pi) + \alpha u(\pi) &= {}^1h \end{aligned} \tag{2.1.4}$$

in sense of $\mathscr{H}^2_{2\pi}$ will be called a solution of the problem $(\mathscr{P}^0_1)$, $(\mathscr{P}^0_2)$ or $(\mathscr{P}^0_3)$, respectively.

The solvability of our problem depends essentially on the number-theoretical character of the parameter ω. With respect to this fact two cases will be investigated separately:

(i) $\omega = p/q$, where p, q are relatively prime natural numbers,

(ii) the number ω satisfies the following assumption with a natural $\varrho \geqq 2$.

$[\mathscr{L}\varrho]$: there exists a constant $C_0 > 0$ such that

$$\left|\frac{1}{\omega} - \frac{m}{n}\right| \geqq C_0 n^{-\varrho} \quad \text{for all} \quad m, n \in \mathscr{N}.$$

Remark 2.1. According to Liouville's theorem (see [4]) the assumption $[\mathscr{L}\varrho]$ is fulfilled e.g. when ω is an algebraic number of the degree ϱ.

Expanding the functions g, ${}^0h$, ${}^1h$ into Fourier series

$$g(x) = \sum_{k \in \mathscr{M}} g_k(x)\, e_k\,, \quad {}^ih = \sum_{k \in \mathscr{M}} {}^ih_k e_k\,, \quad i = 0, 1$$

and assuming the existence of a solution $u \in \mathscr{U}$,

$$u(x) = \sum_{k \in \mathscr{M}} u_k(x)\, e_k\,, \quad x \in \mathscr{I}\,, \tag{2.1.5}$$

the equation (2.1.1) yields the system of differential equations

$$-\omega^2\, u_k''(x) - k^2\, u_k(x) = g_k(x)\,, \quad x \in \mathscr{I},\, k \in \mathscr{M}\,.$$

General solutions of these equations are

$$\begin{aligned} u_k(x) &= {}^0u_k(x) + a_k \cos(kx/\omega) + b_k/k \sin(kx/\omega)\,, \quad k \neq 0\,, \\ u_0(x) &= {}^0u_0(x) + a_0 + b_0 x\,, \qquad x \in \mathscr{I}\,, \end{aligned} \tag{2.1.6}$$

where the particular solutions

$$\begin{aligned} {}^0u_k(x) &= {}^0u_k(g)(x) \equiv -(\omega k)^{-1} \int_0^x g_k(\xi) \sin(k(x - \xi)/\omega)\, d\xi\,, \\ {}^0u_0(x) &= {}^0u_0(g)(x) \equiv -\omega^{-2} \int_0^x g_0(\xi)(x - \xi)\, d\xi \end{aligned} \tag{2.1.7}$$

are chosen to fulfil the relation ${}^0u_k(0) = {}^0u_k'(0) = 0$. Using Schwarz inequality (if $g \in \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})$, then also integrating by parts in (2.1.7)), it is easy to verify that the function ${}^0u(x) = \sum_{k \in \mathscr{M}} {}^0u_k(x)\, e_k$, $x \in \mathscr{I}$, lies in $\mathscr{U}$. Hence, the function u given by the series (2.1.5) belongs to $\mathscr{U}$ if and only if the coefficients a_k, b_k in (2.1.6) satisfy the condition

$$a = \{a_k\}_{-\infty}^{\infty} \in \mathfrak{h}^3\,, \quad b \doteq \{b_k\}_{-\infty}^{\infty} \in \mathfrak{h}^2\,. \tag{2.1.8}$$

Now let us look for such couples $(a, b) \in \mathfrak{h}^3 \times \mathfrak{h}^2$ that the corresponding $u \in \mathscr{U}$ fulfil the boundary conditions required.

2.2. Problem ($\mathscr{P}_1^0$). Substituting (2.1.5) and (2.1.6) for u, the boundary condition (2.1.2$_1$) assumes the form

$$a_k = {}^0h_k\,, \quad k \in \mathscr{M}\,. \tag{2.2.1}$$

In this way, the sequence $a = \{a_k\}_{-\infty}^{\infty} \in \mathfrak{h}^3$ is determined by ${}^0h \in \mathscr{H}_{2\pi}^3$ uniquely and the condition (2.1.2$_2$) gives

$$\begin{aligned} b_k \sin(k\pi/\omega) &= k\, B_k(g, {}^0h, {}^1h)\,, \quad k \in \mathscr{M} \setminus \{0\}\,, \\ b_0 &= \pi^{-1}\, B_0(g, {}^0h, {}^1h)\,, \end{aligned} \tag{2.2.2}$$

where

$$B_k(g, {}^0h, {}^1h) = {}^1h_k - {}^0h_k \cos(k\pi/\omega) - {}^0u_k(g)(\pi)\,, \quad k \in \mathscr{M}\,. \tag{2.2.3}$$

First, let us investigate the rational case with $\omega = p/q$, where p, q are relatively prime natural numbers. Then $\sin(k\pi/\omega) = 0$ if and only if $k \in \mathscr{S}(1) = \{k \in \mathscr{M} \mid k/p \in \mathscr{M}\}$. Therefore, equations (2.2.2) are equivalent to the system

$$\begin{aligned} b_0 &= \pi^{-1}\, B_0(g, {}^0h, {}^1h)\,, \\ b_k &= k(\sin(k\pi q/p))^{-1}\, B_k(g, {}^0h, {}^1h)\,, \quad k \in \mathscr{M} \setminus \mathscr{S}(1)\,, \end{aligned} \tag{2.2.4}$$

$$B_k(g, {}^0h, {}^1h) = 0\,, \quad k \in \mathscr{S}(2) = \mathscr{S}(1) \setminus \{0\}\,. \tag{2.2.5}$$

Here the relations (2.2.4) define the coefficients b_k, $k \in \mathscr{M} \setminus \mathscr{S}(2)$, whereas (2.2.5) represents a solvability condition.

To fulfil (2.1.8) let us assume that $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^2) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^1)$ and ${}^ih \in \mathscr{H}_{2\pi}^3$, $i = 0, 1$. Then, if the condition (2.2.5) is satisfied, all solutions of our problem are given by

$$u = V_1(d) + W_1(g, {}^0h, {}^1h)\,, \tag{2.2.6}$$

where the sequence $d = \{d_k\}_{-\infty}^{\infty}$ ranges over $\mathfrak{h}_{\mathscr{S}(2)}^2$ and the operators V_1, W_1 are defined by

$$V_1(d)(x) = \sum_{k \in \mathscr{S}(2)} k^{-1} d_k \sin(kx/\omega)\, e_k\,, \quad x \in \mathscr{I}\,, \tag{2.2.7}$$

$$\begin{aligned} W_1(g, {}^0h, {}^1h)(x) = &\sum_{k \in \mathscr{M}} [{}^0u_k(g)(x) + {}^0h_k \cos(kx/\omega)]\, e_k + \\ &+ \sum_{k \in \mathscr{M} \setminus \mathscr{S}(1)} B_k(g, {}^0h, {}^1h)\, \frac{\sin(kx/\omega)}{\sin(k\pi/\omega)}\, e_k + \\ &+ \pi^{-1}\, B_0(g, {}^0h, {}^1h)\, x e_0\,, \quad x \in \mathscr{I}\,. \end{aligned} \tag{2.2.8}$$

It is easy to verify that $V_1 \in [\mathfrak{h}_{\mathscr{S}(2)}^2 \to \mathscr{U}]$ and

$$W_1 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^2) \times (\mathscr{H}_{2\pi}^3)^2 \to \mathscr{U}] \cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^1) \times (\mathscr{H}_{2\pi}^3)^2 \to \mathscr{U}]\,.$$

Defining an operator $Z_1 = Z_1(g, {}^0h, {}^1h)$, $Z_1 : (\mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})) \times \times (\mathscr{H}^3_{2\pi})^2 \to \mathscr{H}^3_{2\pi}$ by

$$(2.2.9)\quad Z_1(g, {}^0h, {}^1h) = {}^1h(. + \pi q/p) - {}^0h + + \frac{q^2}{2p^2}\int_0^\pi \int_0^\xi [g(\vartheta)(. + \xi q/p) + g(\vartheta)(. - \xi q/p)]\, d\vartheta\, d\xi$$

and integrating by parts, the condition (2.2.5) may be modified into the form

$$\int_0^{2\pi} Z_1(g, {}^0h, {}^1h)(t)\, e_{-k}(t)\, dt = 0\,, \quad k \in \mathscr{S}(2)\,.$$

