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ON A CLASS OF GENERALIZED JACOBI'S ORTHONORMAL POLYNOMIALS

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1. INTRODUCTION

1,1. Denote by I the closed interval $[-1, 1]$. Let $\alpha > -1$, $\beta > -1$ and let $u(x)$ be a real function integrable and bounded on I. (The integrals in this paper are those of Lebesgue.)

Put

$$J(x) = (1 - x)^{\alpha} (1 + x)^{\beta} \tag{1,1a}$$

and

$$Q(x) = J(x)\, e^{u(x)}\,. \tag{1,1b}$$

Let for $n = 0, 1, 2, \ldots$

$$Q_n(x) = \sum_{k=0}^{n} a_k^{(n)} x^{n-k} \tag{1,1c}$$

with

$$a_0^{(n)} > 0 \tag{1,1d}$$

be the orthonoımal polynomial associated with the function $Q(x)$ on the interval I, i.e.

$$\int_I Q_m(x)\, Q_n(x)\, Q(x)\, \mathrm{d}x = \delta_{m,n}\,. \tag{1,1e}$$

Here

$$m \neq n \Rightarrow \delta_{m,n} = 0\,, \quad \delta_{m,m} = 1\,.$$

The function $Q(x)$ is called the weight of the polynomials $Q_n(x)$.

By a well-known theorem there exists for every n one and only one polynomial $Q_n(x)$ satisfying (1,1e) and (1,1d). (See [5] p. 66.)

1,2. Let

$$(1,2a) \qquad J_n(x) = \sum_{k=0}^{n} b_k^{(n)} x^{n-k}$$

where $b_0^{(n)} > 0$ be the orthonormal polynomial associated with the function $J(x)$ defined by (1,1a) on the interval I.

$J_n(x)$ is a special case of the polynomial $Q_n(x)$ for $u(x) \equiv 0$. $J_n(x)$ is the normalized Jacobi's polynomial. (See [1] p. 42.)

Therefore the polynomials $Q_n(x)$ represent a generalization of the Jacobi's polynomials.

1,3. This paper is the first part of a treatise dealing with the above defined generalized Jacobi's polynomials $Q_n(x)$.

The main object of this paper is to establish the differential equation (4,2b) for a certain class of the polynomials $Q_n(x)$. This equation is a very useful tool for solving many problems connected with the polynomials in question.

J. Korous has derived a differential equation for a more general class of polynomials. His equation, however, is non-homogeneous. (See [2], [3], [4].)

For a class of the polynomials $Q_n(x)$ we derive the inequality (2,9b) which is an extension of the well-known inequality for Jacobi's polynomials.

We shall also establish a relation between $Q_n(x)$ and $Q_n'(x)$. (See Section 3,4.)

2. SOME PROPERTIES OF THE POLYNOMIALS $Q_n(x)$

2,1. Throughout this paper the following notation is used:

1. $n \geqq 0$ is an integer.
2. $\{P\}$ is the degree of the polynomial $P(x)$.
   $\{P\} = -\infty$, if $P(x) \equiv 0$,
   $P(x) = \pi_n$, if $\{P\} \leqq n$.
3. I is the closed interval $[-1, 1]$.
4. c_i $(i = 1, 2, \ldots)$ are positive constants independent of $x \in I$ and of n.
   $c_i(x)$ $(i = 1, 2, \ldots)$ is a function of $x \in I$ and n such that $|c_i(x)| < c_i$.

The numbering of c_i a $c_i(x)$ is independent for every section.

2,2. If $P(x) = \sum_{k=0}^{r} a_k x^k$, $a_r \neq 0$, then by a well-known theorem

$$(2,2a) \qquad P(x) = \sum_{k=0}^{r} p_k Q_k(x),$$

where

$$(2,2b) \qquad p_k = \int_I P(x) Q_k(x) Q(x) \, dx.$$

(See [5] p. 73.)

Hence

(2,2c) $$n > r \Rightarrow \int_I P(x)\, Q_n(x)\, Q(x)\, dx = 0$$

and

(2,2d) $$\int_I P(x)\, Q_r(x)\, Q(x)\, dx = \frac{a_r}{a_0^{(r)}},$$

where $a_0^{(r)} > 0$ is defined by (1,1c).

2,3. Let

(2,3a) $$q_0 = 0\,, \quad n > 0 \Rightarrow q_n = \frac{a_0^{(n-1)}}{a_0^{(n)}}\,.$$

By (2,2d) for $n > 0$

(2,3b) $$q_n = \int_I x\, Q_n(x)\, Q_{n-1}(x)\, Q(x)\, dx$$

and

(2,3c) $$nq_n^{-1} = \int_I Q_n'(x)\, Q_{n-1}(x)\, Q(x)\, dx\,.$$

Hence for $n = 1, 2, \ldots$

(2,3d) $$0 < q_n < 1\,.$$

Proof. From (2,3b) we see at once

$$q_n < \int_I |Q_n(x)\, Q_{n-1}(x)|\, Q(x)\, dx \Rightarrow q_n^2 < \int_I Q_n^2(x)\, Q(x)\, dx \int_I Q_{n-1}^2(x)\, Q(x)\, dx = 1\,.$$

2,4. The equation

(2,4a) $$q_{n+1}\, Q_{n+1}(x) + (j_n - x)\, Q_n(x) + q_n\, Q_{n-1}(x) = 0\,,$$

where

(2,4b) $$j_n = \int_I x\, Q_n^2(x)\, Q(x)\, dx \Rightarrow |j_n| < 1$$

is the well-known recurrence formula for orthonormal polynomials. (See [5] p. 77.)

