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ON ONE CLASS OF QUASIGROUPS

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Let Q be a quasigroup. We shall denote by R_x the right translation by $x \in Q$ and by L_x the left translation. For all $a, b, c \in Q$ there exists just one element d dependent on a, b, c such that $ab . c = a . bd$. Hence we may introduce a mapping $S_{a,b}$ for all $a, b \in Q$ which has the following property:

$$(ab)\, c = a(b\, .\, S_{a,b}(c)) \quad \text{for every} \quad c \in Q\,.$$

It is evident that $S_{a,b} = L_b^{-1}L_a^{-1}L_{ab}$. Similarly we define mappings $T_{a,b} = R_a^{-1}R_b^{-1}R_{ab}$, $V_{a,b} = R_b^{-1}L_a^{-1}R_bL_a$, $X_{a,b} = R_b^{-1}L_a^{-1}R_{ab}$, $Y_{a,b} = L_a^{-1}R_b^{-1}L_{ab}$, $Z_a = R_a^{-1}L_a$. Thus we have for all $a, b, c \in Q$,

$$c(ab) = (T_{a,b}(c)\, .\, a)\, .\, b\,, \quad (ac)\, .\, b = a\, .\, (V_{a,b}(c))\, .\, b)\,, \quad ab = Z_a(b)\, .\, a\,.$$

It is easy to show that the mappings defined above are inner mappings in every loop, moreover, these mappings generate the inner mapping group. Since inner mappings of every group are automorphisms, it is of interest to consider a class of loops or quasigroups whose mappings $S_{a,b}$, $T_{a,b}$, $V_{a,b}$, $X_{a,b}$, $Y_{a,b}$, Z_a (or some of them only) are automorphisms. The theory of such loops was developed by BRUCK and PAIGE in [1] (they called these loops A-loops). Further, in his book ([2]) V. D. BELOUSOV introduced the class of all quasigroups in which $S_{a,b}$ are automorphisms. We shall call such quasigroups SA-quasigroups. Similarly we define TA-quasigroups, etc. A quasigroup will be called an STA-quasigroup if it is simultaneously an SA and TA-quasigroup and, finally, a quasigroup will be called an A-quasigroup if it is SA, TA, XA, YA, VA and ZA-quasigroup. In this paper we shall mainly investigate STA-quasigroups. In the first section we shall prove some structure theorems for STA-quasigroups. In the second section we shall generalize some theorems from [1] and in the third we shall apply the results to some classes of quasigroups.

First we shall make several arrangements concerning our notation. Let Q be a quasigroup. $G(Q)$ will be the group generated by all L_x, R_x. For $x \in Q$ we shall denote by the symbol $I(x)$ the group of all $\alpha \in G(Q)$ such that $\alpha(x) = x$. Id Q will be the set of all idempotents in Q and Aut Q will be the automorphism group of Q. For every $x \in Q$ there exist uniquely determined elements $e(x), f(x)$ such that $x \,.\, e(x) = f(x) \,.\, x = x$. By e and f we shall denote the corresponding mappings.

I

Lemma 1. *Let Q be an SA-quasigroup. Then:*

(i) $L_{e(x)} \in \text{Aut } Q$ *for all* $x \in Q$.

(ii) $\text{Id } Q = e(Q)$, $e(Q) \subseteq f(Q)$.

(iii) Id Q *is a left distributive subquasigroup in* Q.

Proof. Since Q is an SA-quasigroup, $S_{x,e(x)} \in \text{Aut } Q$ for every $x \in Q$. However, $S_{x,e(x)} = L_{e(x)}^{-1} L_x^{-1} L_{x.e(x)} = L_{e(x)}^{-1}$. Hence $L_{e(x)} \in \text{Aut } Q$. Therefore, $e(x) \,.\, (e(x) \,.\, e(x)) = (e(x) \,.\, e(x)) (e(x) \,.\, e(x))$, and hence $e(x) \in \text{Id } Q$. Thus $e(Q) = \text{Id } Q$ and $e(Q) \subseteq f(Q)$.