Hence, denoting by $R_1 = R_1(g, {}^0h, {}^1h)$ the operator given by

$$(2.2.10)\quad R_1(g, {}^0h, {}^1h)(t) = \sum_{j=0}^{p-1} \frac{d}{dt} Z_1(g, {}^0h, {}^1h)(t + 2\pi j/p)\,, \quad t \in (0, 2\pi)\,,$$

$$R_1 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \times (\mathscr{H}^3_{2\pi})^2 \to \mathscr{H}^2_{2\pi}] \cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi}) \times (\mathscr{H}^3_{2\pi})^2 \to \mathscr{H}^2_{2\pi}]$$

and performing the following arrangements

$$\begin{aligned} R_1(g, {}^0h, {}^1h)(t) &= \sum_{k\in\mathscr{M}} (2\pi)^{-1} \int_0^{2\pi} R_1(g, {}^0h, {}^1h)(\tau)\, e^{-ik\tau}\, d\tau\, e_k(t) = \\ &= \sum_{k\in\mathscr{M}} \frac{ik}{2\pi} \sum_{j=0}^{p-1} \int_0^{2\pi} Z_1(g, {}^0h, {}^1h)(\tau + 2\pi j/p)\, e_{-k}(\tau)\, d\tau\, e_k(t) = \\ &= \sum_{k\in\mathscr{M}} \frac{ik}{2\pi} \int_0^{2\pi} Z_1(g, {}^0h, {}^1h)(\eta)\, e_{-k}(\eta)\, d\eta \sum_{j=0}^{p-1} e^{2\pi ikj/p}\, e_k(t) = \\ &= \sum_{k\in\mathscr{S}(2)} \frac{ikp}{2\pi} \int_0^{2\pi} Z_1(g, {}^0h, {}^1h)(\eta)\, e_{-k}(\eta)\, d\eta\, e_k(t)\,, \end{aligned}$$

we obtain the solvability condition (2.2.5) in a more closed form

$$(2.2.11)\qquad R_1(g, {}^0h, {}^1h) = 0\,.$$

Theorem 2.2.1. *Let the problem $(\mathscr{P}^0_1)$ with $\omega = p/q$ be given, where p, q are relatively prime natural numbers. Let $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})$, ${}^ih \in \mathscr{H}^3_{2\pi}$, $i = 0, 1$. Then the problem has a solution if and only if*

$$R_1(g, {}^0h, {}^1h) = 0 \quad (\textit{equality in the space } \mathscr{H}^2_{2\pi})\,.$$

In the affirmative case every solution of $(\mathscr{P}^0_1)$ is given by

$$u = V_1(d) + W_1(g, {}^0h, {}^1h)\,,$$

where d is an arbitrary element of $\mathfrak{h}^2_{\mathscr{S}(2)}$.

Further, let ω be an irrational number satisfying the assumption $[\mathscr{L}\varrho]$ for a natural $\varrho \geqq 2$. Then equations (2.2.2) determine the coefficients b_k, $k \in \mathscr{M}$, uniquely and so the uniqueness of the solution of $(\mathscr{P}_1^0)$ is proved. Supposing $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho+1}) \cup \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho})$ and ${}^i h \in \mathscr{H}_{2\pi}^{\varrho+2}$, $i = 0, 1$, the requirement (2.1.8) is fulfilled as the assumption $[\mathscr{L}\varrho]$ gives the estimate

(2.2.12) $$|\sin(k\pi/\omega)| \geqq 2C_0 k^{1-\varrho}, \quad k \in \mathscr{N}.$$

$$\text{(Indeed, } |\sin(k\pi/\omega)| = \frac{|\sin(k\pi/\omega - n\pi)|}{|k\pi/\omega - n\pi|} k\pi |1/\omega - n/k| \geqq 2kC_0 k^{-\varrho}.)$$

In this case, our problem has a solution

(2.2.13) $$u = W_2(g, {}^0h, {}^1h),$$

where the operator

$$W_2 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho+1}) \times (\mathscr{H}_{2\pi}^{\varrho+2})^2 \to \mathscr{U}] \cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho}) \times (\mathscr{H}_{2\pi}^{\varrho+2})^2 \to \mathscr{U}]$$

is defined by

(2.2.14) $$W_2(g, {}^0h, {}^1h)(x) = \sum_{k \in \mathscr{M}} [{}^0u_k(g)(x) + {}^0h_k \cos(kx/\omega)] e_k + \\ + \sum_{k \in \mathscr{M} \setminus \{0\}} B_k(g, {}^0h, {}^1h) \frac{\sin(kx/\omega)}{\sin(k\pi/\omega)} e_k + \\ + \pi^{-1} B_0(g, {}^0h, {}^1h)\, x e_0, \quad x \in \mathscr{I}.$$

Theorem 2.2.2. *Let the problem* $(\mathscr{P}_1^0)$ *with* ω *satisfying the assumption* $[\mathscr{L}\varrho]$ *for a natural* $\varrho \geqq 2$ *be given. Let* $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho+1}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho})$ *and* ${}^i h \in \mathscr{H}_{2\pi}^{\varrho+2}$, $i = 0, 1$. *Then the problem has a unique solution*

$$u = W_2(g, {}^0h, {}^1h).$$

2.3. Problem $(\mathscr{P}_2^0)$. Inserting (2.1.5) and (2.1.6) into $(2.1.3_1)$ we obtain

(2.3.1) $$b_k = \omega({}^0h_k - \alpha_0 a_k), \quad k \in \mathscr{M} \setminus \{0\}, \\ b_0 = {}^0h_0 - \alpha_0 a_0.$$

Considering these equations as definitions of coefficients b_k, the boundary condition $(2.1.3_2)$ gives

(2.3.2) $$a_0[\alpha_1 - \alpha_0 - \alpha_0\alpha_1\pi] = A_0(g, {}^0h, {}^1h),$$

$$a_k\left[(\alpha_1 - \alpha_0)\cos(k\pi/\omega) - \frac{k^2 + \omega^2\alpha_0\alpha_1}{k\omega} \sin(k\pi/\omega)\right] = A_k(g, {}^0h, {}^1h), \quad k \in \mathscr{M} \setminus \{0\},$$

where

$$(2.3.3)\quad A_0(g, {}^0h, {}^1h) = {}^1h_0 - {}^0h_0(1 + \alpha_1\pi) + \omega^{-2}\int_0^\pi g_0(\pi - \xi)(1 + \alpha_1\xi)\,d\xi\,,$$

$$A_k(g, {}^0h, {}^1h) = {}^1h_k - {}^0h_k[\cos(k\pi/\omega) + \alpha_1\omega k^{-1}\sin(k\pi/\omega)] + \\ + \omega^{-2}\int_0^\pi g_k(\pi - \xi)[\cos(k\xi/\omega) + \\ + \alpha_1\omega k^{-1}\sin(k\xi/\omega)]\,d\xi\,,\quad k \in \mathcal{M} \setminus \{0\}\,.$$

Firstly, let us investigate the particular case when $\alpha_0 = \alpha_1$ and $\omega = p/q$, p, q – natural, relatively prime. Consequently, equations (2.3.2) reduce to

$$(2.3.4)\quad a_0\alpha_0^2 = -\pi^{-1} A_0(g, {}^0h, {}^1h)\,,$$

$$a_k \sin(k\pi/\omega) = \frac{-k\omega}{k^2 + \omega^2\alpha_0^2} A_k(g, {}^0h, {}^1h)\,,\quad k \in \mathcal{M} \setminus \{0\}\,.$$

(A): Let $\alpha_0 = \alpha_1 = 0$. Then the equations (2.3.4) are fulfilled if and only if

$$(2.3.5)\qquad a_k = \frac{-\omega}{k\sin(k\pi/\omega)} A_k(g, {}^0h, {}^1h)\,,\quad k \in \mathcal{M} \setminus \mathcal{S}(1)\,,$$

$$(2.3.6)\qquad A_k(g, {}^0h, {}^1h) = 0\,,\quad k \in \mathcal{S}(1)\,.$$

(Here $\mathcal{S}(1)$ is the same set as in Section 2.2.)

The solvability condition (2.3.6) can be written after certain arrangements (analogous with those used to derive (2.2.11)) as

$$(2.3.6')\qquad R_3(g, {}^0h, {}^1h) \equiv \sum_{j=0}^{p-1} Z_3(g, {}^0h, {}^1h)(. + 2\pi j/p) = 0\,,$$

where the operator $Z_3: (\mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})) \times (\mathscr{H}^2_{2\pi})^2 \to \mathscr{H}^2_{2\pi}$ is defined by

$$(2.3.7)\quad Z_3(g, {}^0h, {}^1h) = {}^1h(. + \pi q/p) - {}^0h + \\ + \frac{1}{2}(q/p)^2\int_0^\pi [g(x)(. - qx/p) + g(x)(. + qx/p)]\,dx\,.$$

Provided that the condition (2.3.6′) is fulfilled, every solution of our problem has the form

$$(2.3.8)\qquad u = V_3(d) + W_3(g, {}^0h, {}^1h)\,,\quad d = \{d_k\}_{-\infty}^{\infty} \in \mathfrak{h}^3_{\mathcal{S}(1)}\,,$$

where

$$(2.3.9)\qquad V_3(d)(x) = \sum_{k\in\mathcal{S}(1)} d_k \cos(kx/\omega)\,e_k\,,\quad x \in \mathscr{I}\,,$$

