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UNIFORM EXTENSION OF LINEAR FUNCTIONALS

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We work in real Banach spaces. If X is a Banach space, $x_n, x \in X$, then $x_n \to x$ or $x_n \xrightarrow{w} x$ means the norm of the weak convergence of the sequence $\{x_n\}$ to x, respectively. Similarly $f_n \xrightarrow{w^*} f$ for the pointwise convergence in X^* (the dual of X). For $r > 0$, $K_r = \{x \in X;\ \|x\| \leqq r\}$, $S_r = \{x \in X;\ \|x\| = r\}$. Analogously K_r^*, S_r^* in X^*. The reals will be denoted by R, the positive integers by N. For a closed linear subspace P of a Banach space $X, X/P$ denotes the quotient space and the codimension of P (codim P) is the dimension of $X/P(\dim X/P)$. For a linear subspace $P \subset X$, $P^\perp$ is the annihilator (or polar) of P at X^*. For a linear subspace $P \subset X, f \in X^*$, $\|f\|_P = \sup_{K_1 \cap P} |f(x)|$. For $M \subset X$, $\overline{M}$ means the norm closure of M in X. If K is a convex subset of X, ext K denotes the set of all extreme points of K. For a topological space S, $C(S)$ denotes the Banach space of all real valued continuous bounded functions on S with the supremum norm. By $\langle x, y \rangle$ we mean the closed line segment with the end points $x, y \in X$.

Definition 1. Let X be a real Banach space, $P \subset X$ a linear subspace of X. We call X *uniformly rotund* (*weakly uniformly rotund*) along P if the following implication is valid:

if $x_n, y_n \in S_1$, $x_n - y_n \in P$, $\|x_n + y_n\| \to 2$, then $x_n - y_n \to 0$ $(x_n - y_n \xrightarrow{w} 0)$. Similarly for the case of X^* and the weak * uniform rotundity along $P \subset X^*$.

Lemma 1. *If $P \subset X$ is a linear subspace of a real Banach space X, then the following properties are equivalent:*

(i) X *is uniformly rotund* (weakly uniformly rotund) *along* P.

(ii) *If* $x_n, y_n \in S_1$, $\inf_{t \in \langle 0,1 \rangle} \|tx_n + (1 - t) y_n\| \to 1$, $x_n - y_n \in P$, *then* $x_n - y_n \to 0$ $(x_n - y_n \xrightarrow{w} 0)$.

(iii) *If* $x_n, y_n \in K$, $\|\frac{1}{2}(x_n + y_n)\| \to 1$, $x_n - y_n \in P$, *then* $x_n - y_n \to 0$ $(x_n - y_n \xrightarrow{w} 0)$.

(iv) *If* $x_n, y_n \in X$, $\{x_n\}$ *bounded*, $x_n - y_n \in P$, $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 \to 0$, *then* $x_n - y_n \to 0$ $(x_n - y_n \xrightarrow{w} 0)$.

Proof. It involves only some computations and is quite similar to that in [6].

Proposition 1. *Suppose $P \subset X$ is a finite dimensional subspace of a Banach space X. Then X is uniformly rotund along P iff X is uniformly rotund along each one dimensional subspace of P.*

