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EMBEDDING THE POLYTOMIC TREE INTO THE n-CUBE

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In the whole paper a "graph" is a nondirected, possibly infinite graph without loops and multiple edges, expressed as an ordered pair $\mathscr{G} = \langle V, E \rangle$, where V is the set of vertices and E is the set of edges, a subset of $V^{(2)}$, the set of all unordered pairs of elements of V. $\mathscr{G}' = \langle V', E' \rangle$ is said to be the subgraph of $\mathscr{G} = \langle V, E \rangle$ induced by V' iff $V' \subset V$, $E' = E \cap V'^{(2)}$. $\mathscr{G}' = \langle V', E' \rangle$ is said to be a partial subgraph of $\mathscr{G} = \langle V, E \rangle$ iff $V' \subset V$, $E' \subset E \cap V'^{(2)}$. (Cf [3].) By $]\,[$ we denote the post-office function.

Definition 1. Let S be a set, by 2^S denote as usual the set of all subsets of S. Put $E(S) = \{(A, B) \mid A \subset S, B \subset S, \operatorname{card}(A \dot{-} B) = 1\}$. $(A \dot{-} B)$ denotes here the symmetric difference of A and B. By the S-cube we understand the graph $\mathscr{K}(S) = \langle 2^S, E(S) \rangle$.

Definition 2. By $\mathfrak{K}(S)$ denote the class of all graphs isomorphic to some partial subgraph of $\mathscr{K}(S)$. If $S = \{1, 2, \ldots, n\}$, write $\mathfrak{K}(S) = \mathfrak{K}_n$. Put $\bar{\mathfrak{K}} = \{\mathscr{G} \mid \exists S, \mathscr{G} \in \mathfrak{K}(S)\}$. By $\mathfrak{K}$ denote the class of all graphs $\mathscr{G}$ such that for any finite partial subgraph $\mathscr{G}'$ of $\mathscr{G}$, $\mathscr{G}' \in \bar{\mathfrak{K}}$.

Trivially, if $\mathscr{G} \in \mathfrak{K}(S)$ and $\mathscr{G}'$ is a partial subgraph of $\mathscr{G}$, then $\mathscr{G}' \in \mathfrak{K}(S)$.

Definition 3. Let $\mathscr{G} = \langle V, E \rangle$ be a graph, F a set. Assume there exists a mapping $\psi : E \to F$ such that

(i) if $(e_1, e_2, \ldots, e_r)$ is the sequence of edges of a finite open path in $\mathscr{G}$, then there is an element of F that appears an odd number of times in the sequence $(\psi(e_1), \psi(e_2), \ldots, \psi(e_r))$.

(ii) if $(f_1, f_2, \ldots, f_s)$ is the sequence of edges of a finite closed path in $\mathscr{G}$, then all the elements of F appear an even number (possibly null) of times in the sequence $(\psi(f_1), \psi(f_2), \ldots, \psi(f_s))$.

Then we call ψ a $\bar{C}$-valuation of $\mathscr{G}$. Let n be a natural number. If card $(\psi(E)) \leqq n$, we call ψ a C_n-valuation of $\mathscr{G}$.

Definition 4. By $\bar{\mathfrak{C}}$ denote the class of all graphs $\mathscr{G}$ such that there exists a $\bar{C}$-valuation of $\mathscr{G}$, by $\mathfrak{C}$ denote the class of all graphs $\mathscr{G}$ such that for any finite partial subgraph $\mathscr{G}'$ of $\mathscr{G}$, $\mathscr{G}' \in \bar{\mathfrak{C}}$. Let n be a natural number. By $\mathfrak{C}_n$ denote the class of all graphs $\mathscr{G}$ such that there exists a C_n-valuation of $\mathscr{G}$.

Remark 1. If $\mathscr{G} \in \bar{\mathfrak{C}}$ is finite, then for some n, $\mathscr{G} \in \mathfrak{C}_n$. Further, $\mathfrak{C}_n \subset \bar{\mathfrak{C}} \subset \mathfrak{C}$.

