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NON-TANGENTIAL LIMITS OF THE DOUBLE LAYER POTENTIALS

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INTRODUCTION

We shall first introduce some fundamental notations, notions and theorems that will be used later.

Let G be a fixed Borel set in the Euclidean m-space R^m, $m \geqq 2$, and suppose that the boundary B of G is compact. Let the points of R^m be identified with m-dimensional vectors. For each $x, y \in R^m$ denote by xy the scalar product of the vectors x, y; denote by $|x|$ the Euclidean norm of the vector x. Further define, for any $y \in R^m$ and $r > 0$,

$$\Omega(y, r) = \{x \in R^m; \; |x - y| < r\} ;$$

the boundary of $\Omega(0, 1)$ denote by Γ. For a natural number α, $\alpha \leqq m$, denote by H_α the Hausdorff α-dimensional measure. Put

$$d_M(y) = \lim_{r \to 0+} \frac{H_m(\Omega(y, r) \cap M)}{H_m(\Omega(y, r))}$$

for any Borel set $M \subset R^m$ provided the limit exists. $d_M(y)$ is called the m-dimensional density of the set M at the point y. The vector $\Theta \in \Gamma$ is called the exterior normal of G at the point $y \in R^m$ in the sense of Federer provided the symmetric difference of G and the half-space

$$\{x \in R^m; \; (x - y)\,\Theta < 0\}$$

has m-dimensional density 0 at y. Since at every point $y \in R^m$ there exists at most one exterior normal in the sense of Federer, we may define a vector-valued function $n(y)$ in this way: we put $n(y) = \Theta$ if there is the exterior normal Θ at y; otherwise $n(y)$ equals the zero vector. Let $\hat{B}$ stand for the reduced boundary of G, i.e. the set of all $y \in R^m$ with $n(y) \neq 0$ (always $\hat{B} \subset B$). It follows from [3], theorem 4.5 that $n(y)$ is a Baire function; in particular, $\hat{B}$ is a Borel set.

$P(G)$ will denote the perimeter of G defined by

$$P(G) = \sup_{v} \int_{G} \operatorname{div} v(x) \, dx \,,$$

where v ranges over all m-dimensional infinitely differentiable vector-valued functions with compact supports in R^m, satisfying $|v(x)| \leqq 1$ for each $x \in R^m$. In what follows we shall assume

$$(0.1) \qquad P(G) < \infty \,.$$

Then (cf. [5]) $H_{m-1}(\hat{B}) < \infty$.

For any $\Theta \in \Gamma$ and $z \in R^m$ put

$$H(\Theta, z) = \{z + r\Theta; \, r > 0\}\,, \quad \mathscr{S}(z) = \{H(\Theta, z); \; \Theta \in \Gamma\}\,.$$

A point $y \in H(\Theta, z)$ is called a hit of $H(\Theta, z)$ on G provided both

$$H(\Theta, z) \cap G \cap \Omega(y, r) \quad \text{and} \quad (H(\Theta, z) - G) \cap \Omega(y, r)$$

have a positive H_1-measure for every $r > 0$. If $n(\Theta, z)$ denotes the total number of all the hits of $H(\Theta, z)$ on G, then according to [5], prop. 1.6 $n(\Theta, z)$ is a non-negative Baire function of the variable $\Theta \in \Gamma$. We may thus define a cyclic variation of G at the point z by

$$v(z) = \int_{\Gamma} n(\Theta, z) \, dH_{m-1}(\Theta)\,.$$

By [5], lemma 2.12 and with respect to assumption (0.1) we have

$$(0.2) \qquad v(z) = \int_{B} \frac{|n(y)\,(y - z)|}{|y - z|^m} \, dH_{m-1}(y)$$

for every $z \in R^m$. Since $H_{m-1}(\hat{B}) < \infty$ and for any fixed $z \notin B$ the integrand in (0, 2) is a bounded function, it is $v(z) < \infty$ (cf. also [5], lemma 2.9). Notice that $v(z) < \infty$ implies the existence of $d_G(z)$ (cf. [5], lemma 2.7).