(2.3.10) $$W_3(g, {}^0h, {}^1h)(x) = \sum_{k\in\mathcal{M}\setminus\{0\}} \left[{}^0u_k(g)(x) + {}^0h_k \frac{p}{kq} \sin(kx/\omega)\right] e_k +$$
$$+ \left[{}^0u_0(g)(x) + {}^0h_0 x\right] e_0 -$$
$$- \sum_{k\in\mathcal{M}\setminus\mathcal{S}(1)} A_k(g, {}^0h, {}^1h) \frac{p\cos(kx/\omega)}{kq\sin(k\pi/\omega)} e_k, \quad x \in \mathcal{I}.$$

Obviously:

$$V_3 \in [\mathfrak{h}^3_{\mathcal{S}(1)} \to \mathcal{U}], \quad W_3 \in [\mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{U}] \cap$$
$$\cap [\mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi}) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{U}], \quad R_3 \in [\mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{H}^2_{2\pi}] \cap$$
$$\cap [\mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi}) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{H}^2_{2\pi}].$$

(B): Let $\alpha_0 = \alpha_1 \neq 0$. Then conditions (2.3.5), (2.3.6) from the case (A) are to be replaced by

(2.3.11) $$a_k = \frac{-k\omega}{k^2 + \omega^2\alpha_0^2} [\sin(k\pi/\omega)]^{-1} A_k(g, {}^0h, {}^1h), \quad k \in \mathcal{M} \setminus \mathcal{S}(1),$$
$$a_0 = -\pi^{-1}\alpha_0^{-2} A_0(g, {}^0h, {}^1h),$$

(2.3.12) $$A_k(g, {}^0h, {}^1h) = 0, \quad k \in \mathcal{S}(2) = \mathcal{S}(1) \setminus \{0\}.$$

Defining the operator Z_4: $(\mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \cup \mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi})) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{H}^1_{2\pi}$ by

(2.3.13) $$Z_4(g, {}^0h, {}^1h)(t) = \frac{\mathrm{d}}{\mathrm{d}t} Z_3(g, {}^0h, {}^1h)(t) +$$
$$+ (\alpha_0\, q/2p) \int_0^\pi [g(\xi)(t - \xi q/p) - g(\xi)(t + \xi q/p)]\, \mathrm{d}\xi,$$

the solvability condition (2.3.12) is equivalent to

(2.3.12′) $$R_4(g, {}^0h, {}^1h) \equiv \sum_{j=0}^{p-1} Z_4(g, {}^0h, {}^1h)(. + 2\pi j/p) = 0.$$

If this condition is fulfilled, every solution of $(\mathcal{P}^0_2)$ has the form

(2.3.14) $$u = V_4(d) + W_4(g, {}^0h, {}^1h), \quad d = \{d_k\}_{-\infty}^{\infty} \in \mathfrak{h}^3_{\mathcal{S}(2)},$$

where

(2.3.15) $$V_4(d)(x) = \sum_{k\in\mathcal{S}(2)} d_k \left[\cos(kx/\omega) - \frac{\alpha_0 p}{kq} \sin(kx/\omega)\right] e_k,$$

$$(2.3.16)\quad W_4(g, {}^0h, {}^1h)(x) = \sum_{k\in\mathcal{M}} {}^0u_k(g)(x)\, e_k + {}^0h_0 x e_0 +$$

$$+ \sum_{k\in\mathcal{M}\setminus\{0\}} {}^0h_k p/(kq) \sin(kx/\omega)\, e_k +$$

$$+ \sum_{k\in\mathcal{M}\setminus\mathcal{S}(1)} \Big\{ -kpq\, A_k(g, {}^0h, {}^1h) \big[\sin(k\pi/\omega)(k^2q^2 +$$

$$+ \alpha_0^2 p^2)\big]^{-1} \Big[\cos(kx/\omega) - \frac{\alpha_0 p}{kq} \sin(kx/\omega)\Big] e_k \Big\} -$$

$$- \pi^{-1}\alpha_0^{-2} A_0(g, {}^0h, {}^1h)(1 - \alpha_0 x)\, e_0\,, \quad x \in \mathcal{I}\,.$$

Obviously:

$$V_4 \in [\mathfrak{h}^3_{\mathcal{S}(2)} \to \mathcal{U}]\,, \quad W_4 \in [\mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{U}] \cap$$

$$\cap [\mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi}) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{U}]\,, \quad R_4 \in [\mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{H}^1_{2\pi}] \cap$$

$$\cap [\mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi}) \times (\mathcal{H}^2_{2\pi})^2 \to \mathcal{H}^1_{2\pi}]\,.$$

Theorem 2.3.1. *Let the problem $(\mathcal{P}_2^0)$ with $\alpha_0 = \alpha_1$ and $\omega = p/q$ be given, where p, q are relatively prime natural numbers. Let $g \in \mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \cup \mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi})$ and ${}^ih \in \mathcal{H}^2_{2\pi}$, $i = 0, 1$.*

(A): *Let $\alpha_0 = \alpha_1 = 0$. Then the problem has a solution if and only if*

$$R_3(g, {}^0h, {}^1h) = 0 \quad (\textit{equality in the space } \mathcal{H}^2_{2\pi})\,.$$

In the affirmative case every solution of $(\mathcal{P}_2^0)$ is given by

$$u = V_3(d) + W_3(g, {}^0h, {}^1h)\,,$$

where d is an arbitrary element of $\mathfrak{h}^3_{\mathcal{S}(1)}$.

(B): *Let $\alpha_0 = \alpha_1 \neq 0$. Then the problem has a solution if and only if*

$$R_4(g, {}^0h, {}^1h) = 0 \quad (\textit{equality in the space } \mathcal{H}^1_{2\pi})\,.$$

If this condition is satisfied, every solution of the problem is given by

$$u = V_4(d) + W_4(g, {}^0h, {}^1h)\,,$$

where d is an arbitrary element of $\mathfrak{h}^3_{\mathcal{S}(2)}$.

Further, let $\alpha_0 = \alpha_1$ and let ω be an irrational number satisfying the assumption $[\mathcal{L}\varrho]$ for a natural $\varrho \geqq 2$. Since in this case $\sin(k\pi/\omega) \neq 0$ for all $k \in \mathcal{M} \setminus \{0\}$, the coefficients a_k, $k \in \mathcal{M} \setminus \{0\}$, are determined by (2.3.4) uniquely.

(A): Let $\alpha_0 = \alpha_1 = 0$. Then equations (2.3.4) give the solvability condition

$$(2.3.17)\qquad R_5(g, {}^0h, {}^1h) \equiv A_0(g, {}^0h, {}^1h) = 0$$

and every solution of the problem can be written as

$$u = V_5(d) + W_5(g, {}^0h, {}^1h), \quad d \in \mathscr{R}, \tag{2.3.18}$$

where the operators V_5 and W_5 are given by

$$V_5(d)(x) = de_0, \quad x \in \mathscr{I}, \tag{2.3.19}$$

$$\begin{aligned} W_5(g, {}^0h, {}^1h)(x) = \sum_{k \in \mathscr{M} \setminus \{0\}} \{ & {}^0u_k(g)(x) + {}^0h_k \omega k^{-1} \sin(kx/\omega) - \\ & - \omega[k \sin(k\pi/\omega)]^{-1} A_k(g, {}^0h, {}^1h) \cos(kx/\omega)\} e_k + \\ & + [{}^0u_0(g)(x) + {}^0h_0 x] e_0, \quad x \in \mathscr{I}. \end{aligned} \tag{2.3.20}$$

Using the estimate (2.2.12), it is easy to verify that

$$W_5 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho+1}) \times (\mathscr{H}_{2\pi}^{\varrho+1})^2 \to \mathscr{U}] \cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho}) \times (\mathscr{H}_{2\pi}^{\varrho+1})^2 \to \mathscr{U}].$$

(B): Let $\alpha_0 = \alpha_1 \neq 0$. Then equations (2.3.4) determine all coefficients a_k and the problems has a unique solution (provided that the functions g, ${}^0h$ and ${}^1h$ are smooth enough):

$$u = W_6(g, {}^0h, {}^1h), \tag{2.3.21}$$

where

$$\begin{aligned} W_6(g, {}^0h, {}^1h)(x) = \sum_{k \in \mathscr{M} \setminus \{0\}} \{ & {}^0u_k(g)(x) + {}^0h_k \omega k^{-1} \sin(kx/\omega) - \\ & - k\omega[(k^2 + \omega^2\alpha_0^2) \sin(k\pi/\omega)]^{-1} A_k(g, {}^0h, {}^1h) \times \\ & \times [\cos(kx/\omega) - \alpha_0 \omega k^{-1} \sin(kx/\omega)]\} e_k - \\ & - \pi^{-1}\alpha_0^{-2} A_0(g, {}^0h, {}^1h)(1 - \alpha_0 x) e_0 + \\ & + [{}^0u_0(g)(x) + {}^0h_0 x] e_0, \quad x \in \mathscr{I}. \end{aligned} \tag{2.3.22}$$

Obviously:

$$W_6 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho+1}) \times (\mathscr{H}_{2\pi}^{\varrho+1})^2 \to \mathscr{U}] \cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho}) \times (\mathscr{H}_{2\pi}^{\varrho+1})^2 \to \mathscr{U}].$$

Theorem 2.3.2. *Let the problem* $(\mathscr{P}_2^0)$ *with* $\alpha_0 = \alpha_1$ *and with* ω *satisfying the assumption* $[\mathscr{L}\varrho]$ *for a natural* $\varrho \geqq 2$ *be given. Let* $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho+1}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^{\varrho})$ *and* ${}^ih \in \mathscr{H}_{2\pi}^{\varrho+1}$, $i = 0, 1$.