Now it remains to prove (iii). Let $x, y \in \text{Id } Q$. By (i) and (ii), $L_x, L_y \in \text{Aut } Q$. But $S_{x,y} \in \text{Aut } Q$. Hence $L_{xy} \in \text{Aut } Q$, and hence $xy \in \text{Id } Q$. Let further $x, xy \in \text{Id } Q$ and $y \in Q$. The mappings $L_x, L_{xy}, L_y^{-1} L_x^{-1} L_{xy}$ are automorphisms. Hence L_y is an automorphism, that is, $y \in \text{Id } Q$. Similarly if $xy, y \in \text{Id } Q$ and $x \in Q$ then $x \in \text{Id } Q$. We have proved that Id Q is a subquasigroup of Q. The left distributive law for Id Q follows from (i) and (ii).

Lemma 2. *Let Q be a TA-quasigroup. Then:*

(i) $R_{f(x)} \in \text{Aut } Q$ *for all* $x \in Q$.

(ii) $\text{Id } Q = f(Q)$, $f(Q) \subseteq e(Q)$.

(iii) Id Q *is a right distributive subquasigroup in* Q.

The proof is dual to that of Lemma 1.

Corollary. *Let Q be an idempotent quasigroup. Then Q is an SA-quasigroup (TA-quasigroup) if and only if Q is left (right) distributive.*

Theorem 1. *Let Q be an STA-quasigroup. Then:*

(i) $L_{e(x)}, R_{e(x)} \in \text{Aut } Q$ *and* $e(x) = f(x)$ *for all* $x \in Q$.

(ii) $\text{Id } Q = e(Q) = f(Q)$.

(iii) Id Q *is a distributive subquasigroup in* Q.

Proof. In view of Lemmas 1,2 we have only to prove that $e(x) = f(x)$ for all $x \in Q$. The element $e(x)$ is idempotent, hence $f(e(x)) = e(x)$. By Lemma 2, $R_{e(x)} \in \in \operatorname{Aut} Q$. Thus

$$f(x) . x = x . e(x) = (f(x) . x) . e(x) = (f(x) . e(x)) (x . e(x)) = (f(x) . e(x)) . x .$$

Therefore $f(x) = f(x) . e(x)$. But $f(x) = f(x) . f(x)$. Hence $e(x) = f(x)$ and the proof of Theorem 1 is complete.

Lemma 3. *Let Q be an SA-quasigroup and $\alpha \in \operatorname{Id} Q$ be arbitrary. By Q_α denote the set of all $x \in Q$ such that $\alpha x = x$. Then Q_α is a subquasigroup in Q; moreover, Q_α is a left loop and α is its left unit. Further, if $\beta, \gamma \in \operatorname{Id} Q$ then there is an automorphism φ of Q such that $\varphi(Q_\beta) = Q_\gamma$ and $\varphi(\beta) = \gamma$.*

Proof. First, Q_α is a subquasigroup in Q. Let $x, y \in Q_\alpha$. By Lemma 1, $\alpha . xy = \alpha x . \alpha y = xy$. Hence $xy \in Q_\alpha$. Let $u, v \in Q$ be such that $xu = vx = y$. We can write $xu = y = \alpha y = \alpha x . \alpha u = x . \alpha u$. Therefore $u = \alpha u$, $u \in Q_\alpha$. Similarly $v \in Q_\alpha$. It is obvious that Q_α is a left loop having α as a left unit. Let $\beta, \gamma \in \operatorname{Id} Q$ be arbitrary. There is $\delta \in \operatorname{Id} Q$ such that $\delta . \beta = \gamma$. If $x \in Q_\beta$ then

$$\gamma . \delta x = \delta\beta . \delta x = \delta . \beta x = \delta x .$$

Hence $\delta x \in Q_\gamma$. Again, let be $y \in Q_\gamma$. There exists $u \in Q$ such that $\delta u = y$. We can write

$$\delta u = y = \gamma y = \delta\beta . \delta u = \delta . \beta u .$$

Thus $u = \beta u$, $u \in Q_\beta$. Therefore $L_\delta(Q_\beta) = Q_\gamma$. But L_δ is an automorphism of Q. Now we can put $L_\delta = \varphi$.