Let C be a space of all continuous functions on B equiped with the supremum norm. Denote C^* the space of all linear continuous functionals on C. Elements of C^* may be interpreted as bounded measures with supports in B (cf. [1]). For $\mu \in C^*$ let μ^+, μ^- and $|\mu|$ be positive, negative and total variations of the measure μ respectively (cf. [1]). It is known that $\mu = \mu^+ - \mu^-$, $|\mu| = \mu^+ + \mu^-$ and the norm of μ equals $|\mu|\,(B)$. We define the integrability and measurability of functions and sets with respect to $\mu \in C^*$ in the same way as in [1].

If φ_M stands for the characteristic function of the set $M \subset R^m$, put, for a Borel set $A \subset B$, $\mu \mid A = \varphi_M \mu$ (for the multiplication of a measure by a function see [1]). For every $\mu \in C^*$ there exists a Borel set $A \subset B$ such that $\mu \mid A = \mu^+$, $\mu \mid (B - A) = \mu^-$. By [1], chap. V, § 5, part 7, corollary of theorem 13 there are actually two disjoint sets $M, N \subset B$ such that μ^+ is concentrated on M and μ^- is concentrated on N. Clearly the set M is μ-measurable (it is μ^+-measurable as $\mu^+(B - M) = 0$ and μ^--measurable as $\mu^-(M) = 0$). Thus there exists a Borel set $A \subset B$ such that $M \subset A$ and $|\mu|(A - M) = 0$. It is evident that A satisfies the above requirements.

Let $\mathscr{B}$ be the system of all bounded Baire functions on B. Assuming

$$(0.3) \qquad v(z) < \infty ,$$

we define the double layer potential for each $f \in \mathscr{B}$, $z \in R^m$ by

$$(0.4) \qquad W(f, z) = \int_B f(y) \frac{n(y)(y - z)}{|y - z|^m} \, \mathrm{d}H_{m-1}(y)$$

(cf. [5], lemma 2.12). Let $\mu \in C^*$. Then we define the double layer potential $W(\mu, z)$ for all $z \notin B$ and for $z \in B$ such that

$$(0.5) \qquad \int_B \frac{|n(y)(y - z)|}{|y - z|^m} \, \mathrm{d}|\mu|(y) < \infty ,$$

by

$$(0.6) \qquad W(\mu, z) = \int_B \frac{n(y)(y - z)}{|y - z|^m} \, \mathrm{d}\mu(y) .$$

For $M \subset R^m$, $y \in R^m$ let us call the contingent of M at y and denote by contg (M, y) the system of all half-lines $H(\Theta, y) \in \mathscr{S}(y)$ for which there is a sequence of points $y_n \in M$ $(n = 1, 2, \ldots)$ with $y_n \neq y$, $y_n \to y$ and

$$\lim_{n \to \infty} \frac{y_n - y}{|y_n - y|} = \Theta .$$

Obviously, contg $(M, y) \neq \emptyset$ if and only if y is an accumulation point of M.

Now we prove the following statement which will be needed later.

0.1 Proposition. *Let* $M \subset R^m$, $S \subset R^m$, $\eta \in R^m$ *and*

$$\operatorname{contg}(M, \eta) \cap \operatorname{contg}(S, \eta) = \emptyset .$$

Then there are $a > 0$, $\delta > 0$ *such that*

$$(0.7) \qquad (M \cap S \cap \Omega(\eta, \delta)) - \{\eta\} = \emptyset$$

and if dist (y, M) *denotes the distance of the point* y *from the set* M, *then*

$$(0.8) \qquad \operatorname{dist}(y, M) \geqq a|y - \eta|$$

holds for each $y \in S \cap \Omega(\eta, \delta)$.

Proof. The relation (0.7) follows from (0.8). Obviously, the statement is true in the case $y \notin \bar{S} \cap \bar{M}$.