Proof. One part of the equivalence is evident. Suppose X is not uniformly rotund along P. Then by (iv) of Lemma 1 there exist $\{x_n\}$ bounded, $y_n \in X$, $x_n - y_n \in P$, $2(\|x_n\|^2 + \|y_n\|^2) - \|x_n + y_n\|^2 \to 0$, $\|x_n - y_n\| \geqq \varepsilon > 0$ for some $\varepsilon > 0$. Suppose without a loss of generality $w_n = x_n - y_n \to w \in P$, $w \neq 0$, for $\{y_n\}$ is also bounded (see for example [6]), and $2(\|y_n\|^2 + \|y_n + w\|^2) - \|2y_n + w\|^2 = 2(\|y_n\|^2 + \|y_n + w_n\|^2) - \|2y_n + w_n\|^2 + 2(\|y_n + w\|^2 - \|y_n + w_n\|^2) + \|2y_n + w_n\|^2 - \|2y_n + w\|^2$. However, $|\|y_n + w\|^2 - \|y_n + w_n\|^2| = |(\|y_n + w\| - \|y_n + w_n\|) \cdot (\|y_n + w\| + \|y_n + w_n\|)| \leqq \|w_n - w\| \cdot K$ where K is some constant, since $\{y_n + w_n\}$ is a bounded sequence. Similarly for the term $\|2y_n + w_n\|^2 - \|2y_n + w\|^2$. Therefore $2(\|y_n\|^2 + \|y_n + w\|^2) - \|2y_n + w\|^2 \to 0$, $\{y_n\}$ bounded. Thus X is not uniformly rotund along the subspace generated by w.

Proposition 2. *Suppose $P \subset X$ is a linear subspace of a Banach space X. Then X is (weakly) uniformly rotund along P iff X is (weakly) uniformly rotund along $\bar{P}$.*

Proof. One part of the equivalence is evident. Suppose X is not weakly uniformly rotund along $\bar{P}$. Then there exists $\{x_n\} \subset X$, $\{x_n\}$ bounded, $f \in S_1^* \subset X^*$, $w_n \in \bar{P}$, $|f(w_n)| \geqq \varepsilon$ for some $\varepsilon > 0$ such that $2(\|x_n\|^2 + \|x_n + w_n\|^2) - \|2x_n + w_n\|^2 \to 0$. We may choose for $n \in N$, $\hat{w}_n \in P$ such that $\|w_n - \hat{w}_n\| \leqq 1/n$ and therefore $w_n - \hat{w}_n \to 0$. Then again $\{w_n\}$ is bounded and $2(\|x_n\|^2 + \|x_n + \hat{w}_n\|^2) - \|2x_n + \hat{w}_n\|^2 = 2(\|x_n\|^2 + \|x_n + w_n\|^2) - \|2x_n + w_n\|^2 + 2(\|x_n + \hat{w}_n\|^2 - \|x_n + w_n\|^2) + \|2x_n + w_n\|^2 - \|2x_n + \hat{w}_n\|^2$. $|\|x_n + \hat{w}_n\|^2 - \|x_n - w_n\|^2| = |(\|x_n + \hat{w}_n\| + \|x_n + w_n\|)(\|x_n + \hat{w}_n\| - \|x_n + w_n\|)|$. The first term on the right hand side of the last equality is bounded. The second one is not greater than $\|w_n - \hat{w}_n\|$, which converges to zero. Furthermore, $f(\hat{w}_n - w_n) \to 0$. Thus for $n \geqq n_0$ we would have $|f(\hat{w}_n)| \geqq \varepsilon/2 > 0$. The other parts of the statement are derived similarly.

In the following, $\varrho(x, y)$ will denote $\|x - y\|$ in X, and for $M \subset X$, $\varrho(x, M) = \inf_{y \in M} \varrho(x, y)$.

Definition 2. A linear subspace P of a Banach space X *is said to be a uniformly Haar subspace* (*a weakly uniformly Haar subspace*) if the following implication is valid: if $\{x_n\}$ is bounded in X, $\{y_n\} \subset P$, $\{z_n\} \subset P$, $\varrho(x_n, P) - \varrho(x_n, y_n) \to 0$, $\varrho(x_n, P) - \varrho(x_n, z_n) \to 0$, then $y_n - z_n \to 0$ ($y_n - z_n \overset{w}{\to} 0$). Similarly for $P \subset X^*$ and the w^*-topology.