Lemma 4. *Let Q be a TA-quasigroup and $\alpha \in \operatorname{Id} Q$ be arbitrary. By Q^α denote the set of all $x \in Q$ such that $x\alpha = x$. Then Q^α is a subquasigroup of Q; moreover, Q^α is a right loop and α is its right unit. Further, if $\beta, \gamma \in \operatorname{Id} Q$ then there is an automorphism φ of Q such that $\varphi(Q^\beta) = Q^\gamma$ and $\varphi(\beta) = \gamma$.*

The proof is dual to that of Lemma 3.

Theorem 2. *Let Q be an STA-quasigroup and $\alpha \in \operatorname{Id} Q$ be arbitrary. Then $Q_\alpha = Q^\alpha$ is a subloop in Q and α is its unit. Further, if $\beta, \gamma \in \operatorname{Id} Q$ then there is an automorphism φ of Q such that $\varphi(Q_\beta) = Q_\gamma$ and $\varphi(\beta) = \gamma$.*

Proof. By Lemmas 3,4 and Theorem 1.

Theorem 3. *Let Q be an STA-quasigroup. Then $\operatorname{Id} Q$ is a normal subquasigroup in Q and $Q/\operatorname{Id} Q$ is an STA-loop. Furthermore, for every $\mu \in \operatorname{Id} Q$ it is $Q_\mu \cong Q/\operatorname{Id} Q$.*

Proof. First we shall prove that $a \,.\, \mathrm{Id}\, Q = \mathrm{Id}\, Q \,.\, a$ for every $a \in Q$. Let $\alpha \in \mathrm{Id}\, Q$ be arbitrary. There is $\beta \in \mathrm{Id}\, Q$ such that $\beta \,.\, e(a) = \alpha$ (since $e(a) \in \mathrm{Id}\, Q$ and $\mathrm{Id}\, Q$ is a subquasigroup). We obtain

$$\beta a = \beta(a \,.\, e(a)) = (\beta a)(\beta \,.\, e(a)) = \beta a \,.\, \alpha = \gamma \,.\, a\alpha \,,$$

where $\gamma = T_{a,\alpha}^{-1}(\beta)$. Since $\mathrm{Id}\, Q$ is a characteristic subquasigroup in Q and $T_{a,\alpha}$ is an automorphism, $\gamma \in \mathrm{Id}\, Q$. There is $\delta \in \mathrm{Id}\, Q$ such that $\delta\gamma = e(a\alpha)$. Hence

$$a\alpha = e(a\alpha) \,.\, a\alpha = \delta\gamma \,.\, a\alpha = \varepsilon\,(\gamma \,.\, a\alpha)\,,$$

where $\varepsilon = T_{\gamma,a\alpha}^{-1}(\delta)$, $\varepsilon \in \mathrm{Id}\, Q$. Thus we have

$$a\alpha = \varepsilon(\gamma \,.\, a\alpha) = \varepsilon \,.\, \beta a = \varphi a\,,$$

for some $\varphi \in \mathrm{Id}\, Q$. Therefore $a \,.\, \mathrm{Id}\, Q \subseteq \mathrm{Id}\, Q \,.\, a$. Similarly $\mathrm{Id}\, Q \,.\, a \subseteq a \,.\, \mathrm{Id}\, Q$, and hence, $a \,.\, \mathrm{Id}\, Q = \mathrm{Id}\, Q \,.\, a$. Now we shall construct a homomorphism σ of Q onto Q_μ, where $\mu \in \mathrm{Id}\, Q$ is fixed but arbitrary. Let $a \in Q$. Then there is an element $\tau(a)$ in $\mathrm{Id}\, Q$ such that $e(a) \,.\, \tau(a) = \mu$. Put $\sigma(a) = a \,.\, \tau(a)$. We can write

$$(a \,.\, \tau(a)) \,.\, \mu = (a \,.\, \tau(a)) \,.\, (e(a) \,.\, \tau(a)) = (a \,.\, e(a)) \,.\, \tau(a) = a \,.\, \tau(a)\,.$$