Theorem 1 in [2] asserts that

(a) $\mathfrak{K}_n \subset \mathfrak{C}_n$
(b) $\mathscr{G} \in \mathfrak{C}_n$ connected $\Rightarrow \mathscr{G} \in \mathfrak{K}_n$
(c) $\mathfrak{C} = \mathfrak{K}$.

Remark 2. Let $\mathscr{T}$ be an arbitrary tree. Then condition (ii) of Def. 3 is empty and moreover, putting $F = E$, ψ the identity map, we have $\mathscr{T} \in \bar{\mathfrak{C}}$ and hence $\mathscr{T} \in \mathfrak{K}$. Also, $\mathscr{T} \in \mathfrak{K}_n \Leftrightarrow \mathscr{T} \in \mathfrak{C}_n$.

In what remains, we shall be concerned with trees only, and with the problem to find to a tree $\mathscr{T}$ the smallest n such that $\mathscr{T} \in \mathfrak{K}_n$. We shall denote this n by dim $(\mathscr{T})$.

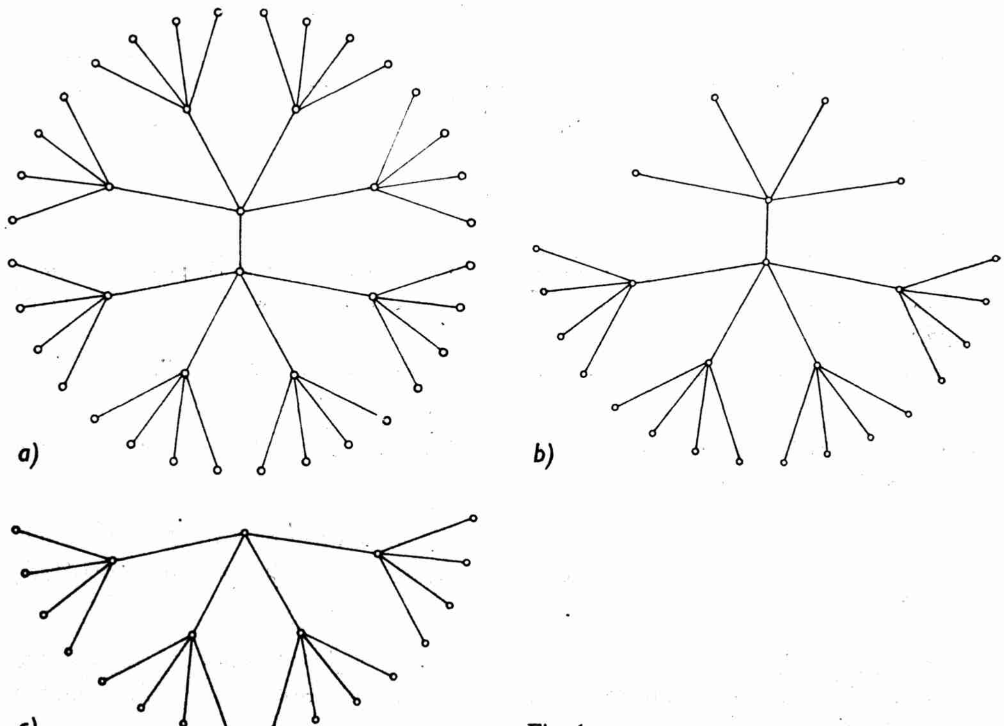

Fig. 1.

To study trees the vertices of which have their degree bounded from above by a given number, we introduce three infinite classes of trees, closely related to each other. $\mathscr{T}_l^{(k)}$, the "polytomic tree", is a straightforward generalization of the dichotomic tree $\mathscr{D}_l$ of [1]. $\mathscr{T}_l^{(k)}$ may be considered to be a star of k rays, each endpoint of a ray being again the center of a new k-star, and this procedure repeated l times. So, there are vertices of "level" 1 to $(l + 1)$, where the (single) vertex of level 1 has degree k, the vertices of the outermost level $(l + 1)$ have degree 1 and the remaining vertices have degree $(k + 1)$. ${}^{\flat}\mathscr{T}_l^{(k)}$ and ${}^{\sharp}\mathscr{T}_l^{(k)}$ arise from $\mathscr{T}_l^{(k)}$ if it is completed in such a way that all its vertices have either degree 1 or degree $(k + 1)$.