If the statement (0.8) were false, we could find, for any sequence $\{a_n\}_{n=1}^{\infty}$ with $0 < a_n < 1$, $a_n \to 0$, two sequences $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ with $y_n \in S \cap \Omega(\eta, a_n) - \{\eta\}$, $z_n \in M$ and

$$|y_n - z_n| < a_n|y_n - \eta| = a_n r_n ,$$

where $|y_n - \eta| = r_n$. Putting $|z_n - \eta| = \bar{r}_n$, we get

$$r_n - a_n r_n \leqq \bar{r}_n \leqq r_n + a_n r_n .$$

Further

$$(0.9) \qquad 0 \leqq \left| \frac{z_n - \eta}{|z_n - \eta|} - \frac{y_n - \eta}{|y_n - \eta|} \right| \leqq \frac{|z_n - y_n|}{\bar{r}_n} + \left| \frac{y_n - \eta}{\bar{r}_n} - \frac{y_n - \eta}{r_n} \right| \leqq$$

$$\leqq \frac{a_n r_n}{\bar{r}_n} + r_n \frac{|r_n - \bar{r}_n|}{r_n \bar{r}_n} \leqq 2 \frac{a_n}{1 - a_n} \to 0$$

as $n \to \infty$. Since the sequence $\{(z_n - \eta)/|z_n - \eta|\}_{n=1}^{\infty}$ is a sequence of points of the compact set Γ, there is a convergent subsequence; we may assume it to have been already extracted. This implies

$$\lim_{n \to \infty} \frac{z_n - \eta}{|z_n - \eta|} = \Theta \in \Gamma .$$

On the other hand, by (0.9) also

$$\lim_{n \to \infty} \frac{y_n - \eta}{|y_n - \eta|} = \Theta .$$

Hence $H(\Theta, \eta) \in \operatorname{contg}(M, \eta) \cap \operatorname{contg}(S, \eta)$ which is the desired contradiction.

The preceding proposition implies that for $\eta \in B$ with $H(\Theta, \eta) \notin \operatorname{contg}(B, \eta)$ a $\delta > 0$ may be found such that the set

$$S = \{\eta + r\Theta;\ 0 < r < \delta\}$$

is included either in the interior of G or in $R^m - G$. Denoting for $\alpha \in \{0, \frac{1}{2}, 1\}$

$$G_\alpha = \{x \in R^m;\ d_G(x) = \alpha\} ,$$

then obviously $G_{1/2} \subset B$, $G_1 \subset \bar{G}$, $R^m - \bar{G} \subset G_0$. We have $S \subset G_1$ or $S \subset G_0$. Further $\hat{B} \subset G_{1/2}$ and by [5], lemma 3.7

$$H_{m-1}(G_{1/2} - \hat{B}) = 0 .$$

In the end let us make a note that the Hausdorff measure of a set is an invariant of the motion (i.e. a translation and a rotation) in R^m. Then also the quantities $v(x)$, $d_G(x)$, $W(f, x)$ are invariants of the motion, as well as the existence of the exterior normal in the sense of Federer; so for example the reduced boundary of the set after a motion is equal to the reduced boundary of the original set G, subjected to the motion.

1.

Recall that the symbol G denotes a fixed Borel set in R^m, $m \geqq 2$ with a compact boundary B and with a finite perimeter.

Now we shall prove this statement:

1.1 Proposition. *Let $S \subset R^m - B$, $\eta \in \bar{S} \cap B$. Then*

$$\limsup_{\substack{x \to \eta \\ x \in S}} W(f, x) < \infty \tag{1.1}$$

holds for every function $f \in C$ (or for every $f \in \mathscr{B}$) if and only if

$$\limsup_{\substack{x \to \eta \\ x \in S}} v(x) < \infty\,. \tag{1.2}$$

If, moreover, there is $\delta > 0$ such that

$$S \cap \Omega(\eta, \delta) \subset G_i \tag{1.3}$$

holds for $i = 0$ or $i = 1$, then the limit

$$\lim_{\substack{x \to \eta \\ x \in S}} W(f, x) \tag{1.4}$$

exists for each function $f \in C$ (or for each $f \in \mathscr{B}$ continuous at the point η) if and only if (1.2) holds. The value of the limit (1.4) is then given by