Therefore $a \,.\, \tau(a) \in Q_\mu$. Since $a \,.\, \mathrm{Id}\, Q = \mathrm{Id}\, Q \,.\, a$, there is $\alpha \in \mathrm{Id}\, Q$ such that $a \,.\, \tau(a) = \alpha a$. Let $b \in Q$ be arbitrary. Then

$$(a \,.\, \tau(a)) \,.\, (b \,.\, \tau(b)) = (\alpha a)(b \,.\, \tau(b)) = \beta(a \,.\, b\tau(b)) = \\ = \beta(ab \,.\, \gamma) = (ab \,.\, \gamma)\,\delta = ab \,.\, \varepsilon\,,$$

where $\beta, \gamma, \delta, \varepsilon \in \mathrm{Id}\, Q$ are suitably chosen. Further we have

$$(ab \,.\, \varepsilon)\,\mu = ((a \,.\, \tau(a))\,(b \,.\, \tau(b)))\,\mu = (a\tau(a) \,.\, \mu)\,(b\tau(b) \,.\, \mu) = \\ = (a \,.\, \tau(a))\,(b \,.\, \tau(b)) = ab \,.\, \varepsilon\,.$$

But

$$(ab \,.\, \varepsilon)\,(e(ab) \,.\, \varepsilon) = (ab \,.\, e(ab))\,\varepsilon = ab \,.\, \varepsilon\,.$$

Hence $\mu = e(ab) \,.\, \varepsilon$, that is, $\tau(ab) = \varepsilon$. Therefore,

$$\sigma(a) \,.\, \sigma(b) = (a \,.\, \tau(a))\,(b \,.\, \tau(b)) = ab \,.\, \varepsilon = ab \,.\, \tau(ab) = \sigma(ab)\,.$$

Hence σ is an endomorphism of Q. It is evident that for every $x \in Q_\mu$ it is $\sigma(x) = x$. Hence σ is an epimorphism onto Q_μ. Let η be the normal congruence relation corresponding to σ. Since $e(\alpha) = \alpha$ for every $\alpha \in \mathrm{Id}\, Q$, $\sigma(\alpha) = \mu$. Conversely, if $\sigma(x) = \mu$ for any $x \in Q$ then $x \,.\, \tau(x) = \mu = e(x) \,.\, \tau(x)$. Hence $x = e(x)$, $x \in \mathrm{Id}\, Q$. Thus $\mathrm{Id}\, Q$ is one class of η. Therefore $\mathrm{Id}\, Q$ is a normal subquasigroup of Q. Further, $Q_\mu \cong Q/\eta = Q/\mathrm{Id}\, Q$ and by Theorem 2, Q_μ is an STA-loop.

Theorem 4. *Let Q be an STA-quasigroup. Define a relation ϱ as follows: For every $a, b \in Q$, $a\varrho b$ if and only if $e(a) = e(b)$. Then the following conditions are equivalent:*

(i) *There are a distributive quasigroup D and an STA-loop K such that $Q \cong \cong D \times K$.*

(ii) *The mapping e is an endomorphism of Q.*

(iii) *The relation ϱ is a normal congruence relation on Q.*

(iv) *The relation ϱ is a congruence relation on Q.*

(v) *There is $\alpha \in \mathrm{Id}\, Q$ such that Q_α is a normal subquasigroup in Q.*

(vi) *For all $\alpha \in \mathrm{Id}\, Q$, Q_α is normal in Q.*

(vii) *For every $\alpha \in \mathrm{Id}\, Q$, it is $Q \cong \mathrm{Id}\, Q \times Q_\alpha$.*

Proof. (i) implies (ii). Without loss of generality we can assume that $Q = D \times K$. Since D is idempotent and K is a loop, the mappings e are their endomorphisms. Hence it follows that e is an endomorphism of Q.

(ii) implies (iii). Evidently, ϱ is the corresponding equivalence relation to e.

(iii) implies (iv). This is obvious.