Definition 5. Let $k \geqq 2$ and $l \geqq 1$ be natural numbers. Define

$$\mathscr{T}_l^{(k)} = \langle V_l^{(k)}, E_l^{(k)} \rangle, \quad {}^{\flat}\mathscr{T}_l^{(k)} = \langle {}^{\flat}V_l^{(k)}, {}^{\flat}E_l^{(k)} \rangle, \quad {}^{\sharp}\mathscr{T}_l^{(k)} = \langle {}^{\sharp}V_l^{(k)}, {}^{\sharp}E_l^{(k)} \rangle$$

as follows:

Put

$$V_l^{(k)} = \{ v_j^{(i)} \mid 1 \leqq i \leqq l + 1,\ 1 \leqq j \leqq k^{i-1} \}$$
$${}^{\flat}V_l^{(k)} = \{ v_j^{(i)} \mid (1 \leqq i \leqq l + 1) \vee (-l \leqq i \leqq -1),\ 1 \leqq j \leqq k^{|i|-1} \}$$
$${}^{\sharp}V_l^{(k)} = \{ v_j^{(i)} \mid 1 \leqq |i| \leqq l + 1,\ 1 \leqq j \leqq k^{|i|-1} \}.$$

Further, for $v_j^{(i)} \in {}^{\sharp}V_l^{(k)}$, $v_{j'}^{(i')} \in {}^{\sharp}V_l^{(k)}$, $(v_j^{(i)}, v_{j'}^{(i')}) \in {}^{\sharp}E_l^{(k)} \Leftrightarrow (|i'| = |i| - 1) \,\&\, (j' =]j/k^2[\vee ((i = 1) \,\&\, (i' = -1))$. Denote $(v_1^{(1)}, v_1^{(-1)})$ by $e_1^{(0)}$ and further $(v_j^{(i)}, v_{j'}^{(i')}) \in E^{(k)}$ by $e_{j'}^{(i)}$, if $|i| < |i'|$. ${}^{\flat}\mathscr{T}_l^{(k)}$ resp. $\mathscr{T}_l^{(k)}$ are defined as the subgraphs of ${}^{\sharp}\mathscr{T}_l^{(k)}$ induced by ${}^{\flat}V_l^{(k)}$ resp. $V_l^{(k)}$.

Fig. 1a, b, c shows ${}^{\sharp}\mathscr{T}_2^{(4)}$, ${}^{\flat}\mathscr{T}_2^{(4)}$ and $\mathscr{T}_2^{(4)}$.

As is seen, ${}^{\sharp}\mathscr{T}_l^{(k)}$ consists of two trees $\mathscr{T}_l^{(k)}$ with their "roots" joined by a new edge whereas ${}^{\flat}\mathscr{T}_l^{(k)}$ arises in a similar manner from one $\mathscr{T}_l^{(k)}$ and one $\mathscr{T}_{l-1}^{(k)}$ (for $l \geqq 2$). As for the number of vertices, card ${}^{\sharp}V_l^{(k)} = 2(k^{l+1} - 1)/(k - 1)$, card ${}^{\flat}V_l^{(k)} = (k^{l+1} + k^l - 2)/(k - 1)$ and card $V_l^{(k)} = (k^{l+1} - 1)/(k - 1)$. In [1], $\mathscr{T}_l^{(2)}$ is denoted by $\mathscr{D}_l$. Theorem 3 of [1] asserts that for $l \geqq 2$, dim $\mathscr{T}_l^{(2)} = l + 2$ (dim $\mathscr{T}_1^{(2)} = 2$ being trivial). Another partial result of the general problem of dim $\mathscr{T}_l^{(k)}$ is supplied by the following theorem. But first a