(A): *Let* $\alpha_0 = \alpha_1 = 0$. *Then the problem has a solution if and only if*

$$R_5(g, {}^0h, {}^1h) = 0 \quad (\textit{equality in } \mathscr{R}).$$

In the affirmative case every solution of $(\mathscr{P}_2^0)$ *is given by*

$$u = V_5(d) + W_5(g, {}^0h, {}^1h),$$

where d *is an arbitrary real number.*

(B): *If* $\alpha_0 = \alpha_1 \neq 0$, *the problem has a unique solution*

$$u = W_6(g, {}^0h, {}^1h).$$

Finally, let $\alpha_0 \neq \alpha_1$ and $\omega = p/q$, where p, q are relatively prime natural numbers. Let us denote

$$(2.3.23)\quad S_3(k) = (\alpha_1 - \alpha_0)\cos(k\pi/\omega) - \frac{k^2 + \omega^2\alpha_0\alpha_1}{k\omega}\sin(k\pi/\omega), \quad k \in \mathcal{M} \setminus \{0\},$$

$$S_3(0) = \alpha_1 - \alpha_0 - \alpha_0\alpha_1\pi.$$

Then $|S_3(k)| = |\alpha_1 - \alpha_0| > 0$ for all $k \in \mathcal{S}(2)$ and the relation $(2.3.23_1)$ can be written for $k \in \mathcal{M} \setminus \mathcal{S}(1)$ as

$$S_3(k) = (\alpha_1 - \alpha_0)\sin(k\pi q/p)\left[\operatorname{cotg}(k\pi q/p) - \frac{k^2 + \omega^2\alpha_0\alpha_1}{k\,\omega(\alpha_1 - \alpha_0)}\right].$$

The first term in the square brackets acquires only $p - 1$ values on the set of $k \in \mathcal{M} \setminus \mathcal{S}(1)$, the second one can assume the same value for at most two different $k \in \mathcal{M}$, the whole expression in the brackets is equal to $k/(\omega(\alpha_0 - \alpha_1))$ asymptotically. Hence, the set $\mathcal{S}(3) = \{k \in \mathcal{M} \mid S_3(k) = 0\}$ contains at most $2p - 1$ numbers and, moreover, the following relations hold:

$$|S_3(k)| \geq C|k|, \qquad k \in \mathcal{M} \setminus \mathcal{S}(1) \setminus \mathcal{S}(3), \quad C \text{ being a constant},$$

$$|S_3(k)| = |\alpha_1 - \alpha_0|, \quad k \in \mathcal{S}(2).$$

Obviously, conditions (2.3.2) are equivalent to

$$(2.3.24)\qquad a_k = (S_3(k))^{-1} A_k(g, {}^0h, {}^1h), \quad k \in \mathcal{M} \setminus \mathcal{S}(3),$$

$$(2.3.25)\qquad A_k(g, {}^0h, {}^1h) = 0, \quad k \in \mathcal{S}(3).$$

Then, to guarantee that $a = \{a_k\}_{-\infty}^{\infty} \in \mathfrak{h}^3$, a higher smoothness of the functions g, ${}^0h$ and ${}^1h$ must be assumed, e.g. $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}^3_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^2_{2\pi})$, ${}^ih \in \mathscr{H}^3_{2\pi}$, $i = 0, 1$ or $g \in \mathscr{C}(\mathscr{I}; [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathcal{S}(2)}) \cup \mathscr{C}^{(1)}(\mathscr{I}; [\mathscr{H}^1_{2\pi}]^{\perp}_{\mathcal{S}(2)})$, ${}^ih \in [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathcal{S}(2)}$, $i = 0, 1$ (in this case $A_k(g, {}^0h, {}^1h) = 0$ for $k \in \mathcal{S}(2)$).

Defining the operator R_7: $(\mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})) \times (\mathscr{H}^2_{2\pi})^2 \to \mathfrak{h}^{r}_{\mathcal{S}(3)}$ by

$$(2.3.26)\quad R_7(g, {}^0h, {}^1h) = \{r_k\}_{-\infty}^{\infty}, \quad r_k = \begin{cases} A_k(g, {}^0h, {}^1h), & k \in \mathcal{S}(3), \\ 0, & k \in \mathcal{M} \setminus \mathcal{S}(3), \end{cases}$$

the solvability condition (2.3.25) can be written as

$$(2.3.25')\qquad R_7(g, {}^0h, {}^1h) = 0.$$

If this condition is fulfilled, every solution of $(\mathscr{P}_2^0)$ is given by

(2.3.27) $$u = V_7(d) + W_7(g, {}^0h, {}^1h), \quad d = \{d_k\}_{-\infty}^{\infty} \in \mathfrak{h}^1_{\mathscr{S}(3)},$$

where

(2.3.28) $$V_7(d)(x) = \sum_{k \in \mathscr{S}(3) \setminus \{0\}} d_k[\cos(kx/\omega) - \alpha_0 \omega\, k^{-1} \sin(kx/\omega)]\, e_k + \\ + d_0(1 - \alpha_0 x)\, e_0, \quad x \in \mathscr{I},$$

(2.3.29) $$W_7(g, {}^0h, {}^1h)(x) = \begin{cases} \tilde{W}_7(g, {}^0h, {}^1h)(x), & \text{if } 0 \in \mathscr{S}(3), \\ \tilde{W}_7(g, {}^0h, {}^1h)(x) + (S_3(0))^{-1} A_0(g, {}^0h, {}^1h) \times \\ \times (1 - \alpha_0 x)\, e_0, & \text{if } 0 \notin \mathscr{S}(3), \end{cases}$$

(2.3.30) $$\tilde{W}_7(g, {}^0h, {}^1h)(x) = \sum_{k \in \mathscr{M} \setminus \mathscr{S}(3) \setminus \{0\}} (S_3(k))^{-1} A_k(g, {}^0h, {}^1h) \times \\ \times [\cos(kx/\omega) - \alpha_0 \omega k^{-1} \sin(kx/\omega)]\, e_k + \\ + \sum_{k \in \mathscr{M} \setminus \{0\}} [{}^0u_k(g)(x) + {}^0h_k \omega k^{-1} \sin(kx/\omega)]\, e_k + \\ + ({}^0u_0(g)(x) + {}^0h_0 x)\, e_0, \quad x \in \mathscr{I}.$$

(As to (2.3.27) let us remind that the sequence $d = \{d_k\}_{-\infty}^{\infty} \in \mathfrak{h}^1_{\mathscr{S}(3)}$ has only a finite number of non-vanishing terms which may assume arbitrary complex values such that $d_{-k} = \bar{d}_k$, $k \in \mathscr{M}$.) Obviously:

$$V_7 \in [\mathfrak{h}^1_{\mathscr{S}(3)} \to \mathscr{U}], \quad W_7 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}^3_{2\pi}) \times (\mathscr{H}^3_{2\pi})^2 \to \mathscr{U}] \cap \\ \cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \times (\mathscr{H}^3_{2\pi})^2 \to \mathscr{U}]$$

and as well

$$W_7 \in [\mathscr{C}(\mathscr{I}; [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(2)}) \times ([\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(2)})^2 \to \mathscr{U}^{\perp}_{\mathscr{S}(2)}].$$

Theorem 2.3.3. *Let the problem $(\mathscr{P}_2^0)$ with $\alpha_0 \neq \alpha_1$ and $\omega = p/q$ be given, where p, q are relatively prime natural numbers. Let $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}^3_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^2_{2\pi})$, ${}^ih \in \mathscr{H}^3_{2\pi}$, $i = 0, 1$ or $g \in \mathscr{C}(\mathscr{I}; [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(2)}) \cup \mathscr{C}^{(1)}(\mathscr{I}; [\mathscr{H}^1_{2\pi}]^{\perp}_{\mathscr{S}(2)})$, ${}^ih \in [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(2)}$, $i = 0, 1$. Then the following assertions hold:*

(A): *If $\mathscr{S}(3)$ is a void set, the problem has a unique solution*

$$u = W_7(g, {}^0h, {}^1h).$$

(B): *If the set $\mathscr{S}(3)$ is non-void, the problem has a solution if and only if*

$$R_7(g, {}^0h, {}^1h) = 0 \quad (\textit{equality in the space } \mathfrak{h}^1_{\mathscr{S}(3)}).$$

In the affirmative case every solution of $(\mathscr{P}_2^0)$ is given by

$$u = V_7(d) + W_7(g, {}^0h, {}^1h),$$

where d is an arbitrary element of $\mathfrak{h}^1_{\mathscr{S}(3)}$.