2,5. For $x \neq t$

(2,5a) $$Q_n(x, t) = \sum_{k=0}^{n} Q_k(x)\, Q_k(t) =$$
$$= (x - t)^{-1}\, q_{n+1}[Q_{n+1}(x)\, Q_n(t) - Q_n(x)\, Q_{n+1}(t)]\,.$$

Similarly for the polynomials $J_n(x)$ defined by (1,2a)

(2,5b) $$J_n(x, t) = \sum_{k=0}^{n} J_k(x)\, J_k(t) = (x - t)^{-1}\, q^*_{n+1}[J_{n+1}(x)\, J_n(t) - J_n(x)\, J_{n+1}(t)]\,,$$

holds where $x \neq t$ and

(2,5c) $$q^*_{n+1} = \frac{b_0^{(n)}}{b_0^{(n+1)}}\,.$$

(2,5a) is the Christoffel's formula. (See [5] p. 79.)

Applying (2,2b) we can write the formula (2,2a) in the form

(2,5d) $$P(x) = \int_I P(t)\, Q_r(x, t)\, Q(t)\, \mathrm{d}t$$

or

(2,5e) $$P(x) = \int_I P(t)\, J_r(x, t)\, J(t)\, \mathrm{d}t\,.$$

2,6. We introduce the following sets of functions:

Let $f_x(t)$ be a real function of t which depends on the parameter $x \in I$ and is defined for all $t \in [-1, 1]$ with the possible exception $t = x$. The functions $f_x(t)$ exist for every value of $x \in I$. Put

(2,6a) $$\gamma = \min(\alpha, \beta)\,.$$

$\mathfrak{F}_\gamma$ denotes the set of the functions $f_x(t)$ such that for $\gamma \geqq -\frac{1}{2}$

(2,6b) $$f_x(t) \in \mathfrak{F}_\gamma \Leftrightarrow \int_I (1 - t^2)^{-1/2}\, |f_x(t)|\, \mathrm{d}t = c_1(x)\,.$$

Here

(2,6c) $$\int_I (1 - t^2)^{-1/2}\, |f_x(t)|\, \mathrm{d}t = \lim_{y \to x-} \int_{-1}^{y} (1 - t^2)^{-1/2}\, |f_x(t)|\, \mathrm{d}t + \lim_{y \to x+} \int_{y}^{1} (1 - t^2)^{-1/2}\, |f_x(t)|\, \mathrm{d}t\,.$$

The integrals in (2,6c) are those of Lebesgue.

If $\gamma < -\frac{1}{2}$, then $f_x(t) \in \mathfrak{F}_\gamma$ if and only if there exists a constant $c > 0$ independent of $x \in I$ and $t \in I$ such that

(2,6d) $$|f_x(t)| < c\,.$$

The inequality (2,6d) implies

(2,6e) $$\overline{\lim_{t\to x}}\, |f_x(t)| \leqq c$$

for every $x \in I$.

Remark. It is easily seen that $\varphi(t) \in \mathfrak{F}_\gamma$ for $\gamma \geqq -\frac{1}{2}$, if $\int_I (1 - t^2)^{-1/2} |\varphi(t)| \, dt < < +\infty$ for we may write $f_x(t) = \varphi(t)$ for every $x \in I$.

Similarly $\varphi(t) \in \mathfrak{F}_\gamma$ for $\gamma < -\frac{1}{2}$, if $\varphi(t)$ is bounded on I.

Clearly, if $\gamma_1 < -\frac{1}{2}$ and $\gamma_2 \geqq -\frac{1}{2}$, then $\mathfrak{F}_{\gamma_1} \subset \mathfrak{F}_{\gamma_2}$.

2,7. Let $\varphi(t)$ be a real function defined on I. Then we shall use the following notation

(2,7a) $$\Delta_x \varphi(t) = (x - t)^{-1}[\varphi(x) - \varphi(t)] .$$

It is easily seen that $\Delta_x \varphi(t) \in \mathfrak{F}_\gamma \Rightarrow \varphi(t) \in \mathfrak{F}_\gamma$.

2,8. *In the notation of Sections* 1,2 *and* 2,1,

(2,8a) $$\gamma = \min(\alpha, \beta) \geqq -\tfrac{1}{2} \Rightarrow \sqrt[4]{((1 - x^2)\, J^2(x))}\, J_n(x) = c_1(x) .$$

Proof. See e.g. [2] p. 9. In this paper (2,8a) is proved for $\gamma > \frac{1}{2}$ but a slight modification of the proof establishes (2,8a) also for $\gamma = -\frac{1}{2}$.