Remark 3. ${}^{\sharp}\mathscr{T}_l^{(k)} \in \mathfrak{K}_n \Rightarrow {}^{\flat}\mathscr{T}_l^{(k)} \in \mathfrak{K}_n \Rightarrow \mathscr{T}_l^{(k)} \in \mathfrak{K}_n \Rightarrow {}^{\sharp}\mathscr{T}_l^{(k)} \in \mathfrak{K}_{n+1}$. The first two implications being trivial, consider for the third the two constituent $\mathscr{T}_l^{(k)}$ of ${}^{\sharp}\mathscr{T}_l^{(k)}$ as having a C_n-valuation with the same F and the joining edge being assigned a new element f_{n+1}.

Theorem 1.

$$\dim({}^{\sharp}\mathscr{T}_2^{(2p)}) = \dim({}^{\flat}\mathscr{T}_2^{(2p)}) = \dim(\mathscr{T}_2^{(2p)}) = 3p + 1,$$
$$\dim({}^{\sharp}\mathscr{T}_2^{(2p+1)}) = \dim({}^{\flat}\mathscr{T}_2^{(2p+1)}) = 3p + 3,$$
$$\dim(\mathscr{T}_2^{(2p+1)}) = 3p + 2.$$

Proof. In view of Remark 3, it is sufficient to prove

$$^{\sharp}\mathscr{T}_2^{(2p)} \in \mathfrak{R}_{3p+1}, \quad \mathscr{T}_2^{(2p+1)} \in \mathfrak{R}_{3p+2}, \quad \mathscr{T}_2^{(2p)} \notin \mathfrak{R}_{3p}, \quad \mathscr{T}_2^{(2p+1)} \notin \mathfrak{R}_{3p+1},$$
$$^{\flat}\mathscr{T}_2^{(2p+1)} \notin \mathfrak{R}_{3p+2}.$$

1. To construct a C_{3p+1}-valuation ψ of $^{\sharp}\mathscr{T}_2^{(2p)}$, put

$$F = \{a'_{p+1}, a'_{p+2}, \ldots, a'_{2p}, a_1, a_2, \ldots, a_{2p+1}\}.$$

Further define

$$(*) \qquad \begin{aligned} \psi(e_1^{(0)}) &= a_{2p+1}, \\ \psi(e_j^{(1)}) &= a_j \quad (1 \leqq j \leqq 2p), \\ \psi(e_j^{(-1)}) &= a''_j \quad (1 \leqq j \leqq 2p), \end{aligned}$$

where we write for short

$$a''_t = a_t \ (1 \leqq t \leqq p), \quad a''_t = a'_t \ (p+1 \leqq t \leqq 2p), \quad a''_{2p+1} = a_{2p+1}.$$

Instead of proceeding by defining explicitly $\psi(e_j^{(2)})$ and $\psi(e_j^{(-2)})$, observe that the edges $e_j^{(2)}$ and $e_j^{(-2)}$ are classified naturally into groups of $2p$ by the j of the $e_j^{(1)}$ they are adjacent to:

$$\begin{aligned} G_j^{(1)} &= \{e_t^{(2)} \mid 2p(j-1) + 1 \leqq t \leqq 2pj\}, \quad 1 \leqq j \leqq 2p, \\ G_j^{(-1)} &= \{e_t^{(-2)} \mid 2p(j-1) + 1 \leqq t \leqq 2pj\}, \quad 1 \leqq j \geqq 2p. \end{aligned}$$

Obviously a permutation of the valuation ψ inside one group is immaterial. So, we define merely a set of $2p$ values for each group putting

$$(**) \qquad \begin{aligned} \psi(G_j^{(1)}) &= \{a_t \mid j + 1 \leqq t \leqq \min((j+p), (2p+1))\} \cup \\ &\cup \{a_t \mid 1 \leqq t \leqq j - p - 1\} \cup \{a'_t \mid p + 1 \leqq t \leqq 2p\}, \\ \psi(G_j^{(-1)}) &= \{a''_t \mid j + 1 \leqq t \leqq \min((j+p), (2p+1))\} \cup \\ &\cup \{a''_t \mid 1 \leqq t \leqq j - p - 1\} \cup \{a_t \mid p + 1 \leqq t \leqq 2p\}. \end{aligned}$$