Remark 2.2. In the case when $\alpha_0 \neq \alpha_1$ and ω is an irrational number the investigation of existence of solutions in integers of the equation $S_3(k) = 0$ represents a fairly difficult number-theoretical problem. Therefore this case is omitted in this paper.

2.4. Problem $(\mathscr{P}_3^0)$. Identically with the problem $(\mathscr{P}_1^0)$, the boundary condition $(2.1.4_1)$ gives

(2.4.1) $$a_k = {}^0h_k\,, \quad k \in \mathscr{M}\,.$$

Then the relations (2.1.5) and (2.1.6) inserted into $(2.1.4_2)$ yield

(2.4.2) $$b_k[\omega^{-1}\cos(k\pi/\omega) + \alpha k^{-1}\sin(k\pi/\omega)] = D_k(g, {}^0h, {}^1h)\,, \quad k \in \mathscr{M} \setminus \{0\}\,,$$
$$b_0(1 + \alpha\pi) = D_0(g, {}^0h, {}^1h)\,,$$

where

(2.4.3) $$D_0(g, {}^0h, {}^1h) = {}^1h_0 - \alpha{}^0h_0 + \omega^{-2}\int_0^\pi g_0(\pi - \xi)(1 + \alpha\xi)\,d\xi\,,$$
$$D_k(g, {}^0h, {}^1h) = {}^1h_k + {}^0h_k[k\omega^{-1}\sin(k\pi/\omega) - \alpha\cos(k\pi/\omega)] +$$
$$+ \omega^{-1}\int_0^\pi g_k(\pi - \xi)\,[\omega^{-1}\cos(k\xi/\omega) +$$
$$+ \alpha k^{-1}\sin(k\xi/\omega)]\,d\xi\,, \quad k \in \mathscr{M} \setminus \{0\}\,.$$

Firstly, let us investigate the simpler case, when $\alpha = 0$ and $\omega = p/q$, p, q – natural, relatively prime. Then expression in the brackets in (2.4.2) reduce to $\omega^{-1}\cos(k\pi q/p)$.

(A): Let p be an odd number. Then $\cos(k\pi q/p) \neq 0$ for all $k \in \mathscr{M}$ and so equations (2.4.2) give

(2.4.4) $$b_k = \omega[\cos(k\pi/\omega)]^{-1} D_k(g, {}^0h, {}^1h)\,, \quad k \in \mathscr{M} \setminus \{0\}\,,$$
$$b_0 = D_0(g, {}^0h, {}^1h)\,.$$

Thus, if $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^2) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^1)$, ${}^0h \in \mathscr{H}_{2\pi}^3$ and ${}^1h \in \mathscr{H}_{2\pi}^2$, the problem $(\mathscr{P}_3^0)$ has a unique solution

(2.4.5) $$u = W_8(g, {}^0h, {}^1h)\,,$$

where

$$W_8 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}_{2\pi}^2) \times \mathscr{H}_{2\pi}^3 \times \mathscr{H}_{2\pi}^2 \to \mathscr{U}] \cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}_{2\pi}^1) \times \mathscr{H}_{2\pi}^3 \times \mathscr{H}_{2\pi}^2 \to \mathscr{U}]$$

is defined by

(2.4.6) $$W_8(g, {}^0h, {}^1h)(x) = \sum_{k\in\mathcal{M}\setminus\{0\}} \{{}^0u_k(g)(x) + {}^0h_k \cos(kx/\omega) + \omega[k\cos(k\pi/\omega)]^{-1} D_k(g, {}^0h, {}^1h)\sin(kx/\omega)\} e_k + [{}^0u_0(g)(x) + {}^0h_0 + D_0(g, {}^0h, {}^1h)\,x]\, e_0\,, \quad x \in \mathcal{I}.$$

(B): Let $p = 2m$ be an even number. Then $\cos(k\pi q/p) = 0$ if and only if $k \in \mathcal{S}(4) = \{k \in \mathcal{M} \mid k/m = \text{odd number}\}$ and so conditions (2.4.2) give the system

(2.4.7) $$b_k = \omega[\cos(k\pi q/p)]^{-1} D_k(g, {}^0h, {}^1h)\,, \quad k \in \mathcal{M} \setminus \mathcal{S}(4) \setminus \{0\}\,,$$
$$b_0 = D_0(g, {}^0h, {}^1h)\,,$$

(2.4.8) $$D_k(g, {}^0h, {}^1h) = 0\,, \quad k \in \mathcal{S}(4)\,.$$

The solvability condition (2.4.8) can be written after certain arrangements in the form

(2.4.8′) $$R_9(g, {}^0h, {}^1h) \equiv \sum_{j=0}^{2m-1} (-1)^j Z_9(g, {}^0h, {}^1h)(\,.\, + \pi j/m) = 0\,,$$

where the operator Z_9: $(\mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \cup \mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi})) \times \mathcal{H}^3_{2\pi} \times \mathcal{H}^2_{2\pi} \to \mathcal{H}^2_{2\pi}$ is defined by

(2.4.9) $$Z_9(g, {}^0h, {}^1h)(t) = {}^1h(t + \pi q/p) + (q/p)\frac{\mathrm{d}}{\mathrm{d}t}\,{}^0h(t) + \tfrac{1}{2}(q/p)^2 \int_0^\pi [g(\xi)(t + \xi q/p) - g(\xi)(t - \xi q/p)]\,\mathrm{d}\xi\,, \quad t \in (0, 2\pi)\,.$$

Hence, $R_9 \in [\mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \times \mathcal{H}^3_{2\pi} \times \mathcal{H}^2_{2\pi} \to \mathcal{H}^2_{2\pi}] \cap [\mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi}) \times \mathcal{H}^3_{2\pi} \times \mathcal{H}^2_{2\pi} \to \mathcal{H}^2_{2\pi}]$.

Thus, supposing that the functions $g \in \mathcal{C}(\mathcal{I}; \mathcal{H}^2_{2\pi}) \cup \mathcal{C}^{(1)}(\mathcal{I}; \mathcal{H}^1_{2\pi})$, ${}^0h \in \mathcal{H}^3_{2\pi}$ and ${}^1h \in \mathcal{H}^2_{2\pi}$ fulfil the condition (2.4.8′), every solution of our problem has the form

(2.4.10) $$u = V_9(d) + W_9(g, {}^0h, {}^1h)\,, \quad d = \{d_k\}_{-\infty}^{\infty} \in \mathfrak{h}^2_{\mathcal{S}(4)}\,,$$

where

(2.4.11) $$V_9(d)(x) = \sum_{k\in\mathcal{S}(4)} d_k k^{-1} \sin(kx/\omega)\, e_k\,, \quad x \in \mathcal{I}\,,$$

(2.4.12) $$W_9(g, {}^0h, {}^1h)(x) = \sum_{k\in\mathcal{M}} [{}^0u_k(g)(x) + {}^0h_k\cos(kx/\omega)]\, e_k + \sum_{k\in\mathcal{M}\setminus\{0\}\setminus\mathcal{S}(4)} D_k(g, {}^0h, {}^1h)\frac{\omega\sin(kx/\omega)}{k\cos(k\pi/\omega)}\, e_k + D_0(g, {}^0h, {}^1h)\, x e_0\,, \quad x \in \mathcal{I}\,.$$

Obviously:

$$V_9 \in [\mathfrak{h}^2_{\mathscr{S}(4)} \to \mathscr{U}], \quad W_9 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \times \mathscr{H}^3_{2\pi} \times \mathscr{H}^2_{2\pi} \to \mathscr{U}] \cap$$
$$\cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi}) \times \mathscr{H}^3_{2\pi} \times \mathscr{H}^2_{2\pi} \to \mathscr{U}].$$

Theorem 2.4.1. *Let the problem $(\mathscr{P}^0_3)$ with $\alpha = 0$ and $\omega = p/q$ be given, where p, q are relatively prime natural numbers. Let $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})$, ${}^0h \in \mathscr{H}^3_{2\pi}$ and ${}^1h \in \mathscr{H}^2_{2\pi}$. Then the following assertions hold:*

(A): *If p is an odd number, the problem has a unique solution*

$$u = W_8(g, {}^0h, {}^1h).$$

(B): *Let $p = 2m$ be an even number. Then the problem has a solution if and only if*

$$R_9(g, {}^0h, {}^1h) = 0 \quad (\text{equality in the space } \mathscr{H}^2_{2\pi}).$$

In the affirmative case every solution of $(\mathscr{P}^0_3)$ is given by

$$u = V_9(d) + W_9(g, {}^0h, {}^1h),$$

where d is an arbitrary element of $\mathfrak{h}^2_{\mathscr{S}(4)}$.