2,9. *Let in the notation of Sections* 2,7 *and* 2,6

(2,9a) $$\gamma \geqq -\tfrac{1}{2} \quad \textit{and} \quad \Delta_x u(t) \in \mathfrak{F}_\gamma .$$

Then

(2,9b) $$\sqrt[4]{((1 - x^2)\, Q^2(x))}\, Q_n(x) = c_1(x) .$$

Proof. We shall use a method of J. Korous. (See [2] p. 9.) Applying (2,5e) we deduce that

(1) $$Q_n(x) = a_n\, J_n(x) + R_n(x) .$$

Here

$$a_n = \int_I Q_n(t)\, J_n(t)\, J(t)\, dt .$$

Hence

(2) $$a_n^2 \leqq \int_I e^{-u(t)}\, Q_n^2(t)\, Q(t)\, dt \int_I J_n^2(t)\, J(t)\, dt < c_1 .$$

Since

$$J_{n-1}(x, t) = \pi_{n-1}$$

with respect to t,

(3)

$$R_n(x) = \int_I Q_n(t)\, J_{n-1}(x, t)\, J(t)\, \mathrm{d}t = \int_I Q_n(t)\, J_{n-1}(x, t)\, [J(t) - e^{-u(x)}\, Q(t)]\, \mathrm{d}t\,.$$

Applying (2,5b) we obtain for the integrated function $L_n(t, x)$ in the second integral in (3)

(4) $$|L_n(t, x)| < q_n^*|Q_n(t)|\, |x - t|^{-1}\, |1 - \exp[u(t) - u(x)]|\,. \\ .\{|J_n(x)\, J_{n-1}(t)| + |J_{n-1}(x)\, J_n(t)|\}\, J(t)\,.$$

It is easily seen that

(5) $$|1 - \exp[u(t) - u(x)]| < c_2|x - t|\, \Delta_x u(t)\,.$$

Let

(6) $$s = \sup_{x\in I} \sqrt[4]{((1 - x^2)\, J^2(x))}\, |Q_n(x)|$$

and $x_0 \in I$ a point in which the above function assumes the value s.

Further let $\delta > 0$ and

(7) $$I_0 = (x_0 - \delta,\ x_0 + \delta) \cap I\,.$$

Since $u(x)$ is bounded in I (see Section 1,1) $\Delta_{x_0} u(t)$ is bounded on the interval $I - I_0$.

Making use of (4), (5), (2,8a) and (2,3d) we deduce that

(8) $$\sqrt[4]{((1 - x_0^2)\, J^2(x_0))} \int_{I-I_0} |L_n(t, x_0)|\, \mathrm{d}t < c_3 \int_I |Q_n(t)|\, \{|J_{n-1}(t)| + \\ + |J_n(t)|\}\, J(t)\, \mathrm{d}t < c_4 \left[\int_I Q_n^2(t)\, \mathrm{e}^{-u(t)}\, Q(t)\, \mathrm{d}t\right]^{1/2} \left\{\int_I [J_{n-1}^2(t) + J_n^2(t)]\, J(t)\, \mathrm{d}t\right\}^{1/2} < c_5\,.$$

Further, (3), (4), (5), (2,8a), (2,9a) and (2,6b) yield

(9) $$\sqrt[4]{((1 - x_0^2)\, J^2(x_0))} \int_{I_0} |L_n(t, x_0)|\, \mathrm{d}t < c_6 s \int_{I_0} (1 - t^2)^{-1/2}\, |\Delta_x u(t)|\, \mathrm{d}t < \frac{s}{2}$$

if we choose δ in (7) sufficiently small.

It follows from (8) and (9) that

(10) $$\sqrt[4]{((1 - x_0^2)\, J^2(x_0))}\, |R_n(x_0)| < c_7 + \frac{s}{2}\,.$$

(1), (2), (10) and (2,8a) yield

$$s = \sqrt[4]{((1 - x_0^2)\, J^2(x_0))}\, |Q_n(x_0)| \leqq \{|a_n|\, |J_n(x_0)| + $$
$$+ |R_n(x_0)|\} \sqrt[4]{((1 - x_0^2)\, J^2(x_0))} < c_8 + \frac{s}{2}.$$

Hence

(11) $$s < c_9 .$$

As for $x \in I$

$$Q(x) = e^{u(x)}\, J(x) \leqq c_{10}\, J(x),$$

(2,9b) follows from (11).

3. LEMMAS

3,1. We shall use the following notation:

If $\varphi(t)$ is integrable on I, then for $m = 0, 1, \ldots, n = 0, 1, \ldots$

(3,1a) $$I_{m,n}[\varphi(t)] = \int_I \varphi(t)\, Q_m(t)\, Q_n(t)\, Q(t)\, \mathrm{d}t .$$

Remark. t on the left-hand side of (3,1a) indicates that the integration variable is t.

3,2. *Let in the notation of Section* 2,6

(3,2a) $$f_x(t) \in \mathfrak{F}_\gamma .$$

Then for $m \leqq n$

(3,2b) $$I_{m,n}[f_x(t)] = c_1(x) .$$

Remark 1. The integral (3,2b) is meant in the sense of (2,6e).

Remark 2. If $\varphi(t)$ does not depend on x we put

$$f_x(t) = \varphi(t),$$

so that

(3,2c) $$|I_{m,n}[\varphi(t)]| < c_1$$

provided that $\varphi(t) \in \mathfrak{F}_\gamma$.

Proof. 1. If $\gamma \geqq -\frac{1}{2}$, we may apply (2,9b). It is for $t \in (-1, 1)$

$$|Q_m(t)\, Q_n(t)\, Q(t)| < c_2(1 - t^2)^{-1/2}$$

so that

$$|I_{m,n}[f_x(t)]| < c_3 \int_I (1 - t^2)^{-1/2} |f_x(t)| \, dt < c_4$$

in virtue of (2,6b).