$$W(f, \eta) + f(\eta)\, H_{m-1}(\Gamma)\, (i - d_G(\eta))\,. \tag{1.5}$$

Proof. First we shall prove that the condition (1.2) is necessary and sufficient for (1.1) to be true for each $f \in C$. If this were false, we could find $x_k \in S$ ($k = 1, 2, \ldots$), $x_k \to \eta$, $v(x_k) \to \infty$. The point $x \in R^m$ being fixed, the quantity $W(f, x)$ determines a linear functional on the space C, whose norm is equal to $v(x)$ (cf. [5], relation (2.5)). It follows from (1.1) by Banach-Steinhaus theorem that there are two numbers k_0 and c such that $v(x_k) \leqq c$ for each $k > k_0$. This is the desired contradiction.

Let (1.2) hold. Hence we have $v(\eta) < \infty$ as the function $v(x)$ is lower semicontinuous with respect to $x \in R^m$ according to the statement 2.9 in [5]. Further, this implies that the density $d_G(\eta)$ at the point η exists (cf. [5], lemma 2.7).

. Taking into account (0.2) and (0.4), we get that the condition (1.1) is satisfied for each function $f \in \mathscr{B}$. Now suppose that (1.3) holds and prove the existence of the limit (1.4) for any $f \in \mathscr{B}$ continuous at the point η. According to (1.2) there is δ_1, $0 < \delta_1 < \delta$ such that

$$c = \sup \{v(x);\ x \in S \cap \Omega(\eta, \delta_1)\} < \infty .$$

From the lower semicontinuity of $v(x)$ we obtain

$$c = \sup \{v(x);\ x \in \bar{S} \cap \Omega(\eta, \delta_1)\} .$$

First assume that $f(x) = 1$ for all $x \in B$. This (by [5], lemma 2.5, provided $v(z) < \infty$) implies

$$W(f, z) = H_{m-1}(\Gamma)\, d_G(z)$$

if G is bounded and

$$W(f, z) = H_{m-1}(\Gamma)\,(1 - d_G(z))$$

if G is unbounded. By the assumption (1.3) just one of the following cases occurs: either $d_G(z) = 1$ for each $x \in S \cap \Omega(\eta, \delta)$ or $d_G(z) = 0$ for each $x \in S \cap \Omega(\eta, \delta)$. Moreover, comparing the values $W(f, \eta)$ and $W(f, z)$ for $z \in S \cap \Omega(\eta, \delta)$, we arrive at

$$\lim_{\substack{x \to \eta \\ x \in S}} W(f, x) = W(f, \eta) + H_{m-1}(\Gamma)\,(i - d_G(\eta)) .$$

Now let $f \in \mathscr{B}$, f continuous at the point η and $f(\eta) = 0$. Certainly there exists a function h continuous on R^m such that $0 \leqq h \leqq 1$, $h(x) = 1$ for each $x \in \Omega(0, \frac{1}{2})$ and $h(x) = 0$ for each $x \in R^m - \Omega(0, 1)$. Putting

$$g_r(x) = f(x)\, h\left(\frac{1}{r}(x - \eta)\right), \quad f_r(x) = f(x) - g_r(x)$$

for any $r > 0$, we have $g_r(x) = 0$ on $B - \Omega(\eta, r)$ and

$$\lim_{r \to 0+} \sup \{|g_r(x)|;\ x \in B\} = 0 .$$

Since $f_r(x) = 0$ on $B \cap \Omega(\eta, r/2)$, the function $W(f_r, x)$ is continuous on $\Omega(\eta, r/2)$. To prove

$$\lim_{\substack{x \to \eta \\ x \in S}} W(f, x) = W(f, \eta) ,$$

we shall prove that $W(g_r, x)$ tends to zero uniformly on $\bar{S} \cap \Omega(\eta, \delta_1)$ as $r \to 0+$. This will be sufficient because