(One such valuation ψ is shown for $p = 2$ on Fig. 2, where for transparency we write 1 for a_1, 3′ for a'_3 etc.) (Observe that considering the valuation induced by ψ on $^{\flat}\mathscr{T}_2^{(2p)}$ and looking at $e_1^{(0)}$ as "$e_{2p+1}^{(1)}$" and at $\{e_j^{(-1)} \mid 1 \leqq j \leqq 2p\}$ as "$G_{2p+1}^{(1)}$", ψ on them meets the rules $(*)$ and $(**)$.)

Let us now show that ψ so defined is a C-valuation. For paths of odd length the condition (i) of Def. 3 holds trivially, so we concern ourselves only with paths of length 2 or 4 in $\mathscr{T}_2^{(2p)}$. The paths of length 2 being well valuated by inspection, assume there is a path $\boldsymbol{p}$ of length 4 such that two elements of F, say x and y, appear on it twice each. The center of any path of length 4 in $\mathscr{T}_2^{(2p)}$ is either in $\boldsymbol{v}_1^{(1)}$ or in $\boldsymbol{v}_1^{(-1)}$. Assume for $\boldsymbol{p}$ the former happens. Hence x and y must be both unprimed a's, say a_r and a_s.

So it must simultaneously be $a_r \in G_s^{(1)}$, $a_s \in G_r^{(1)}$, with possible $r = k + 1$ or $s = k + 1$. That however is impossible by definition of $\psi(G_j^{(1)})$. What concerns the case that the center of $\mathfrak{p}$ is in $\mathfrak{v}_1^{(-1)}$, observe the symmetry in ψ which permits us to repeat the former argument with interchange of a_j and a'_j $(p + 1 \leqq j \leqq 2p)$. Q.E.D.

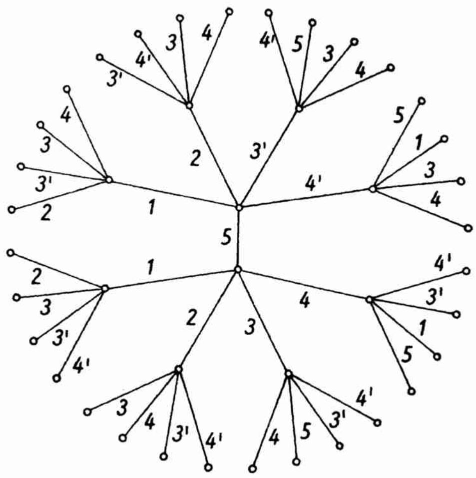

Fig. 2.

2. To construct a C_{3p+2}-valuation of $\mathscr{T}_2^{(2p+1)}$, consider the valuation used for ${}^\sharp\mathscr{T}_2^{(2p)}$, specifically that induced on ${}^\flat\mathscr{T}_2^{(2p)}$. $\mathscr{T}_2^{(2p+1)}$ arises from ${}^\flat\mathscr{T}_2^{(2p)}$ by adding one $e_j^{(2)}$ in each $G_j^{(1)}$. The desired C_{3p+2}-valuation is simply obtained by modifying ψ in the way that to each mentioned new $e_j^{(2)}$ the new value a'_{2p+1} is assigned. Obviously this does not spoil the property (i) of Def. 3. Q.E.D.