(iv) implies (ii). Let $a, b \in Q$ be arbitrary. We have $e(a) = e(e(a))$, hence $a \varrho e(a)$. Since ϱ is a congruence, $ab \varrho e(a) . e(b)$. From this, $e(ab) = e(e(a) . e(b))$. But $e(a) . e(b) \in \mathrm{Id}\, Q$, hence $e(e(a) . e(b)) = e(a) . e(b)$. Thus e is an endomorphism.

(ii) implies (vii). Let $\mu \in \mathrm{Id}\, Q$ be fixed but arbitrary. By Theorem 3 (and its proof) there is an epimorphism σ of Q onto Q_μ such that $\sigma(a) = a . \tau(a)$, where $e(a) . \tau(a) = = \mu$ for all $a \in Q$. Define φ of Q into $\mathrm{Id}\, Q \times Q_\mu$ as follows: $\varphi(a) = (e(a), \sigma(a))$ for every $a \in Q$. Since e and σ are homomorphisms, φ is a homomorphism. Let $a, b \in Q$ be such that $\varphi(a) = \varphi(b)$. Then $e(a) = e(b)$, $a . \tau(a) = b . \tau(b)$. But $e(a) . \tau(a) = = \mu = e(b) . \tau(b)$. Hence $\tau(a) = \tau(b)$ and $a = b$. Thus φ is one-to-one. Let $\alpha \in \mathrm{Id}\, Q$ and $x \in Q_\mu$ be arbitrary elements. There are $\beta \in \mathrm{Id}\, Q$, $a \in Q$ such that $\alpha\beta = \mu$ and $a\beta = x$. Therefore

$$a\beta = x = x\mu = a\beta . \alpha\beta = a\alpha . \beta .$$

Hence $a\alpha = a$, $e(a) = \alpha$. Finally, $\tau(a) = \beta$ and $\sigma(a) = x$. Thus $\varphi(a) = (\alpha, x)$ and φ is on $\mathrm{Id}\, Q \times Q_\mu$.

(iii) implies (vi) by Theorem 2.

(v) implies (iii). Let π be a normal congruence relation such that Q_α is one of its classes. We shall prove that $\varrho = \pi$. But first we shall prove that $x\pi e(x)$ for every $x \in Q$. There are $\beta \in \mathrm{Id}\, Q$, $y \in Q$ such that $\alpha\beta = e(x)$, $y\beta = x$. We can write

$$y\beta = x = x . e(x) = y\beta . e(x) = y\beta . \alpha\beta = y\alpha . \beta .$$

Hence $y = y\alpha$, $y \in Q_\alpha$. Thus $y\pi\alpha$. Since π is a congruence, $y\beta\pi\alpha\beta$. That is, $x\pi e(x)$. Let now $a, b \in Q$ be such that $a \varrho b$. Then $e(a) = e(b)$. But $a\pi e(a)$, $b\pi e(b)$. Hence $a\pi b$. Let, conversely, $a\pi b$. Then $a . e(a) \pi b . e(b)$. Since π is normal, $e(a) \pi e(b)$. Let $\gamma \in \operatorname{Id} Q$ be such that $e(a) . \gamma = \alpha$. We have $e(a) . \gamma\pi e(b) . \gamma$, which means that $\alpha\pi e(b) . \gamma$. Thus $e(b) . \gamma \in Q_\alpha$. But $e(b) . \gamma \in \operatorname{Id} Q$. Hence $\alpha = e(b) . \gamma$ and consequently, $e(a) = e(b)$. Therefore $a \varrho b$.

(vii) implies (i). It is sufficient to put $D = \operatorname{Id} Q$, $K = Q_\alpha$ for some $\alpha \in \operatorname{Id} Q$.

II

If M is a subset of a quasigroup Q, $\{M\}$ will be the subquasigroup generated by M. Further we shall say that M is commutative if $ab = ba$ for all $a, b \in M$ and we shall say that M is associative if $a . bc = ab . c$ for all $a, b, c \in M$.