Further, let $\alpha = 0$ and let ω be an irrantional number satisfying the assumption $[\mathscr{L}\varrho]$ for a natural $\varrho \geqq 2$. Hence the estimate

(2.4.13) $$|\cos(k\pi/\omega)| \geqq 2^{1-\varrho} C_0 k^{1-\varrho}, \quad k \in \mathscr{N}$$

follows and so

$$W_8 \in [\mathscr{C}(\mathscr{I}; \mathscr{H}^{\varrho+1}_{2\pi}) \times \mathscr{H}^{\varrho+2}_{2\pi} \times \mathscr{H}^{\varrho+1}_{2\pi} \to \mathscr{U}] \cap$$
$$\cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^{\varrho}_{2\pi}) \times \mathscr{H}^{\varrho+2}_{2\pi} \times \mathscr{H}^{\varrho+1}_{2\pi} \to \mathscr{U}].$$

Since $\cos(k\pi/\omega) \neq 0$ for all $k \in \mathscr{M}$, equations (2.4.2) give again (2.4.4) and the function $u = W_8(g, {}^0h, {}^1h)$ is the unique solution of the problem.

Theorem 2.4.2. *Let the problem $(\mathscr{P}^0_3)$ with $\alpha = 0$ and with ω satisfying the assumption $[\mathscr{L}\varrho]$ for a natural $\varrho \geqq 2$ be given. Let $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}^{\varrho+1}_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^{\varrho}_{2\pi})$, ${}^0h \in \mathscr{H}^{\varrho+2}_{2\pi}$ and ${}^1h \in \mathscr{H}^{\varrho+1}_{2\pi}$. Then the problem has a unique solution*

$$u = W_8(g, {}^0h, {}^1h).$$

Finally, let $\alpha \neq 0$ and $\omega = p/q$, where p, q are relatively prime natural numbers. Let us introduce the set $\mathscr{S}(5) = \{k \in \mathscr{M} \mid S_5(k) = 0\}$, where

(2.4.14) $$S_5(k) = k\omega^{-1}\cos(k\pi/\omega) + \alpha\sin(k\pi/\omega), \quad k \in \mathscr{M} \setminus \{0\},$$
$$S_5(0) = 1 + \alpha\pi.$$

Thus, if $\cos(k\pi q/p) = 0$, then $|S_5(k)| = |\alpha| > 0$, if not, we can write

$$S_5(k) = \cos(k\pi q/p)\,[kq/p + \alpha\,\mathrm{tg}\,(k\pi q/p)]\,, \quad k \neq 0\,.$$

Since $\mathrm{tg}\,(k\pi q/p)$ assumes only p values on the set of $k \in \mathcal{M}$, the set $\mathcal{S}(5)$ contains at most p numbers. Moreover, for all $k \in \mathcal{M} \setminus \mathcal{S}(5)$ such that $\cos(k\pi q/p) \neq 0$ the estimate $|S_5(k)| \geqq C|k|$ is valid, where C is a suitable positive constant.

Thus, conditions (2.4.2) are equivalent to

$$\text{(2.4.15)} \qquad b_k = k[S_5(k)]^{-1}\, D_k(g, {}^0h, {}^1h)\,, \quad k \in \mathcal{M} \setminus \mathcal{S}(5) \setminus \{0\}\,,$$

$$b_0 = [S_5(0)]^{-1}\, D_0(g, {}^0h, {}^1h)\,, \quad \text{when} \quad 0 \notin \mathcal{S}(5)\,,$$

$$\text{(2.4.16)} \qquad D_k(g, {}^0h, {}^1h) = 0\,, \quad k \in \mathcal{S}(5)\,.$$

The solvability condition (2.4.16) may be also written as

$$\text{(2.4.16')} \qquad R_{10}(g, {}^0h, {}^1h) = 0\,,$$

where the operator R_{10}: $(\mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})) \times \mathscr{H}^3_{2\pi} \times \mathscr{H}^2_{2\pi} \to \mathfrak{h}^1_{\mathcal{S}(5)}$

is defined by

$$\text{(2.4.17)} \quad R_{10}(g, {}^0h, {}^1h) = \{r_k\}^{\infty}_{-\infty}\,, \quad r_k = \begin{cases} D_k(g, {}^0h, {}^1h)\,, & \text{if} \quad k \in \mathcal{S}(5)\,, \\ 0\,, & \text{if} \quad k \in \mathcal{M} \setminus \mathcal{S}(5). \end{cases}$$

If this condition is satisfied, every solution of the problem has the form

$$\text{(2.4.18)} \qquad u = V_{10}(d) + W_{10}(g, {}^0h, {}^1h)\,, \quad d = \{d_k\}^{\infty}_{-\infty} \in \mathfrak{h}^1_{\mathcal{S}(5)}\,,$$

where

$$\text{(2.4.19)} \quad W_{10}(g, {}^0h, {}^1h)(x) = \begin{cases} \tilde{W}_{10}(g, {}^0h, {}^1h)(x)\,, & \text{if} \quad 0 \in \mathcal{S}(5)\,, \\ \tilde{W}_{10}(g, {}^0h, {}^1h)(x) + (S_5(0))^{-1} \times \\ \times\, D_0(g, {}^0h, {}^1h)\, x e_0\,, & \text{if} \quad 0 \notin \mathcal{S}(5)\,, \end{cases}$$

$$\text{(2.4.20)} \; \tilde{W}_{10}(g, {}^0h, {}^1h)(x) = \sum_{k \in \mathcal{M}} [{}^0u_k(g)(x) + {}^0h_k \cos(kx/\omega)]\, e_k \, +$$

$$+ \sum_{k \in \mathcal{M} \setminus \{0\} \setminus \mathcal{S}(5)} (S_5(k))^{-1}\, D_k(g, {}^0h, {}^1h) \sin(kx/\omega)\, e_k\,, \quad x \in \mathscr{I}$$

and

$$\text{(2.4.21)} \qquad V_{10}(d)(x) = \sum_{k \in \mathcal{S}(5) \setminus \{0\}} d_k \sin(kx/\omega)\, e_k + d_0 x e_0\,, \quad x \in \mathscr{I}\,.$$

Obviously: $V_{10} \in [\mathfrak{h}^1_{\mathcal{S}(5)} \to \mathscr{U}]$.

Properties of the operator W_{10} are rather complicated for the equality $|S_5(k)| = |\alpha|$ (when $\cos(k\pi/\omega) = 0$) leads to the requirement of a higher smoothness of g, ${}^0h$ and ${}^1h$. It is necessary to distinguish two cases:

(i): The number p is odd. Then $W_{10} \in [\mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \times \mathscr{H}^3_{2\pi} \times \mathscr{H}^2_{2\pi} \to \mathscr{U}] \cap$ $\cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi}) \times \mathscr{H}^3_{2\pi} \times \mathscr{H}^2_{2\pi} \to \mathscr{U}]$.

(ii): The number p is even, $p = 2m$. Then $W_{10} \in [\mathscr{C}(\mathscr{I}; \mathscr{H}^3_{2\pi}) \times \mathscr{H}^4_{2\pi} \times \mathscr{H}^3_{2\pi} \to$ $\to \mathscr{U}] \cap [\mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \times \mathscr{H}^4_{2\pi} \times \mathscr{H}^3_{2\pi} \to \mathscr{U}]$ as well as $W_{10} \in [\mathscr{C}(\mathscr{I}; [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(4)}) \times$ $\times [\mathscr{H}^3_{2\pi}]^{\perp}_{\mathscr{S}(4)} \times [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(4)} \to \mathscr{U}^{\perp}_{\mathscr{S}(4)}]$.

Theorem 2.4.3. *Let the problem* $(\mathscr{P}^0_3)$ *with* $\alpha \neq 0$ *and* $\omega = p/q$ *be given, where* p, q *are relatively prime natural numbers. Let one of the following assumptions be fulfilled:*

(i) *p is an odd number,* $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})$, ${}^0h \in \mathscr{H}^3_{2\pi}$ *and* ${}^1h \in \mathscr{H}^2_{2\pi}$;

(ii) $p = 2m$ *is an even number and* $g \in \mathscr{C}(\mathscr{I}; \mathscr{H}^3_{2\pi}) \cup \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^2_{2\pi})$, ${}^0h \in \mathscr{H}^4_{2\pi}$, ${}^1h \in \mathscr{H}^3_{2\pi}$ *or* $g \in \mathscr{C}(\mathscr{I}; [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(4)}) \cup \mathscr{C}^{(1)}(\mathscr{I}; [\mathscr{H}^1_{2\pi}]^{\perp}_{\mathscr{S}(4)})$, ${}^0h \in [\mathscr{H}^3_{2\pi}]^{\perp}_{\mathscr{S}(4)}$, ${}^1h \in [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(4)}$.

Then the following assertions hold:

(A): *If* $\mathscr{S}(5)$ *is a void set, the problem has a unique solution*

$$u = W_{10}(g, {}^0h, {}^1h)\,.$$

(B): *If the set* $\mathscr{S}(5)$ *is non-void, the problem has a solution if and only if*

$$R_{10}(g, {}^0h, {}^1h) = 0 \quad (\textit{equality in the space } \mathfrak{h}^1_{\mathscr{S}(5)})\,.$$

In the affirmative case every solution of $(\mathscr{P}^0_3)$ *is given by*

$$u = V_{10}(d) + W_{10}(g, {}^0h, {}^1h)\,,$$

where d is an arbitrary element of $\mathfrak{h}^1_{\mathscr{S}(5)}$.