2. If $\gamma < -\frac{1}{2}$, we have by (2,6d) and (2,6e)

$$|I_{m,n}[f_x(t)]| < c_5 \int_I |Q_m(t) \, Q_n(t)| \, Q(t) \, dt <$$

$$< c_6 \left[\int_I Q_m^2(t) \, Q(t) \, dt \int_I Q_m^2(t) \, Q(t) \, dt \right]^{1/2} = c_6 \, .$$

3,3. *Let $\psi(t)$ be integrable in I. We put*

(3,3a) $$A(x, \psi) = q_n^{-1} \int_I \psi(t) \, Q_n(t) \, Q_{n-1}(x, t) \, Q(t) \, dt \, ,$$

where $Q_{n-1}(x, t)$ is defined by (2,5a).

Further put

(3,3b) $$\lambda_1(t) = 1 \, , \quad \lambda_2(t) = t \, , \quad \lambda_3(t) = 1 - t^2 \, ,$$
$$\psi_i(t) = \lambda_i(t) \, \varphi(t) \quad (i = 1, 2, 3) \, .$$

Let

(3,3c) $$\Delta_x \varphi(t) \in \mathfrak{F}_\gamma \, .$$

Then for $i = 1, 2, 3$

(3,3d) $$A(x, \psi_i) = \{\alpha_i(x) + I_{n,n}[\lambda_i(t) \, \Delta_x \varphi(t)]\} \, Q_{n-1}(x) +$$
$$+ \{\beta_i(x) - I_{n,n-1}[\lambda_i(t) \, \Delta_x \varphi(t)]\} \, Q_n(x) \, .$$

Here

(3,3e) $$\alpha_1(x) = \beta_1(x) = \beta_2(x) = 0 \, ,$$
$$\alpha_2(x) = \varphi(x) \, , \quad \alpha_3(x) = -(x + j_n) \, \varphi(x) \, , \quad \beta_3(x) = q_n \, \varphi(x) \, ,$$

where j_n is defined by (2,4b).

Further

(3,3f) $$A(x, \psi_i) = c_1(x) \, Q_{n-1}(x) + c_2(x) \, Q_n(x) \, .$$

Proof. 1. The existence of the integrals on the right-hand side of (3,3d) is made evident by (3,2b). Further, (3,3d) and (3,2b) verify (3,3f) provided that (3,3e) is true.

2. Since

$Q_{n-1}(x, t) = \pi_{n-1}$ with respect to the variable t we have for

$$\psi(t) = 1\,, \quad \psi^*(t) = \varphi(t) - \varphi(x)\,, \quad \psi^{**}(t) = x - t$$

the equations

$$A(x, \psi) = 0\,, \quad A(x, \varphi) = A(x, \psi^*) \tag{1}$$

and in virtue of (2,5a),

$$A(x, \psi^{**}) = -Q_{n-1}(x)\,. \tag{2}$$

3. As a consequence of (2,5a), (1) yields

$$A(\psi_1) = A(x, \varphi) = I_{n,n}[\Delta_x\varphi(t)]\, Q_{n-1}(x) - I_{n,n-1}[\Delta_x\varphi(t)]\, Q_n(x)\,. \tag{3}$$

4. Since

$$\psi_2(t) = \psi_2(x) + (t - x)\,\varphi(x) + t[\varphi(t) - \varphi(x)]$$

we deduce from (1) and (2) that (3,3e) holds also for $i = 2$.

5. It can be easily seen that

$$\psi_3(t) = \psi_3(x) + (1 - t^2)\,[\varphi(t) - \varphi(x)] + (x^2 - t^2)\,\varphi(x)\,. \tag{4}$$

Hence, if we put $\varphi_1(t) = t$, $\varphi_2(t) = x + t$ then owing to (1),

$$A(x, \psi_3) = I_{n,n}[\lambda_3(t)\,\Delta_x\varphi(t)]\, Q_{n-1}(x) -$$
$$- I_{n,n-1}[\lambda_3(t)\,\Delta_x\varphi(t)]\, Q_n(x) + [I_{n,n-1}[\varphi_1(t)]\, Q_n(x) - I_{n,n}[\varphi_2(t)]\, Q_{n-1}(x)]\,\varphi(x)\,.$$

From this equation (3,3e) follows for $i = 3$ by applying (2,3b) and (2,4b).

3,4. *Let*

$$\Delta_x u'(t) \in \mathfrak{F}_\gamma \quad \text{and} \quad \frac{\partial}{\partial x}[\Delta_x u'(t)] \in \mathfrak{F}_\gamma\,. \tag{3,4a}$$

For $\nu = 0, 1, \ldots, n$

$$\gamma_\nu = -I_{n,\nu}[\lambda_3(t)\, u'(t)]\,, \tag{3,4b}$$

where $\lambda_3(t)$ *is defined by* (3,3b).

$$[1 + e_n(x)]^{-1} = 1 + (2n)^{-1}\{\alpha + \beta + 1 + (j_n + x)\, u'(x) - \tag{3,4c}$$
$$- I_{n,n}[\lambda_3(t)\,\Delta_x u'(t)]\}\,,$$

where j_n *is defined by* (2,4b).

$$d_n(x) = \tfrac{1}{2}[1 + e_n(x)]\,\{x - j_n - (2n)^{-1}\,[(\alpha + \beta + 2)\, j_n + \tag{3,4d}$$
$$+ \alpha - \beta + \gamma_n - 2q_n^2\, u'(x) + 2q_n\, I_{n,n-1}[\lambda_3(t)\,\Delta_x u'(t)]\}\,.$$