Theorem 5. *Let Q be a ZA-quasigroup and M a commutative subset in Q. Then $\{M\}$ is a commutative subquasigroup in Q.*

Proof. Let $a \in Q$ be arbitrary. P_a will be the set of all $b \in Q$ such that $ab = ba$, that is, $Z_a(b) = b$. It is evident that P_a is a subquasigroup in Q. Let $a \in M$. Since $M \subseteq P_a$, $\{M\} \subseteq P_a$. Let $b \in \{M\}$. Then $b \in P_a$ for all $a \in M$ and hence $M \subseteq P_b$. Thus $\{M\} \subseteq P_b$.

Theorem 6. *Let Q be a ZA-quasigroup. Then Q is powercommutative. If $x, y \in Q$ then $xy . x = x . yx$.*

Proof. Let $x, y \in Q$. The set consisting of x only is commutative. Hence, by Theorem 5, $\{x\}$ is commutative. Further,

$$R_x^{-1}L_x(yx) = Z_x(yx) = Z_x(y) . Z_x(x) = R_x^{-1}L_x(y) . x = L_x(y) .$$

Hence $x . yx = xy . x$.

Theorem 7. *Let Q be an STA-quasigroup and M an associative subset in Q. Then $\{M\}$ is a group.*

Proof. $G(a, b)$ let be the set of all $x \in Q$ such that $S_{a,b}(x) = x$; $a, b \in M$ fixed but arbitrary. Since $S_{a,b}$ is an automorphism, $G(a, b)$ is a subquasigroup. By hypothesis, $M \subseteq G(a, b)$. Hence $\{M\} \subseteq G(a, b)$. Thus $a . bx = ab . x$ for all $x \in \{M\}$. From this we get $T_{b,x}(a) = a$.

For every $x \in \{M\}$, let $H(b, x)$ denote the set of all $y \in Q$ such that $T_{b,x}(y) = y$. Since $M \subseteq H(b, x)$ and $H(b, x)$ is a subquasigroup, $\{M\} \subseteq H(b, x)$. Thus $y . bx =$ $= yb . x$ for all $x, y \in \{M\}$ and all $b \in M$. Hence $M \subseteq N\{M\}$, where $N\{M\}$ is the middle nucleus of $\{M\}$. But, as it is well known, $N\{M\}$ is a subgroup of $\{M\}$. Hence $\{M\} = N\{M\}$ and the proof is now complete.

Similar theorems can be proved for SVA and for TVA-quasigroups.

Theorem 8. *Every A-quasigroup is power-associative.*

Proof. By Theorems 6, 7.

Theorem 9. *Let Q be an STA- and ZA-quasigroup. Then Q is an A-quasigroup.*

Proof. First, Q is an XA-quasigroup. Let $a, b \in Q$ be arbitrary. The following mappings are automorphisms:

$$\alpha = L_b^{-1} L_a^{-1} L_{ab}, \quad \beta = R_b^{-1} L_b, \quad \gamma = L_{ab}^{-1} R_{ab}.$$

Hence $\beta\alpha\gamma$ is an automorphism. However,

$$\beta\alpha\gamma = R_b^{-1} L_b L_b^{-1} L_a^{-1} L_{ab} L_{ab}^{-1} R_{ab} = R_b^{-1} L_a^{-1} R_{ab} = X_{a,b}.$$

Similarly, Q is a YA-quasigroup.

Finally, Q is a VA-quasigroup. Let $a, b \in Q$. The mappings

$$\alpha = R_b^{-1} L_a^{-1} R_{ab}, \quad \beta = R_{ab}^{-1} R_b R_a, \quad \gamma = R_a^{-1} L_a$$

are automorphisms. Hence

$$\alpha\beta\gamma = R_b^{-1} L_a^{-1} R_{ab} R_{ab}^{-1} R_b R_a R_a^{-1} L_a = R_b^{-1} L_a^{-1} R_b L_a = V_{a,b}$$

is an automorphism.

Theorem 10. *Let Q be an SA or TA-quasigroup. If Q is commutative then Q is an A-quasigroup.*

Proof. Since Q is commutative, $S_{a,b} = T_{a,b}$ and $Z_a = 1$. Now we can use Theorem 9.