Remark 2.3. The problem with $\alpha \neq 0$ and an irrational ω leads to difficulties analogous to those met with in the irrational case of $(\mathscr{P}^0_2)$ with $\alpha_0 \neq \alpha_1$ and so this problem is omitted as well.

3. THE WEAKLY NONLINEAR PROBLEM

3.1. General considerations. Taking into account that the linear problem was solved in the previous paragraph, the linear parts g, ${}^0h$ and ${}^1h$ of the right-hand sides in (0.1) and (0.3)–(0.5) may be omitted without loss of generality. Therefore we can formulate the weakly nonlinear problem as follows:

Let the operators $F = F(u, \varepsilon)$, $F : \mathscr{U} \times \langle 0, \varepsilon_0 \rangle \to \mathscr{C}(\mathscr{I}; \mathscr{H}^1_{2\pi})$, ${}^iX = {}^iX(u, \varepsilon)$, ${}^iX : \mathscr{U} \times \langle 0, \varepsilon_0 \rangle \to \mathscr{H}^2_{2\pi}$, $i = 0, 1$ and real numbers $\omega > 0$, α_0, α_1, α be given. The mapping $u^* = u^*(\varepsilon)$, continuous on an interval $\langle 0, \varepsilon^* \rangle \subset \langle 0, \varepsilon_0 \rangle$ into $\mathscr{U}$, is called a solution of the problem $(\mathscr{P}_1)$, $(\mathscr{P}_2)$ or $(\mathscr{P}_3)$, if for each $\varepsilon \in \langle 0, \varepsilon^* \rangle$ the function $u = u^*(\varepsilon) \in \mathscr{U}$ satisfies the equation

$$-\omega^2 u''(x) + \Delta_t u(x) = \varepsilon F(u, \varepsilon)(x), \quad x \in \mathscr{I} \tag{3.1.1}$$

and the boundary conditions

$$u(0) = \varepsilon\, {}^0X(u, \varepsilon), \quad u(\pi) = \varepsilon\, {}^1X(u, \varepsilon) \tag{3.1.2}$$

or

$$u'(0) + \alpha_0 u(0) = \varepsilon\, {}^0X(u, \varepsilon), \quad u'(\pi) + \alpha_1 u(\pi) = \varepsilon\, {}^1X(u, \varepsilon) \tag{3.1.3}$$

or

$$u(0) = \varepsilon\, {}^0X(u, \varepsilon), \quad u'(\pi) + \alpha\, u(\pi) = \varepsilon\, {}^1X(u, \varepsilon), \tag{3.1.4}$$

respectively.

Remark 3.1. The most frequent case is that F, ${}^0X$ and ${}^1X$ are Njemyckij-operators, i.e.

$$F(u, \varepsilon)(x)(t) \equiv f(t, x, u(x, t), u_t(x, t), u_x(x, t), \varepsilon),$$

$${}^jX(u, \varepsilon)(t) \equiv {}^j\chi(t, u(0, t), u_t(0, t), u_x(0, t), u(\pi, t), u_t(\pi, t), u_x(\pi, t), \varepsilon),$$

$$j = 0, 1, \quad x \in \mathscr{I}, \quad t \in \langle 0, 2\pi \rangle.$$

Then the continuity of the operators F, ${}^0X$, ${}^1X$ and their G-derivatives F'_u, ${}^0X'_u$, ${}^1X'_u$ is guaranteed by a sufficient smoothness of the functions $f = f(t, x, u_0, u_1, u_2, \varepsilon)$, ${}^j\chi = {}^j\chi(t, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \varepsilon)$, $j = 0, 1$. For example: if the derivatives $\partial^n f / \partial t^i\, \partial u_0^j\, \partial u_1^k\, \partial u_2^m$, $n = i + j + k + m \leqq 3$, $i < 3$ exist, are 2π-periodic in t and continuous on the domain

$$\{(t, x, u_0, u_1, u_2, \varepsilon) |\ t \in \langle 0, 2\pi \rangle, x \in \langle 0, \pi \rangle, u_0, u_1, u_2 \in (-\infty, +\infty), \varepsilon \in \langle 0, \varepsilon_0 \rangle\},$$

then the operators F and F'_u are continuous from $\mathscr{U} \times \langle 0, \varepsilon_0 \rangle$ into $\mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi})$. However, if e.g. $F'_u : \mathscr{U} \times \langle 0, \varepsilon_0 \rangle \to \mathscr{C}(\mathscr{I}; \mathscr{H}^3_{2\pi})$ is required, it is already necessary to assume that $\partial f / \partial u_i \equiv 0$, $i = 1, 2$, which means that $f = f(t, x, u_0, \varepsilon)$. Similarly the requirement $F'_u : \mathscr{U} \times \langle 0, \varepsilon_0 \rangle \to \mathscr{C}(\mathscr{I}; \mathscr{H}^4_{2\pi})$ can be fulfilled only if $\partial f / \partial u_i \equiv 0$, $i = 0, 1, 2$, i.e. when $f = f(t, x, \varepsilon)$. Properties of the operators ${}^0X$, ${}^1X$ are quite analogous.

The weakly nonlinear problems will be solved by the following standard procedure, based on the application of Lemmas 1.4 and 1.5 to the results obtained in the linear case.

Let us denote the weakly nonlinear problems $(\mathscr{P}_1)$, $(\mathscr{P}_2)$, $(\mathscr{P}_3)$ by a common symbol $(\mathscr{P})$, the linear problem corresponding to $(\mathscr{P})$ (and having the same parameters ω, α_0, α_1, α) by $(\mathscr{P}^0)$. In accordance with the previous paragraph we shall distinguish the two following cases.

$[\mathscr{C}1]$: There exist Banach spaces $\mathscr{F}$, $\mathscr{F}_0$, $\mathscr{F}_1$, $\mathscr{D}_1$, $\mathscr{D}_2$ and linear continuous operators $R \in [\mathscr{F} \times \mathscr{F}_0 \times \mathscr{F}_1 \to \mathscr{D}_2]$, $V \in [\mathscr{D}_1 \to \mathscr{U}]$, $W \in [\mathscr{F} \times \mathscr{F}_0 \times \mathscr{F}_1 \to \mathscr{U}]$ such that it holds:

The problem $(\mathscr{P}^0)$ given by the right-hand sides $g \in \mathscr{F}$, ${}^0h \in \mathscr{F}_0$ and ${}^1h \in \mathscr{F}_1$ has a solution if and only if $R(g, {}^0h, {}^1h) = 0$.
If this condition is satisfied, every solution of $(\mathscr{P}^0)$ is given by the relation $u = V(d) + W(g, {}^0h, {}^1h)$, where d is an arbitrary element of $\mathscr{D}_1$.

$[\mathscr{C}2]$: There exist Banach spaces $\mathscr{F}$, $\mathscr{F}_0$, $\mathscr{F}_1$ and a linear continuous operator $W \in [\mathscr{F} \times \mathscr{F}_0 \times \mathscr{F}_1 \to \mathscr{U}]$ such that the problem $(\mathscr{P}^0)$ given by $g \in \mathscr{F}$, ${}^0h \in \mathscr{F}_0$ and ${}^1h \in \mathscr{F}_1$ has a unique solution $u = W(g, {}^0h, {}^1h)$.

The two following assertions, corresponding to the cases $[\mathscr{C}1]$ and $[\mathscr{C}2]$, respectively, can be easily obtained by the succesive application of Lemmas 1.4 and 1.5.

Assertion $[\mathscr{A}1]$. *Let $\mathscr{F}$, $\mathscr{F}_0$, $\mathscr{F}_1$, $\mathscr{D}_1$, $\mathscr{D}_2$ and R, V, W be the Banach spaces and the operators from the case $[\mathscr{C}1]$. Let the operators $F = F(u, \varepsilon)$, $F : \mathscr{U} \times \langle 0, \varepsilon_0 \rangle \to \mathscr{F}$, ${}^iX = {}^iX(u, \varepsilon)$, ${}^iX : \mathscr{U} \times \langle 0, \varepsilon_0 \rangle \to \mathscr{F}_i$, $i = 0, 1$, be continuous and have continuous G-derivatives F'_u, ${}^0X'_u$, ${}^1X'_u$. Let ${}^0d$ be an element of the space $\mathscr{D}_1$ and let $U = U(d, \varepsilon)$, $U : \mathscr{B}({}^0d; \delta_0; \mathscr{D}_1) \times \langle 0, \hat{\varepsilon}_0 \rangle \to \mathscr{U}$ be a continuous operator having the G-derivative U'_d continuous (in d and ε) and such that the function $u = U(d, \varepsilon)$ solves the equation*

$$u = V(d) + \varepsilon\, W(F(u, \varepsilon), {}^0X(u, \varepsilon), {}^1X(u, \varepsilon))$$

for arbitrarily chosen $d \in \mathscr{B}({}^0d; \delta_0; \mathscr{D}_1)$ and $\varepsilon \in \langle 0, \hat{\varepsilon}_0 \rangle$. (According to Lemma 1.4 such operator U exists and it is unique.) Defining the operator $P : \mathscr{B}({}^0d; \delta_0; \mathscr{D}_1) \times \langle 0, \hat{\varepsilon}_0 \rangle \to \mathscr{D}_2$ by

$$P(d, \varepsilon) = R(F(U(d, \varepsilon), \varepsilon), {}^0X(U(d, \varepsilon), \varepsilon), {}^1X(U(d, \varepsilon), \varepsilon))\,,$$

let the following assumptions be fulfilled:

(i) $$P({}^0d, 0) = 0\,,$$

(ii) there exists an operator $Q = [P'_d({}^0d, 0)]^{-1} \in [\mathscr{D}_2 \to \mathscr{D}_1]$.