Remarks. Evidently, these conditions are not weaker than the Haar (i.e. Chebyshev) property of the closed P in a reflexive space X, and coincide with it if P is a subspace of a finite dimensional space X. If X is a uniformly rotund (= uniformly convex) space, then it is uniformly rotund along each subspace. Moreover, the space $C\langle 0, 1\rangle$ with the equivalent norm $\sqrt{(\|x\|^2_{C\langle 0,1\rangle} + \|x\|^2_{L_2\langle 0,1\rangle})}$ provides ([6]) an example of a space uniformly rotund in each direction which is nevertheless not even weakly uniformly rotund in the sense of V. ŠMULJAN ([5]), (i.e. whenever $\|x_n\| = \|y_n\| = 1$, $\|x_n + y_n\| \to 2$, then $x_n - y_n \overset{w}{\to} 0$). It is also elementary to construct in three dimensions an example of a space which is not uniformly rotund exactly in one direction.

Proposition 3. *Suppose P is a linear subspace of a Banach space X. If X is uniformly rotund (weakly uniformly rotund) along P, then P is a uniformly Haar (a weakly uniformly Haar) subspace of X.*

Proof. It reduces only to some easy considerations which may be found in [6, Prop. 9].

Remark. If $\dim P \geqq 2$, then it is very easy to construct a Haar proper subspace along which the space is not uniformly rotund.

Definition 3. Suppose $P \subset X$ is a closed linear subspace of a Banach space X. We say P has *the uniform extension property* (*the weakly* uniform extension property*) if the following implication is valid: whenever $f_n \in X^*$, $g_n \in X^*$, $f_n - g_n \in P^\perp$, $\|f_n\|_P \to 1$, $\|f_n\| \to 1$, $\|g_n\| \to 1$, then $f_n - g_n \to 0$ ($f_n - g_n \overset{w^*}{\to} 0$).

Remark. It is evident that the condition in Definition 3 is equivalent to the following one: if $f_n \in X^*$, $g_n \in X^*$, $f_n - g_n \in P^\perp$, $\|f_n\|_P = 1$, $\|f_n\| \to 1$, $\|g_n\| \to 1$, then $f_n - g_n \to 0$ ($f_n - g_n \overset{w^*}{\to} 0$).

Proposition 4. *Suppose $P \subset X$ is a closed linear subspace of a Banach space X. Then P has the uniform extension property (the weakly* uniformly extension property) iff $P^\perp$ has the uniform Haar property (the weakly* uniform Haar property).*

Proof. Suppose $P^\perp$ does not have weakly* uniformly Haar property.

Then there exists $\{f_n\} \subset X^*$, $\{f_n\}$ bounded, $\{h_n\} \subset P^\perp$, $\{g_n\} \subset P^\perp$ and $x \in S_1 \subset X$ such that $\varrho(f_n, P^\perp) - \varrho(f_n, h_n) \to 0$, $\varrho(f_n, P^\perp) - \varrho(f_n, g_n) \to 0$, $|(h_n - g_n)(x)| \geqq \varepsilon > 0$. As it was pointed out in [4], for every $f \in X^*$ and each subspace $Y \subset X$, $\varrho(f, Y^\perp) = \|f\|_Y$. If $\|f_{n_k}\| \to 0$ for some subsequence n_k, then it would follow from our assumptions $h_{n_k} - g_{n_k} \to 0$ which contradicts our assumptions. Therefore we may suppose without any loss of generality $\|f_n\|_P \to a \neq 0$. Denote $F_n = f_n - h_n$, $G_n = f_n - g_n$, $n \in N$. Then $\|F_n\| \to a$, $\|G_n\| \to a$, $\|F_n\|_P = \|G_n\|_P = \|f_n\|_P \to a$. $F_n - G_n = g_n - h_n \in P^\perp$, $|(F_n - G_n)(x)| = |(h_n - g_n)(x)| \geqq \varepsilon > 0$. Therefore P does not have the weakly* uniform extension property.