Then

$$(3,4e)\qquad q_n Q_{n-1}(x) = (2n)^{-1}[1 + e_n(x)](1 - x^2) Q_n'(x) + d_n(x) Q_n(x),$$

$$(3,4f)\qquad e_n(x) = n^{-1} c_1(x),$$

$$(3,4g)\qquad d_n(x) = c_2(x),$$

$$(3,4h)\qquad e_n'(x) = n^{-1} c_3(x).$$

Proof. 1. The existence of integrals on the right-hand sides of (3,4b), (3,4c) and (3,4d) as well as the existence of

$$\frac{\mathrm{d}}{\mathrm{d}x} I_{n,n}[\lambda_3(t)\,\Delta_x u'(t)]$$

is a consequence of (3,4a) and (3,2b). Since by (3,4a) $u''(x) \in \mathfrak{F}_\gamma$, $e_n'(x)$ exists in the interval I.

(3,4f), (3,4g) and (3,4h) follow then from (3,2b).

2. It is easily seen that

$$U_n(x) = (1 - x^2) Q_n'(x) + nx\, Q_n(x) = \pi_n .$$

Hence by (2,2a)

$$(1)\qquad U_n(x) = \sum_{\nu=0}^{n} \alpha_\nu Q_\nu(x),$$

where by (2,2b)

$$(2)\qquad \alpha_\nu = \int_I [(1 - t^2) Q_n'(t) + nt\, Q_n(t)]\, Q_\nu(t)\, Q(t)\,\mathrm{d}t .$$

Integrating by parts we obtain

$$(3)\qquad \alpha_\nu = -\int_I (1 - t^2) Q_n(t)\, Q_\nu'(t)\, Q(t)\,\mathrm{d}t + I_{n\nu}[\psi_4(t)],$$

where

$$\psi_4(t) = (\alpha - \beta) + (\alpha + \beta + 2 + n)\,t - (1 - t^2)\,u'(t).$$

3. (3) enables us to establish the following results:

$$(4)\qquad \nu < n - 1 \Rightarrow \alpha_\nu = \gamma_\nu ,$$

where γ_ν is defined by (3,4b).

Adding (2) to (3) for $\nu = n$ we obtain

$$(5)\qquad \alpha_n = \tfrac{1}{2}[(\alpha + \beta + 2n + 2)\,j_n + \alpha - \beta + \gamma_n].$$

Since

$$x^2 Q'_{n-1}(x) = (n-1)\,x\,Q_{n-1}(x) + \pi_{n-1}\,,$$

(3) yields

(6) $$\alpha_{n-1} = (2n + \alpha + \beta + 1)\,q_n + \gamma_{n-1}\,.$$

4. By means of (2,5d) we obtain

(7) $$(1 - x^2)\,Q'_n(x) = \alpha_{n-1}\,Q_{n-1}(x) + (\alpha_n - nx)\,Q_n(x) - q_n A[x, \lambda_3 u']\,.$$

(3,4e) is a consequence of (7), (5), (6) and (3,3d).

3,5. *Provided that* (3,4a) *holds,*

(3,5a) $$\left|\int_I (1 - x^2)^2\, Q'^2_n(x)\, Q(x)\,\mathrm{d}x\right| < c_1 n^2\,.$$

Proof. (3,5a) is a consequence of (3,4e).

3,6. *The following equation holds*:

(3,6a) $$K_n = q_n \int_I (x + t)\, Q'_n(t)\, Q_{n-1}(t)\, Q(t)\,\mathrm{d}t = nx + s_1^{(n)}\,.$$

Here $s_1^{(n)}$ *is the sum of the zeros of the polynomial* $Q_n(x)$. *Since all these zeros are contained in the interval* $(-1, 1)$, *it is*

(3,6b) $$|s_1^{(n)}| < n\,.$$

Proof. 1.

(1) $$\begin{aligned} x^2 Q'_n(x) &= nx[a_0^{(n)}x^n + a_1^{(n)}x^{n-1}] - \\ &- a_1^{(n)}x^n + \pi_{n-1} = nx[Q_n(x) + \pi_{n-2}] + \\ &+ s_1^{(n)}[a_0^{(n)}x^n + \pi_{n-1}] = nx\,Q_n(x) + \pi_{n-1} + \\ &+ s_1^{(n)}[Q_n(x) + \pi_{n-1}] = [nx + s_1^{(n)}]\,Q_n(x) + \pi_{n-1}\,. \end{aligned}$$

2. Making use of (2,4a) and (1) we deduce that

$$\begin{aligned} K_n &= \int_I (x + t)\, Q'_n(t)\,[(t - j_n)\,Q_n(t) - q_{n+1}\,Q_{n+1}(t)]\,Q(t)\,\mathrm{d}t = \\ &= nx + \int_I \{[nt + s_1^{(n)}]\,Q_n(t) - j_n t\,Q'_n(t)\}\,Q_n(t)\,Q(t)\,\mathrm{d}t = \\ &= n(x + j_n) + s_1^{(n)} - nj_n = nx + s_1^{(n)}\,. \end{aligned}$$