3. We proceed now to show that ${}^\flat\mathscr{T}_2^{(2p+1)} \notin \mathfrak{R}_{3p+2}$. Assume the contrary. Consider ${}^\flat\mathscr{T}_2^{(2p+1)}$ as a partial subgraph of $\mathscr{K}_{3p+2}$. Without loss of generality assume $\mathfrak{v}_1^{(1)}$ is in the vertex $\emptyset$ of $\mathscr{K}_{3p+2}$, and the $2p + 2$ neighbours of $\mathfrak{v}_1^{(1)}$ in ${}^\flat\mathscr{T}_2^{(2p+1)}$ are in the vertices $\{j\}$ for $1 \leqq j \leqq 2p + 2$ of $\mathscr{K}_{3p+2}$. It is now necessary to place the $(2p + 1)(2p + 2) = 4p^2 + 6p + 2$ vertices of degree 1 of the ${}^\flat\mathscr{T}_2^{(2p+1)}$ into the $\binom{3p+2}{2} - \binom{p}{2} = 4p^2 + 5p + 1$ vertices $\{i, j\}$ of $\mathscr{K}_{3p+2}$ with $1 \leqq i \leqq 3p + 2$, $1 \leqq j \leqq 3p + 2$, $i \neq j$, such that not both i and j are $> 2p + 2$. As this is not possible by reason of numbers, the proof is complete.

4. To complete the proof of the whole theorem, we have to show $\mathscr{T}_2^{(2p)} \notin \mathfrak{R}_{3p}$, $\mathscr{T}_2^{(2p+1)} \notin \mathfrak{R}_{3p+1}$. To that purpose we show that from $\mathscr{T}_2^{(k)} \in \mathfrak{R}_n$ follows $2n \geqq 3k + 1$. Indeed, if $\mathscr{T}_2^{(k)}$ is a partial subgraph of $\mathscr{K}_n$, there are certain k^2 vertices of $\mathscr{T}_2^{(k)}$ to be placed into $\binom{n}{2} - \binom{n-k}{2}$ vertices of $\mathscr{K}_n$, hence $k^2 \leqq \binom{n}{2} - \binom{n-k}{2}$ and the desired inequality follows.

To be able to derive statements about much wider classes of trees than $\mathcal{T}_l^{(k)}$, ${}^b\mathcal{T}_l^{(k)}$, ${}^\#\mathcal{T}_l^{(k)}$, we observe that ${}^b\mathcal{T}_l^{(k)}$ and ${}^\#\mathcal{T}_l^{(k)}$ are in a sense the most general trees with given diameter and given maximum degree of the vertices. Strictly speaking, the following holds:

Lemma 1. *Let the maximum degree of the vertices of the tree $\mathcal{T}$ be $k + 1$. If the diameter of $\mathcal{T}$ equals $2l$ resp. $(2l + 1)$, then $\mathcal{T}$ is a partial subgraph of ${}^b\mathcal{T}_l^{(k)}$ resp. ${}^\#\mathcal{T}_l^{(k)}$.*

Proof is obvious.

Corollary 1. *Suppose the maximum degree of the vertices of the tree $\mathcal{T}$ is $d \geqq 1$ and the diameter of $\mathcal{T}$ is $\leqq 5$. If $d = 2a$ then $\dim \mathcal{T} \leqq 3a$, if $d = 2a + 1$ then $\dim \mathcal{T} \leqq 3a + 1$. There is, on the other hand, to any $d \geqq 1$ a tree $\mathcal{T}$ with maximum degree of the vertices equal d and diameter $\leqq 4$ such that $\dim \mathcal{T} = 3a$ for $d = 2a$ resp. $\dim \mathcal{T} = 3a + 1$ for $d = 2a + 1$.*

Proof. The inequalities follow, for $d \geqq 3$, from L 1 and Th 1. On the other hand observe that $\mathcal{T}_2^{(k)}$ has diameter 4 and maximal degree of its vertices $(k + 1)$. The cases $d = 1$ and $d = 2$ are trivial.

For $\mathcal{T}_l^{(2)}$ and $\mathcal{T}_2^{(k)}$ the results obtained are exact. For $k > 2$, $l > 2$ we are only able to give bounds for $\dim \mathcal{T}_l^{(k)}$. From one side, we only succeeded in finding trivial bounds:

Remark 4. $\dim \mathcal{T}_l^{(k)} \leqq kl$. The proof of this rests on the following C_{kl}-valuation of $\mathcal{T}_l^{(k)}$. For the edges of each level of $\mathcal{T}_l^{(k)}$, k different elements of F are reserved and distributed in such a way that adjacent edges are assigned different values. In fact, an insubstantially better bound is obtained by using Th 1. for the first two levels, and applying a slightly finer reasoning to the remaining ones. For $k > 2$, $l > 2$ it holds that $\dim \mathcal{T}_l^{(k)} \leqq 3/2k + 1 + (l - 2)(k - 1)$.