On the other hand, suppose P does not have the weakly* uniform extension property. Then there exist $F_n, G_n \in X^*$, $F_n - G_n = h_n \in P^\perp$, $x \in S_1 \subset X$, $|(F_n - G_n)(x)| \geqq \geqq \varepsilon > 0$, $\|F_n\| \to 1$, $\|G_n\| \to 1$, $\|F_n\|_P \to 1$. Then $\varrho(F_n, P^\perp) = \|F_n\|_P \to 1$, $\varrho(G_n, P^\perp) = \|G_n\|_P \to 1$, $\{F_n\}$ is bounded. Therefore we have $\varrho(F_n, P^\perp) - \varrho(F_n, 0) \to 0$, $\varrho(F_n, P^\perp) - \varrho(F_n, h_n) \to 0$, $\{F_n\}$ bounded, $|h_n(x)| \geqq \varepsilon > 0$. Thus $P^\perp$ would not have the weakly* uniform Haar property.

Proposition 5. *If $P \subset X$ is a one dimensional subspace of a Banach space X, then P has the uniform extension property (the weakly* uniform extension property) iff the norm of X is Fréchet (Gâteaux) differentiable at z where $P \cap S_1 = \{z, -z\}$.*

Proof. By Šmuljan's theorem ([5]) the last two properties are equivalent to the following properties respectively: whenever $f_n \in S_1^*$, $g_n \in S_1^*$, $f_n(z) \to 1$, $g_n(z) \to 1$, then $f_n - g_n \to 0$ ($f_n - g_n \xrightarrow{w^*} 0$). Take $f_n' = f_n / f_n(z)$, $g_n' = g_n / g_n(z)$. Then $f_n' = g_n'$ on P and $\|f_n'\| \to 1$, $\|g_n'\| \to 1$, $\|f_n'\|_P = 1$. Therefore if P has for example the uniform extension property, we have $f_n' - g_n' \to 0$ whenever $f_n(z) \to 1$, $g_n(z) \to 1$, $\|f_n\| = \|g_n\| = 1$. On the other hand, if the norm of X is Fréchet differentiable at $z \in S_1 \cap P$ where P is generated by z, and if we suppose $f_n = g_n$ on P and $\|f_n\|_P \to 1$, $\|f_n\| \to 1$, $\|g_n\| \to 1$, then $|f_n(z)| = \|f_n\|_P \to 1$. Therefore $(\operatorname{sign} f_n(z)) \cdot f_n(z) \to 1$, $(\operatorname{sign} f_n(z)) \cdot g_n(z) \to 1$. Denote $f_n' = (\operatorname{sign} f_n(z)) \cdot f_n$, $g_n' = (\operatorname{sign} f_n(z)) \cdot g_n$. Then $f_n', g_n' \in X^*$, $f_n'(z) = g_n'(z) \to 1$, $\|f_n'\| \to 1$, $\|g_n'\| \to 1$. Denote $f_n'' = f_n' / \|f_n'\|$, $g_n'' = g_n' / \|g_n'\|$. Then we have $f_n''(z) \to 1$, $g_n''(z) \to 1$, $\|f_n''\| = \|g_n''\| = 1$. Therefore by Fréchet differentiability of the norm of X we have $f_n'' - g_n'' \to 0$. Thus

$$\|f_n - g_n\| = \| \|f_n\| \cdot f_n'' - \|g_n\| \cdot g_n''\| \to 0 .$$

Hence P has the uniform extension property.

Corollary 1. *If $x \in S_1 \subset X$, then the norm of X is Gâteaux (Fréchet) differentiable at x iff $P^\perp$ is a weakly* uniformly Haar (a uniformly Haar) subspace of X^*, where P is a one dimensional subspace of X generated by x.*

Now, we are going to investigate the dual property to that of the uniform rotundity along subspaces.