III

Let Q be a quasigroup. Q is called an SF-quasigroup (TF-quasigroup) if $S_{a,b} = S_{a,c}$ ($T_{b,a} = T_{c,a}$) in Q for all $a, b, c \in Q$. In [2] it is proved that Q is an SF-quasigroup if and only if $a \,.\, bc = (ab)(e(a) \,.\, c)$ for every $a, b, c \in Q$. Similarly, Q is a TF-quasigroup if and only if $bc \,.\, a = (b \,.\, f(a))(ca)$ for all $a, b, c \in Q$. A quasigroup Q is called an F-quasigroup if it is simultaneously an SF and TF-quasigroup.

Theorem 11. *Let Q be an SF-quasigroup (TF-quasigroup). Then Q is an SA-quasigroup (TA-quasigroup) if and only if $e(x) \in \mathrm{Id}\, Q$ ($f(x) \in \mathrm{Id}\, Q$) for all $x \in Q$.*

Proof. Let Q be an SF-quasigroup. Let $e(x) \in \text{Id } Q$ for every $x \in Q$. Hence $e(e(x)) = e(x)$, and hence, $e(x) . ab = (e(x) . a)(e(e(x)) . b) = (e(x) . a)(e(x) . b)$ for all $a, b \in Q$. Thus $L_{e(x)} \in \text{Aut } Q$. Let $y \in Q$ be arbitrary. Since Q is an SF-quasigroup, $S_{x,y} = S_{x,e(x)}$. But

$$S_{x,e(x)} = L_{e(x)}^{-1} L_x^{-1} L_{x.e(x)} = L_{e(x)}^{-1}$$

is an automorphism.

Theorem 12. *Let Q be a quasigroup. Then the following conditions are equivalent:*

(i) *Q is an STA and SF-quasigroup.*
(ii) *Q is an STA and TF-quasigroup.*
(iii) *Q is an F-quasigroup and $e(x), f(x) \in \text{Id } Q$ for all $x \in Q$.*
(iv) *Q is an A and F-quasigroup.*
(v) *There are a distributive quasigroup D and a group G such that $Q \cong D \times G$.*

Proof. (i) implies (v). Q is an SF-quasigroup and hence $a . bc = (ab)(e(a) c)$. In particular $ab = a . be(b) = (ab)(e(a) . e(b))$. Thus e is an endomorphism of Q. By Theorem 4, $Q \cong D \times G$, where D is distributive and G is an STA-loop. But G is evidently an SF-loop, hence a group.

(iii) Implies (i) and (ii). By Theorem 11.

(ii) implies (v). Similar as for (i).

The other implications are evident.

A quasigroup Q is called Abelian, if $ab . cd = ac . bd$ for every $a, b, c, d \in Q$. A quasigroup is called di-Abelian (tri-Abelian), if its every subquasigroup that is generated by two (by three) elements is Abelian.

Theorem 13. *Let Q be a quasigroup. Then the following conditions are equivalent:*

(i) *Q is a di-Abelian STA-quasigroup.*
(ii) *There are a distributive quasigroup D and a commutative Moufang loop G such that $Q \cong D \times G$.*

Proof. (i) implies (ii). Since Q is di-Abelian, e is an endomorphism of Q. By Theorem 4, $Q \cong D \times G$ where D is a distributive qusigroup and G is an STA-loop.

But G is di-Abelian. Hence G is a commutative di-associative STA-loop. Furthermore, by Theorem 10 G is an A-loop. However, Osborn ([3]) proved that every commutative di-associative A-loop is a Moufang loop.

(ii) implies (i). By Moufang's Theorem, G is di-associative and hence di-Abelian. Further, Belousov ([2]) proved that every distributive quasigroup is tri-Abelian. From this we can deduce that $D \times G$ is di-Abelian. Since D is distributive and G is a commutative Moufang loop, $D \times G$ is an A-quasigroup.