3,7. *Let*

$$\Delta_x u'(t) \in \mathfrak{F}_\gamma . \tag{3,7a}$$

Then

$$L_n = q_n \int_I (1 - t^2)\, \Delta_x u'(t)\, Q_n'(t)\, Q_{n-1}(t)\, Q(t)\, \mathrm{d}t = n\, c_1(x) . \tag{3,7b}$$

Proof. 1. Let $\gamma \geqq -\frac{1}{2}$. Then in virtue of (3,4e), (3,4f) and (3,4g),

$$|(1 - x^2)\, Q_n'(x)| < c_1 n[|Q_{n-1}(x)| + |Q_n(x)|] .$$

Hence by (2,9b)

$$(1 - x^2)\, |Q_n'(x)|\, \sqrt{[Q(x)]} < c_2 n(1 - x^2)^{-1/4} . \tag{1}$$

From (3,2b) and (1) it follows that

$$|L_n| < c_3 n \int_I (1 - t^2)^{-1/2}\, |\Delta_x u'(t)|\, \mathrm{d}t < c_4 n .$$

2. Let $\gamma < -\frac{1}{2}$. Then in virtue of (3,5a) and of (2,6d),

$$|L_n| < c_5 \left[\int_I (1 - t^2)^2\, Q_n'^2(t)\, Q(t)\, \mathrm{d}t \int_I Q_{n-1}^2(t)\, Q(t)\, \mathrm{d}t\right]^{1/2} < c_6 n .$$

3,8. *Let*

$$\Delta_x u'(t) \in \mathfrak{F}_\gamma \quad \text{and} \quad (1 - t^2)\frac{\partial}{\partial t}\Delta_x u'(t) \in \mathfrak{F}_\gamma . \tag{3,8a}$$

For the sake of brevity, put

$$\varphi_1(t) = Q^{-1}(t)\frac{\mathrm{d}}{\mathrm{d}t}[(1 - t^2)\, \Delta_x u'(t)\, Q(t)] = \tag{3,8b}$$

$$= (1 - t^2)\frac{\partial}{\partial t}\Delta_x u'(t) - [(\alpha + \beta + 2)\, t + \alpha - \beta + (1 - t^2)\, u'(t)]\, \Delta_x u'(t) .$$

Then in the notation (3,7b)

$$B(x) = \int_I (1 - t^2)\, u'(t)\, Q_n'(t)\, Q_{n-1}(x, t)\, Q(t)\, \mathrm{d}t = (1 - x^2)\, u'(x)\, Q_n'(x) - \tag{3,8c}$$

$$- \{n\, u'(x) + \tfrac{1}{2} I_{n,n}[\varphi_1(t)]\}\, q_n\, Q_{n-1}(x) + \{[nx + s_1^{(n)}]\, u'(x) - L_n\}\, Q_n(x) .$$

Here $s_1^{(n)}$ is defined in Section 3,6.

Proof. 1. As a consequence of (3,8a) there exists $u''(x)$ in the interval I. Therefore $u'(x)$ is continuous on I and consequently

$$(1) \qquad u'(t)\,\Delta_x u'(t) \in \mathfrak{F}_\gamma\,.$$

By (1) combined with (3,2b) the existence of $I_{n,n}[\varphi_1(t)]$ is made evident.

2. Clearly

$$(2) \qquad (1 - t^2)\,u'(t) = (1 - x^2)\,u'(x) + (x^2 - t^2)\,u'(x) + \\ + (1 - t^2)\,[u'(t) - u'(x)]\,.$$

Making use of (2,5a), (3,6a) and (3,7b) we may write

$$B(x) = (1 - x^2)\,u'(x)\,Q_n'(x) + [K_n\,u'(x) - L_n]\,Q_n(x) - \\ - q_n\,Q_{n-1}(x)\int_I [t\,u'(x) - (1 - t^2)\,\Delta_x u'(t)]\,Q_n(t)\,Q_n'(t)\,Q(t)\,\mathrm{d}t\,.$$

Integrating by parts, we deduce that the last integral is equal to

$$(3) \qquad nx\,u'(x) + \tfrac{1}{2}I_{n,n}[\varphi_1(t)]\,.$$

(3) and (3,6a) complete the proof.

4. THE DIFFERENTIAL EQUATION OF THE POLYNOMIAL $Q_n(x)$

4,1. 1. *If $\Delta_x u'(t) \in \mathfrak{F}_\gamma$ and $u''(x)$ exists in the interval $[-1, 1]$, then*

$$(4,1a) \qquad \Delta_x u'^2(t) \in \mathfrak{F}_\gamma\,.$$

2. *If $(\partial/\partial x)\,[\Delta_x u'(t)] \in \mathfrak{F}_\gamma$ and $u'''(x)$ exists in the interval $[-1, 1]$, then*

$$(4,1b) \qquad \frac{\partial}{\partial x}\,[\Delta_x u'^2(t)] \in \mathfrak{F}_\gamma\,.$$

Proof. 1. Clearly

$$\Delta_x u'^2(t) = [u'(t) + u'(x)]\,\Delta_x u'(t) \in \mathfrak{F}_\gamma$$

as $u'(t)$ is continuous on I in virtue of the existence of $u''(x)$.