Proposition 6. *Suppose $P \subset X$ is a finite dimensional subspace of a Banach space X. Then if $P^\perp$ has the Haar property (i.e. $P^\perp$ is a Čebyšev subspace of X^*), then $P^\perp$ has the weakly* uniform Haar property.*

Proof. If $P^\perp$ does not have the weakly* uniform Haar property then P does not have the weakly* uniform extension property, by Proposition 4. It means there exist $f_n \in X^*$, $g_n \in X^*$, $f_n = g_n$ on P, $\|f_n\|_P \to 1$, $\|f_n\| \to 1$, $\|g_n\| \to 1$ and $|(f_n - g_n)(x)| \geqq \varepsilon > 0$ for some $x \in S_1 \subset X$. Take subnets f_{n_ν}, g_{n_ν} of f_n and g_n respectively such that $f_{n_\nu} \xrightarrow{w^*} f_0 \in K_1^*$, $g_{n_\nu} \xrightarrow{w^*} g_0 \in K_1^*$. Then the restrictions f_{n_ν}/P and g_{n_ν}/P have the property that $\sup_{x \in S_1 \cap P} |(f_{n_\nu} - f_0)(x)| \underset{\nu}{\to} 0$ and analogously for g_{n_ν} and g_0. Thus because of $\|f_n\|_P \to 1$, $\|f_0\|_P = 1$, $\liminf_\nu \|f_{n_\nu}\| \geqq \|f_0\| \geqq \|f_0\|_P$, $\liminf \|f_{n_\nu}\| = 1$ (for $\|f_n\| \to 1$), we have $\|f_0\| = 1$. Analogously for g_0. Since $|(f_{n_\nu} - g_{n_\nu})(x)| \geqq \varepsilon > 0$, we have $f_0 \neq g_0$. Therefore P does not have the unique extension property and thus $P^\perp$ does not have the Haar property ([4]).

Proposition 7. *If $P \subset X$ is a linear subspace of a Banach space X then X is uniformly rotund along P (weakly uniformly rotund along P) iff the following conditions are satisfied respectively: whenever $\{x_n\}$ is a bounded sequence in X, $\{P_n\}$ a sequence of one dimensional subspaces of P, $y_n, z_n \in P_n$ for $n \in N$ and $\varrho(x_n, P_n) - \varrho(x_n, y_n) \to 0$, $\varrho(x_n, P_n) - \varrho(x_n, z_n) \to 0$ then $y_n - z_n \to 0$ ($y_n - z_n \xrightarrow{w} 0$).*

Proof. Suppose P does not have the property of our assertion in the weak sense. Then there exist a bounded sequence $\{x_n\}$, one dimensional subspaces $P_n \subset P$ and $y_n, z_n \in P_n$ such that $\varrho(x_n, P_n) - \varrho(x_n, y_n) \to 0$, $\varrho(x_n, P_n) - \varrho(x_n, z_n) \to 0$, but $|f(y_n - z_n)| \geqq \varepsilon > 0$ for some $f \in S_1^* \subset X^*$ and $\varepsilon > 0$. If for some subsequence n_k, $\varrho(x_{n_k}, P_{n_k}) \to 0$, then our assumptions imply $\varrho(y_{n_k}, z_{n_k}) \to 0$, a contradiction. Therefore since $\varrho(x_n, P_n) \leqq \|x_n\|$, we may suppose $\varrho(x_n, P_n) \to k \neq 0$. Denote $s_n = x_n - y_n$, $t_n = x_n - z_n$. Then $\|s_n\| \to k$, $\|t_n\| \to k$, $\frac{1}{2}\|s_n\| + \frac{1}{2}\|t_n\| \geqq \|\frac{1}{2}(s_n + t_n)\| = \|x_n - \frac{1}{2}(y_n + z_n)\| \geqq \varrho(x_n, P_n)$. Both sides of this inequalities converge to k and therefore since $s_n - t_n = z_n - y_n \in P_n \subset P$, we have that X would not be weakly uniformly rotund along P for we have simultaneously $\|s_n\| \to k$, $\|t_n\| \to k$, $\|\frac{1}{2}(s_n + t_n)\| \to k$.