Then there exist a number $\varepsilon_1 > 0$ and a continuous mapping $d^* = d^*(\varepsilon)$, d^*: $\langle 0, \varepsilon_1 \rangle \to \mathscr{B}({}^0d; \delta_0; \mathscr{D}_1)$ such that $d^*(0) = {}^0d$ and the equality $P(d^*(\varepsilon), \varepsilon) = 0$ holds for all $\varepsilon \in \langle 0, \varepsilon_1 \rangle$. The transformation $u^* = u^*(\varepsilon)$, defined on $\langle 0, \varepsilon_1 \rangle$ into $\mathscr{U}$ by the relation $u^*(\varepsilon) = U(d^*(\varepsilon), \varepsilon)$, is a unique solution of the problem $(\mathscr{P})$ continuous on $\langle 0, \varepsilon_1 \rangle$ and such that $u^*(0) = V({}^0d)$.

Assertion $[\mathscr{A}2]$. *Let $\mathscr{F}$, $\mathscr{F}_0$ and $\mathscr{F}_1$ be the Banach spaces from the case $[\mathscr{C}2]$. Let the operators $F = F(u, \varepsilon)$, $F : \mathscr{U} \times \langle 0, \varepsilon_0 \rangle \to \mathscr{F}$, ${}^iX = {}^iX(u, \varepsilon)$, ${}^iX : \mathscr{U} \times \langle 0, \varepsilon_0 \rangle \to \mathscr{F}_i$, $i = 0, 1$, be continuous and have continuous G-derivatives F'_u, ${}^0X'_u$, ${}^1X'_u$. Then there exist a number $\varepsilon_1 > 0$ and a unique mapping $u^* = u^*(\varepsilon)$*

continuous on $\langle 0, \varepsilon_1 \rangle$ *into* $\mathscr{U}$ *such that* $u^*(0) = 0$ *and* u^* *solves the weakly nonlinear problem* $(\mathscr{P})$.

Using these general assertions, we can reduce the following sections to the formulations of theorems holding in concrete problems $(\mathscr{P}_1)$, $(\mathscr{P}_2)$ and $(\mathscr{P}_3)$.

3.2. Problem $(\mathscr{P}_1)$. **Theorem 3.2.1.** *Let the problem* $(\mathscr{P}_1)$ *with* $\omega = p/q$ *be given, where* p, q *are relatively prime natural numbers. Let* $\mathscr{S}(2)$ *denote the set* $\{k \in \mathscr{M} \mid k/p \in \mathscr{M} \setminus \{0\}\}$. *Then the assertion* $[\mathscr{A}1]$ *is valid, where* $\mathscr{F} = \mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi})$ *or* $\mathscr{F} = \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})$, $\mathscr{F}_i = \mathscr{H}^3_{2\pi}$, $i = 0, 1$, $\mathscr{D}_1 = \mathfrak{h}^2_{\mathscr{S}(2)}$, $\mathscr{D}_2 = \mathscr{H}^2_{2\pi}$, $V = V_1$, $W = W_1$ *and* $R = R_1$.

Theorem 3.2.2. *Let the problem* $(\mathscr{P}_1)$ *be given, where the number* ω *satisfies the assumption* $[\mathscr{L}\varrho]$ *for a natural* $\varrho \geqq 2$. *Then the assertion* $[\mathscr{A}2]$ *is valid, where* $\mathscr{F} = \mathscr{C}(\mathscr{I}; \mathscr{H}^{\varrho+1}_{2\pi})$ *or* $\mathscr{F} = \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^{\varrho}_{2\pi})$ *and* $\mathscr{F}_i = \mathscr{H}^{\varrho+2}_{2\pi}$, $i = 0, 1$.

Remark 3.2. If F, ${}^0X$ and ${}^1X$ are Njemyckij-operators, the above theorem is useful only with $\varrho = 2$. According to Remark 3.1, our problem with $\varrho \geqq 3$ must inevitably be a linear problem.

3.3. Problem $(\mathscr{P}_2)$. **Theorem 3.3.1.** *Let the problem* $(\mathscr{P}_2)$ *with* $\alpha_0 = \alpha_1$ *and* $\omega = p/q$ *be given, where* p, q *are relatively prime natural numbers. Let* $\mathscr{S}(1) = \{k \in \mathscr{M} \mid k/p \in \mathscr{M}\}$ *and* $\mathscr{S}(2) = \mathscr{S}(1) \setminus \{0\}$.

(A): *If* $\alpha_0 = \alpha_1 = 0$, *the assertion* $[\mathscr{A}1]$ *is valid, where* $\mathscr{F} = \mathscr{C}(\mathscr{I}; \mathscr{H}^2_{2\pi})$ *or* $\mathscr{F} = \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^1_{2\pi})$, $\mathscr{F}_i = \mathscr{H}^2_{2\pi}$, $i = 0, 1$, $\mathscr{D}_1 = \mathfrak{h}^3_{\mathscr{S}(1)}$, $\mathscr{D}_2 = \mathscr{H}^2_{2\pi}$, $V = V_3$, $W = W_3$ *and* $R = R_3$.

(B): *If* $\alpha_0 = \alpha_1 \neq 0$, *the assertion* $[\mathscr{A}1]$ *is valid, where* $\mathscr{F}$, $\mathscr{F}_0$ *and* $\mathscr{F}_1$ *are the same spaces as above and* $\mathscr{D}_1 = \mathfrak{h}^3_{\mathscr{S}(2)}$, $\mathscr{D}_2 = \mathscr{H}^1_{2\pi}$, $V = V_4$, $W = W_4$ *and* $R = R_4$.

Theorem 3.3.2. *Let the problem* $(\mathscr{P}_2)$ *with* $\alpha_0 = \alpha_1$ *be given, where the number* ω *satisfies the assumption* $[\mathscr{L}\varrho]$ *for a natural* $\varrho \geqq 2$.

(A): *Let* $\alpha_0 = \alpha_1 = 0$. *Then the assertion* $[\mathscr{A}1]$ *is valid, where* $\mathscr{F} = \mathscr{C}(\mathscr{I}; \mathscr{H}^{\varrho+1}_{2\pi})$ *or* $\mathscr{F} = \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^{\varrho}_{2\pi})$, $\mathscr{F}_i = \mathscr{H}^{\varrho+1}_{2\pi}$, $i = 0, 1$, $\mathscr{D}_1 = \mathscr{D}_2 = \mathscr{R}$, $V = V_5$, $W = W_5$ *and* $R = R_5$.

(B): *If* $\alpha_0 = \alpha_1 \neq 0$, *then the assertion* $[\mathscr{A}2]$ *holds, where* $\mathscr{F}$, $\mathscr{F}_0$ *and* $\mathscr{F}_1$ *are the same spaces as above.*

Theorem 3.3.3. *Let the problem* $(\mathscr{P}_2)$ *with* $\alpha_0 \neq \alpha_1$ *and* $\omega = p/q$ *be given, where* p, q *are relatively prime natural numbers. Let* $\mathscr{S}(3)$ *denote the set* $\{k \in \mathscr{M} \mid S_3(k) = 0\}$, *where* $S_3(k)$ *is defined by* (2.3.23). *Putting either* $\mathscr{F} = \mathscr{C}(\mathscr{I}; \mathscr{H}^3_{2\pi})$ (*or* $\mathscr{F} = \mathscr{C}^{(1)}(\mathscr{I}; \mathscr{H}^2_{2\pi})$), $\mathscr{F}_i = \mathscr{H}^3_{2\pi}$, $i = 0, 1$, *or* $\mathscr{F} = \mathscr{C}(\mathscr{I}; [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(2)})$ (*or* $\mathscr{F} = \mathscr{C}^{(1)}(\mathscr{I}; [\mathscr{H}^1_{2\pi}]^{\perp}_{\mathscr{S}(2)})$), $\mathscr{F}_i = [\mathscr{H}^2_{2\pi}]^{\perp}_{\mathscr{S}(2)}$, $i = 0, 1$, *where* $\mathscr{S}(2) = \{k \in \mathscr{M} \mid k/p \in \mathscr{M} \setminus \{0\}\}$, *the following propositions hold:*