2. It is easily seen that

$$(1) \qquad \frac{\partial}{\partial x}\,[\Delta_x u'^2(t)] = u''(x)\,\Delta_x u'(t) + [u'(t) + u'(x)]\,\frac{\partial}{\partial x}\,[\Delta_x u'(t)]\,.$$

There exists ξ between x and t such that

$$\Delta_x u'(t) = u''(\xi).$$

In virtue of the existence of $u'''(x)$, the function $u''(x)$ is continuous on I and consequently

(2) $$|\Delta_x u'(t)| < c_1 \Rightarrow \Delta_x u'(t) \in \mathfrak{F}_\gamma .$$

From (1) and (2) it follows that (4,1b) is true.

4,2. *Let*

(4,2a) $$(1 - t^2)\,\Delta_x u''(t),\ (1 - t^2)\frac{\partial}{\partial t}\Delta_x u'(t) \quad and \quad \Delta_x u'(t)$$

be elements of $\mathfrak{F}_\gamma$.

Then

(4,2b) $$Q^{-1}(x)\frac{d}{dx}[(1 - x^2)\,Q_n'(x)\,Q(x)] + (1 - x^2)\,b_n(x)\,Q_n'(x) + \\ + [\lambda_n^2 + a_n(x)]\,Q_n(x) = 0 .$$

Here

(4,2c) $$\lambda_n = \sqrt{[n(n + \alpha + \beta + 1)]} ,$$

(4,2d) $$a_n(x) = n\,c_1(x) ,$$

(4,2e) $$b_n(x) = n^{-1}\,c_2(x) .$$

If (4,2a) *is true and, moreover, the functions*

(4,2f) $$(1 - t^2)\frac{\partial}{\partial x}[\Delta_x u''(t)] ,\ (1 - t^2)\frac{\partial^2}{\partial x\,\partial t}[\Delta_x u'(t)] ,\ \frac{\partial}{\partial x}\Delta_x u'(t)$$

are elements of $\mathfrak{F}_\gamma$, *then* $b_n'(x)$ *exists in the interval* $[-1, 1]$ *and*

(4,2g) $$b_n'(x) = n^{-1}\,c_3(x) .$$

Proof. 1. It is easily seen that

$$D_n(x) = Q^{-1}(x)\frac{\mathrm{d}}{\mathrm{d}x}[(1 - x^2)\,Q_n'(x)\,Q(x)] - (1 - x^2)\,u'(x)\,Q_n'(x) + \lambda_n^2\,Q_n(x) = \pi_{n-1} .$$

By (2,2a)

(1) $$D_n(x) = \sum_{\nu=0}^{n-1}\beta_\nu\,Q_\nu(x) .$$

Making use of (2,2b) and integrating by parts we obtain

$$\beta_\nu + \int_I (1 - x^2)\, u'(x)\, Q_n'(x)\, Q_\nu(x)\, Q(x)\, \mathrm{d}x =$$

$$= - \int_I (1 - x^2)\, Q_n'(x)\, Q_\nu'(x)\, Q(x)\, \mathrm{d}x = \int_I Q_n(x)\, d[(1 - x^2)\, Q_\nu'(x)\, Q(x)] =$$

$$= \int_I (1 - x^2)\, u'(x)\, Q_n(x)\, Q_\nu'(x)\, Q(x)\, \mathrm{d}x\,.$$

Hence

$$\beta_\nu = \int_I (1 - x^2)\, u'(x)\, Q_n^2(x)\, Q(x)\, \mathrm{d}\left[\frac{Q_\nu(x)}{Q_n(x)}\right] = \tag{2}$$

$$= -2 \int_I (1 - x^2)\, u'(x)\, Q_n'(x)\, Q_\nu(x)\, Q(x)\, \mathrm{d}x - \int_I Q_n(x)\, Q_\nu(x)\, \mathrm{d}[(1 - x^2)\, u'(x)\, Q(x)]\,.$$

For the sake of brevity, put

$$\psi_0(t) = Q^{-1}(t) \frac{\mathrm{d}}{\mathrm{d}t} [(1 - t^2)\, u'(t)\, Q(t)] =$$

$$= (1 - t^2)\, [u''(t) + u'^2(t)] - [(\alpha + \beta + 2)\, t + \alpha - \beta]\, u'(t)\,.$$

From (1), (2), (2,5d), (3,3a) and (3,8c) it follows that

$$D_n(x) = - q_n A(x, \psi_0) - 2\, B(x)\,. \tag{3}$$

2. For the sake of brevity, put

$$\varphi_2(t) = (1 - t^2)\, \Delta_x[u''(t) + u'^2(t)] - [(\alpha + \beta + 2)\, t + (\alpha - \beta)]\, \Delta_x u'(t) \tag{4}$$

and let $\varphi_1(t)$ be defined by (3,8b).

Making use of (3,3d) and (3,8c) we obtain from (3)

$$D_n(x) = -2(1 - x^2)\, u'(x)\, Q_n'(x) + 2n\, u'(x) q_n\, Q_{n-1}(x) + \tag{5}$$
$$+ q_n\, \varrho_1(x)\, Q_{n-1}(x) + \varrho_2(x)\, Q_n(x)\,.$$

Here

$$\varrho_1(x) = (x + j_n)\, [u''(x) + u'^2(x)] + (\alpha + \beta + 2)\, u'(x) + I_{n,n}[\varphi_1(t) - \varphi_2(t)] \tag{6}$$

and

$$\varrho_2(x) = -q_n^2[u''(x) + u'^2(x)] - 2[nx + s_1^{(n)}]\, u'(x) + q_n\, I_{n,n-1}[\varphi_2(t)] + 2L_n\,, \tag{7}$$

where $s_1^{(n)}$ is defined in Section 3,6 and L_n by (3,7a).