On the other hand, suppose X is not weakly uniformly rotund along P. Then there exist $\{x_n\}, \{y_n\} \subset S_1 \subset X$, $x_n - y_n \in P$ such that $\inf_{t \in \langle 0,1 \rangle} \|tx_n + (1 - t) y_n\| = 1 - \varepsilon_n \to 1$ and $|f(x_n - y_n)| \geqq \varepsilon > 0$ for some $f \in S_1^* \subset X^*$ and $\varepsilon > 0$. Consider $y_n = \frac{1}{2}(x_n + y_n) - x_n$, $q_n = \frac{1}{2}(x_n + y_n) - y_n$, $r_n = \frac{1}{2}(x_n + y_n)$, P_n the one dimensional subspace of P generated by $x_n - y_n$, $n \in N$. Then $\|r_n - p_n\| = \|x_n\| = \|y_n\| = \|r_n - q_n\| = 1$, and $\varrho(r_n, P_n) \leqq \|r_n\| \leqq 1$. For a fixed $n \in N$ take an arbitrary element $z \in P_n$. Then there exists $\alpha_n \in R$ such that $z = \alpha_n p_n + (1 - \alpha_n) q_n$. Then $z - r_n = -(\alpha_n x_n + (1 - \alpha_n) y_n)$. It is evident that if $\alpha_n \notin \langle 0, 1 \rangle$, then $\|\alpha_n x_n + (1 - \alpha_n) y_n\| \geqq 1$ and for $\alpha_n \in \langle 0, 1 \rangle$, $\|\alpha_n x_n + (1 - \alpha_n) y_n\| \geqq 1 - \varepsilon_n$. Therefore for each $z \in P_n$, $\|z - r_n\| \geqq 1 - \varepsilon_n \to 1$. Thus $1 \geqq \varrho(r_n, P_n) \geqq 1 - \varepsilon_n$. Hence we have obtained $\varrho(r_n, p_n) - \varrho(r_n, P_n) \to 0$, $\varrho(r_n, q_n) - \varrho(r_n, P_n) \to 0$, $p_n, q_n \in P_n$, $|f(p_n - q_n)| \geqq \varepsilon > 0$, $\|r_n\| \leqq 1$. Therefore then X would not satisfy the conditions of our Proposition in the weak sense.

Proposition 8. *Let $P \subset X$ be a closed linear subspace of a Banach space X. Then X^* is uniformly rotund along $P^\perp$ (weakly* uniformly rotund along $P^\perp$) iff the following conditions are satisfied respectively: if $f_n, g_n \in X^*$, $X \supset P_n \supset P$,* P_n closed, codim $P_n = 1$, *$h_n = f_n - g_n \in P_n^\perp$ for $n \in N$ are such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $\|f_n\|_{P_n} \to 1$, then $f_n - g_n \to 0$ ($f_n - g_n \overset{w^*}{\to} 0$).*

Proof. Suppose P does not have the property of our assertion for the weak* case. Then there exist $f_n, g_n \in X^*$, $P_n^\perp$ one dimensional subspaces of $P^\perp$ such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $h_n = f_n - g_n \in P_n^\perp$, $\|f_n\|_{P_n} \to 1$, and such that $|(f_n - g_n)(x)| \geqq \varepsilon > 0$ for some $x \in S_1 \subset X$ and $\varepsilon > 0$. Then again $\varrho(f_n, P_n^\perp) = \|f_n\|_{P_n} \to 1$, $\varrho(g_n, P_n^\perp) = \|g_n\|_{P_n} \to 1$. Thus $\varrho(f_n, P_n^\perp) - \varrho(f_n, 0) \to 0$, $\varrho(f_n, P_n^\perp) - \varrho(f_n, h_n) \to 0$, $\{f_n\}$ bounded and $|h_n(x)| \geqq \varepsilon > 0$. Therefore by Proposition 7, X^* is not weakly* uniformly rotund along $P^\perp$.