Theorem 2. $\dim \mathcal{T}_l^{(k)} > kl/e$ *where* $e = 2{,}71 \ldots$

Proof. Assume $\mathcal{T}_l^{(k)}$ to be isomorphic to some partial subgraph of $\mathcal{K}_n$. Then comparing the number of vertices, $2^n \geqq \operatorname{card} V_l^{(k)} > k^l$ and hence

$$n > l \log_2 k\,. \tag{1}$$

Consider first $2 \leqq k \leqq 8$. Here we have $e \log_2 k > k$ and hence $n > l \log_2 k > kl/e$ and the desired inequality holds. Assume now $k > 8$. It follows from (1) that

$$n > 3l\,. \tag{2}$$

The isomorphism may be assumed such that to the vertex $v_1^{(1)}$ of $\mathcal{T}_l^{(k)}$ the vertex $\emptyset$ of $\mathcal{K}_n$ corresponds. Then to the k^l vertices of distance l from $v_1^{(1)}$ in $\mathcal{T}_l^{(k)}$ there must correspond vertices of $\mathscr{K}_n$ whose cardinalities are either l or less than l by an even number, hence

$$(3) \qquad k^l < \binom{n}{l} + \binom{n}{l+2} + \binom{n}{l-4} + \ldots$$

where the sum at the right is finite, ending either with n or 1 depending on the parity of l. As

$$\binom{n}{p-2} \Big/ \binom{n}{p} \leqq \binom{n}{l-2} \Big/ \binom{n}{l} = q$$

for $p \leqq l$, we may write

$$(4) \quad \binom{n}{l} + \binom{n}{l-2} + \binom{n}{l-4} + \ldots < \binom{n}{l} (1 + q + q^2 + \ldots) = \binom{n}{l} \Big/ (1 - q) .$$

Using (2) we have, however,

$$q = l(l-1)/((n-l+1)(n-l+2)) < l(l-1)/((2l+1)(2l+2)) < 1/4$$

and this yields together with (3) and (4)

$$(5) \qquad k^l < \frac{4}{3} \binom{n}{l} .$$

For estimating $\binom{n}{l}$ we use the trivial $n(n-1) \ldots (n-l+1) < n^l$ and Stirling's formula

$$l! = \sqrt{(2\pi l)}\, (l/e)^l \exp(\theta_l)$$

where $|\theta_l| < 1/(12l)$ and get from (5)

$$k^l < \tfrac{4}{3} \exp(-\theta_l)\, (ne/l)^l\, (2\pi l)^{-1/2} .$$

Finally

$$\left(\frac{ne}{kl}\right)^l > \tfrac{3}{4} \sqrt{(2\pi l)} \exp(\theta_l) = \sqrt{[9/8\pi l \exp(2\theta_l)]} > \sqrt{[9/8\pi l \exp(-1/6)]} > 1 ,$$

Q.E.D.

Corollary 2. *Suppose the maximum degree of the vertices of the tree $\mathscr{T}$ is $d \geqq 3$ and the diameter of $\mathscr{T}$ is $D > 5$. Then* $\dim \mathscr{T} \leqq \frac{1}{2}(d-1) D$. *On the other hand, given $d \geqq 3$ and $D > 5$, there is a tree $\mathscr{T}$ with maximum degree of the vertices equal d and of diameter $\leqq D$ such that* $\dim \mathscr{T} >](D-1)/2[\, . (d-1)/e$.

Proof. The first inequality follows from Lemma 1, Remark 4 and Remark 3. The proof of the second statement follows by observing that for the tree $\mathscr{T}$ we may take $\mathscr{T}_l^{(k)}$ for $l =](D-1)/2[$ and $k = d - 1$.