Theorem 14. *Let Q be a quasigroup. Then the following conditions are equivalent*:

(i) *For all $x \in Q$ it is $e(x), f(x) \in \mathrm{Id}\, Q$, $I(e(x)) \subseteq \mathrm{Aut}\, Q$, $I(f(x)) \subseteq \mathrm{Aut}\, Q$.*

(ii) *There are a distributive quasigroup D and an A-loop G such that $Q \cong D \times G$.*

Proof. (i) implies (ii). Since $e(x), f(x) \in \mathrm{Id}\, Q$, $R_{e(x)} \in I(e(x))$ and $L_{f(x)} \in I(f(x))$. Hence, by hypothesis, $R_{e(x)} \in \mathrm{Aut}\, Q$ and $L_{f(x)} \in \mathrm{Aut}\, Q$. Let $a, b \in Q$ be arbitrary. Then

$$(ab)\, e(a) = R_{e(a)}(ab) = (a\,.\,e(a))\,(b\,.\,e(a)) = a(b\,.\,e(a))\,;$$

that is, $R_{e(a)}^{-1}(a\,.\,(b\,.\,e(a)) = ab$. Now put $\beta = L_b^{-1} L_a^{-1} L_c$, where $c = R_{e(a)}^{-1}(a\,.\,be(a))$. It is evident that $\beta \in I(e(a))$. Hence $\beta \in \mathrm{Aut}\, Q$. But $\beta = L_b^{-1} L_a^{-1} L_{ab} = S_{a,b}$. Thus Q is an SA-quasigroup. Similarly Q is a TA-quasigroup.

Let $\alpha \in \mathrm{Id}\, Q$ be arbitrary. We shall prove that Q_α is normal in Q. To this purpose it is sufficient to prove (as it is well known) that $\varphi(Q_\alpha) \subseteq Q_\alpha$ for all $\varphi \in I(\alpha)$. Let $x \in Q_\alpha$ and $\varphi \in I(\alpha)$ arbitrary. Since $\varphi \in \mathrm{Aut}\, Q$ and $\varphi(\alpha) = \alpha$,

$$\varphi(x)\,.\,\alpha = \varphi(x)\,.\,\varphi(\alpha) = \varphi(x\alpha) = \varphi(x)\,.$$

Hence $\varphi(x) \in Q_\alpha$. Now, using Theorem 4, we get $Q \cong D \times G$; D a distributive quasigroup and G an STA-loop. But every inner mapping of G is an automorphism and hence G is an A-loop.

(ii) implies (i). This implication is an easy exercise.

Theorem 15. *Let Q be a quasigroup. Then the following conditions are equivalent*:

(i) *For every $x \in Q$ it is $I(x) \subseteq \mathrm{Aut}\, Q$.*

(ii) *Q is a tri-Abelian STA-quasigroup.*

(iii) *There are a distributive quasigroup D and an Abelian group G such that $Q \cong D \times G$.*

Proof. (i) implies (iii). Since $L_{f(x)}(x) = R_{e(x)}(x) = x$, $L_{f(x)} \in I(x)$ and $R_{e(x)} \in I(x)$. Hence $L_{f(x)}, R_{e(x)} \in \mathrm{Aut}\, Q$, and hence $e(x), f(x) \in \mathrm{Id}\, Q$. By Theorem 14, $Q \cong D \times G$, D distributive and G a loop. Evidently G has the property that $I(x) \subseteq \mathrm{Aut}\, G$ for every $x \in G$. Now we shall prove that any loop having this property is an Abelian group. Let $a, b, c \in G$. Put $\beta = L_b^{-1} L_a^{-1} L_d$, where $d = R_c^{-1}(a\,.\,bc)$. Since $\beta \in I(c)$, $\beta(j) = j$, j being the unit of G. From this, $a\,.\,bc = ab\,.\,c$. Hence G is a group. Further $R_a^{-1} L_g \in I(b)$, where $g = R_b^{-1}(ba)$. Therefore $R_a^{-1} L_g(j) = j$. Hence $ba = ab$. Thus G is an Abelian group.

(ii) implies (iii). Considering that every tri-Abelian loop is an Abelian group, we can proceed by Theorem 13 and its proof.

(iii) implies (i) and (ii). This implication is evident.