On the other hand, if X^* is not weakly* uniformly rotund along $P^\perp$, by Proposition 7 there exist $\{f_n\} \subset X^*$, $\{f_n\}$ bounded, one dimensional subspaces $P_n^\perp$ of $P^\perp$ and $g_n, h_n \in P_n^\perp$ such that $\varrho(f_n, P_n^\perp) - \varrho(f_n, g_n) \to 0$, $\varrho(f_n, P_n^\perp) - \varrho(f_n, h_n) \to 0$ and $|(g_n - h_n)(x)| \geqq \varepsilon > 0$ for some $x \in S_1 \subset X$ and $\varepsilon > 0$. Suppose again without any loss of generality $\varrho(f_n, P_n^\perp) \to k \neq 0$ ($\varrho(f_n, P_n^\perp) \leqq \|f_n\|$). Then denoting $F_n = f_n - g_n$, $G_n = f_n - h_n$, we have $\|F_n\| \to k$ $\|G_n\| \to k$, $\|F_n\|_{P_n} = \|G_n\|_{P_n} = \|f_n\|_{P_n} = \varrho(f_n, P_n^\perp) \to k$, $|(F_n - G_n)(x)| = |(g_n - h_n)(x)| \geqq \varepsilon > 0$. Therefore the conditions of our statement are not satisfied, since we have simultaneously $P_n \supset P$ (we take P_n closed).

Proposition 9. *A Banach space X has a uniformly Fréchet (uniformly Gâteaux) differentiable norm iff the following conditions are satisfied respectively: whenever P_n are one dimensional subspaces of X and $f_n, g_n \in X^*$ such that $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - g_n \in P_n^\perp$, $\|f_n\|_{P_n} \to 1$, then $f_n - g_n \to 0$ ($f_n - g_n \overset{w^*}{\to} 0$).*

Proof. The property of X to have a uniformly Gâteaux differentiable norm is, by Šmuljan's theorem ([5]), equivalent to the following: whenever $f_n, g_n \in S_1^*$, $z_n \in S_1$ such that $f_n(z_n) \to 1$, $g_n(z_n) \to 1$, then $f_n - g_n \overset{w^*}{\to} 0$ in X^*. Suppose now the property of our assertion is satisfied. Take $f_n, g_n \in S_1^*$, $z_n \in S_1$ such that $f_n(z_n) \to 1$, $g_n(z_n) \to 1$. For $n \in N$ denote P_n the one dimensional subspace of X generated by z_n. Then $1 \geqq \|f_n\|_{P_n} \geqq f_n(z_n) \to 1$, $1 \geqq \|g_n\|_{P_n} \geqq g_n(z_n) \to 1$; denoting $f_n' = f_n/f_n(z_n)$, $g_n' = g_n/g_n(z_n)$, we have $f_n' - g_n' \in P_n^\perp$ and $\|f_n'\| \to 1$, $\|g_n'\| \to 1$, $\|f_n'\|_{P_n} \to 1$. Therefore by our property $f_n' - g_n' \overset{w^*}{\to} 0$ and thus $f_n - g_n \overset{w^*}{\to} 0$.

On the other hand, let X has a uniformly Gâteaux differentiable norm. Suppose $f_n, g_n \in X^*$, P_n one dimensional subspaces of X, $\|f_n\| \to 1$, $\|g_n\| \to 1$, $f_n - g_n \in P_n^\perp$, $\|f_n\|_{P_n} \to 1$. For $n \in N$ take $P_n \ni z_n \in S_1$ such that $f_n(z_n) = \|f_n\|_{P_n}$. Then evidently $g_n(z_n) = \|g_n\|_{P_n}$. Take $f_n' = f_n/\|f_n\|$, $g_n' = g_n/\|g_n\|$. Then $\|f_n'\| = \|g_n'\| = 1$, $f_n'(z_n) = \|f_n\|_{P_n}/\|f_n\|$, $g_n'(z_n) = \|g_n\|_{P_n}/\|g_n\|$. Furthermore, $\|f_n\|_{P_n}/\|f_n\| \to 1$ since $\|f_n\|_{P_n} \to 1$