Theorem 16. *Let Q be a quasigroup. Then the following conditions are equivalent:*

(i) *Q is an SA or TA-quasigroup and Q is totally symmetric.*

(ii) *There are a symmetric distributive quasigroup D and an Abelian group G whose every element has order 2, such that $Q \cong D \times G$.*

Proof. (i) implies (ii). First we shall prove that e is an endomorphism of Q. Since Q is symmetric,

$$L_a = R_a\,, \quad L_a^2 = 1\,, \quad S_{a,b} = L_b L_a L_{ab}\,, \quad e(a) = aa$$

for all $a, b \in Q$. The mapping $S_{a,b}$ is an automorphism and hence $S_{a,b}(ab) = S_{a,b}(a) \,.\, S_{a,b}(b)$. But

$$S_{a,b}(a) = b(a(ab \,.\, a)) = a\,, \quad S_{a,b}(b) = b(a(ab \,.\, b)) = b \,.\, aa\,.$$

Thus $b(a(ab \,.\, ab)) = a(b \,.\, aa)$. From this we get $ab \,.\, ab = a(b(a(b \,.\, aa))) = L_a L_b L_a L_b(aa) = V_{b,a}(aa)$. By Theorem 10, Q is an A-quasigroup. Therefore $V_{b,a}$ is an automorphism and $ab \,.\, ab = V_{b,a}(a) \,.\, V_{b,a}(a)$. However, $V_{b,a}(a) = a(b(a \,.\, ba)) = a \,.\, bb = a \,.\, e(b)$. Thus we have

$$ab \,.\, ab = e(ab) = (a \,.\, e(b))\,(a \,.\, e(b)) = e(a \,.\, e(b))\,.$$

But $(a \,.\, e(b))\,(e(a) \,.\, e(b)) = (a \,.\, e(a))\, e(b) = a \,.\, e(b)$ (Theorem 1). Hence $e(a \,.\, e(b)) = e(a) \,.\, e(b)$ and hence, $e(ab) = e(a) \,.\, e(b)$. Now Theorem 4 may be used and we get an isomorphism $Q \cong D \times G$, D being a distributive quasigroup and G an A-loop. But both D and G are totally symmetric. Let j be the unit of G. We have $aa = e(a) = j$ for all $a \in G$. Further, $a \,.\, ab = b$, $ab = ba$ for every $a, b \in G$. From this we can deduce that any two-element subset of Q is associative. By Theorem 7, G is di-associative. But every di-associative symmetric A-loop is an Abelian group (see [3]).
(ii) implies (i). This part is obvious.

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