The existence of the integrals in (6) and (7) is guaranteed by (4,2a) combined with (4,1a).

3. Replacing $q_n\, Q_{n-1}(x)$ by the right-hand side of (3,4e) we may write (5) in the form

$$D_n(x) = -(1 - x^2)\,[u'(x) + b_n(x)]\, Q_n'(x) - a_n(x)\, Q_n(x)\,. \tag{8}$$

Here

$$b_n(x) = -\frac{1}{2n}\,[1 + e_n(x)]\,\varrho_1(x) - e_n(x)\, u'(x) \tag{9}$$

and

$$a_n(x) = -\varrho_2(x) - [2n\, u'(x) + \varrho_1(x)]\, d_n(x)\,. \tag{10}$$

4. (4,2d) and (4,2e) may be derived from (6), (7), (8), (9) and (10) by employing (3,4f) and (3,4g).

If (4,2f) is true, then (4,1b) holds and $b_n'(x)$ exists. (4,2g) is then deduced similarly as (4,2e) if we take (3,4h) into consideration.

4,3. Sufficient conditions for (4,2a) and (4,2f).

I. *Let* $\gamma \geqq -\frac{1}{2}$.

1. (4,2a) *holds, if there exists* $\varepsilon > 0$ *such that*

$$x \in I\,, \quad t \in I \Rightarrow |u''(x) - u''(t)| < c_1 |x - t|^{\varepsilon}\,. \tag{4,3a}$$

2. (4,2f) *holds, if*

$$x \in I\,, \quad t \in I \Rightarrow |u'''(x) - u'''(t)| < c_2 |x - t|^{\varepsilon}\,. \tag{4,3b}$$

II. *If* $\gamma < -\frac{1}{2}$, *the assertion is true if we put in* (4,3a) *and* (4,3b) $\varepsilon = 1$.

Proof. I. 1. Let (4,3a) be satisfied. Then $u''(x)$ is continuous on the interval I and consequently

$$|\Delta_x u'(t)| \leqq \sup_{x \in I} |u''(x)| \Rightarrow \Delta_x u'(t) \in \mathfrak{F}_\gamma\,. \tag{1}$$

Further

$$|\Delta_x u''(t)| < c_3 |x - t|^{-1-\varepsilon} \Rightarrow \int_{-1}^{1} (1 - t^2)^{3/2}\, \}|\Delta_x u''(t)|\, dt <$$

$$< \frac{1}{\varepsilon}\, c_4[(x + 1)^{\varepsilon} + (1 - x)^{\varepsilon}] < c_5\,. \tag{2}$$

There exists τ between the numbers x and t such that

$$\frac{\partial}{\partial t}\Delta_x u'(t) = \frac{(t-x)\,u''(t) - [u'(t) - u'(x)]}{(t-x)^2} = (t-x)^{-1}\,[u''(t) - u''(\tau)]\,.$$

By (4,3a)

$$\left|\frac{\partial}{\partial t}\Delta_x u'(t)\right| < c_6\,|t-x|^{-1}\,|t-\tau|^{\varepsilon} < c_7\,|t-x|^{-1+\varepsilon}$$

Hence

(3) $$\int_I (1-t^2)^{3/2}\left|\frac{\partial}{\partial t}\Delta_x u'(t)\right| \mathrm{d}t < c_8\,.$$

(1), (2) and (3) show that $\Delta_x u'(t)$, $(1-t^2)\,\Delta_x u''(t)$ and $(1-t^2)\,(\partial/\partial t)\,\Delta_x u'(t)$ are elements of $\mathfrak{F}_\gamma$

2. Let (4,3b) be satisfied. Then there exists ξ_i $(i = 1, 2, \ldots)$ between x and t such that

$$\frac{\partial}{\partial x}[\Delta_x u'(t)] = \frac{(x-t)\,u''(x) - [u'(x) - u'(t)]}{(x-t)^2} = \frac{1}{2}u'''(\xi_1)\,.$$

Hence

(4) $$\left|\frac{\partial}{\partial x}\Delta_x u'(t)\right| \leqq \frac{1}{2}\sup_{x\in I}|u'''(x)|\,.$$

Further

(5) $$\left|\frac{\partial}{\partial x}[\Delta_x u''(t)]\right| = \left|\frac{u'''(x) - u'''(\xi_2)}{x-t}\right| <$$
$$< c_9|x-t|^{-1}\,|x-\xi_2|^{\varepsilon} < c_{10}|x-t|^{-1+\varepsilon}$$

and

(6) $$\left|\frac{\partial^2}{\partial x\,\partial t}[\Delta_x u'(t)]\right| = \left|\frac{u'''(\xi_4) - u'''(\xi_3)}{x-t}\right| <$$
$$< c_{11}|x-t|^{-1}\,|\xi_4 - \xi_3|^{\varepsilon} < c_{12}|x-t|^{-1+\varepsilon}\,.$$

From (4), (5) and (6) we may derive that

$$\frac{\partial}{\partial x}\Delta_x u'(t)\,,\quad (1-t^2)\frac{\partial}{\partial x}\Delta_x u''(t) \quad\text{and}\quad (1-t^2)\frac{\partial^2}{\partial x\,\partial t}\Delta_x u'(t)$$

are elements of $\mathfrak{F}_\gamma$.