$$W(f, x) = W(f_r, x) + W(g_r, x)$$

holds on $\bar{S} \cap \Omega(\eta, \delta_1)$. We have for each $x \in \bar{S} \cap \Omega(\eta, \delta_1)$

$$|W(g_r, x)| = \left| \int_B g_r(y) \frac{n(y)(y - x)}{|y - x|^m} \, dH_{m-1}(y) \right| \leqq$$

$$\leqq \sup \{|g_r(z)|; z \in B\} \int_B \frac{|n(y)(y - x)|}{|y - x|^m} \, dH_{m-1}(y) \leqq$$

$$\leqq c \sup \{|g_r(z)|; z \in B\} \to 0$$

as $r \to 0+$. If now $f \in \mathscr{B}$, f continuous at the point η, we may express this function f in the form of a sum of two functions, a constant function on B and a function lying in $\mathscr{B}$ continuous and vanishing at η. As $W(f, x)$ for a fixed x is linear with respect to f, the proof is complete.

Now we shall establish conditions for the validity of (1.2). Let us prove first the following auxiliary statement.

1.2 Lemma. *Let* $S \subset R^m - B$, $\eta \in \bar{S} \cap B$,

$$\operatorname{contg}(S, \eta) \cap \operatorname{contg}(\hat{B}, \eta) = \emptyset$$

and suppose

$$\sup_{r>0} \frac{H_{m-1}(\Omega(\eta, r) \cap \hat{B})}{r^{m-1}} = k < \infty .$$

Then there are $\delta > 0$, $c < \infty$ *such that for each* $z \in S \cap \Omega(\eta, \delta)$ *and each* $r > 0$

$$\frac{H_{m-1}(\Omega(z, r) \cap \hat{B})}{r^{m-1}} \leqq c . \tag{1.6}$$

Proof. Proposition 0.1 implies that there are $\delta > 0$, $a > 0$ such that for every $z \in S \cap \Omega(\eta, \delta)$

$$\operatorname{dist}(z, \hat{B}) \geqq a|z - \eta| . \tag{1.7}$$

Put $r_1 = |z - \eta|$ and $r = r_1 b$ for $b > 0$. Certainly the relation (1.6) holds for that r for which its corresponding value b satisfies $b < a$ because in that case $\Omega(z, r) \cap \hat{B} = \emptyset$ and thus also $H_{m-1}(\Omega(z, r) \cap \hat{B}) = 0$. For that r for which its corresponding value b satisfies $b \geqq a$ we have the following estimate:

$$\frac{H_{m-1}(\Omega(z, r) \cap \hat{B})}{r^{m-1}} \leqq \frac{H_{m-1}(\Omega(\eta, r_1 + r) \cap \hat{B})}{r^{m-1}} =$$

$$= \frac{H_{m-1}(\Omega(\eta, (1 + b) r_1) \cap \hat{B})}{(r_1(1 + b))^{m-1}} \frac{(1 + b)^{m-1}}{b^{m-1}} \leqq k \frac{(1 + b)^{m-1}}{b^{m-1}} \leqq k \frac{(1 + a)^{m-1}}{a^{m-1}} .$$

Now it is sufficient to put $c = k[(1 + a)^{m-1}/a^{m-1}]$.

1.3 Theorem. *Let* $S \subset R^m - B$, $\eta \in \bar{S} \cap B$ *and*

$$\operatorname{contg}(S, \eta) \cap \operatorname{contg}(\hat{B}, \eta) = \emptyset\,.$$

Further suppose

$$v(\eta) + \sup_{r>0} \frac{H_{m-1}(\Omega(\eta, r) \cap \hat{B})}{r^{m-1}} < \infty\,. \tag{1.8}$$

Then

$$\limsup_{\substack{z \to \eta \\ z \in S}} v(z) < \infty\,. \tag{1.9}$$

Proof. By the statements 0.1 and 1.2 we determine the constants a, δ, c such that (1.7) and (1.6) hold in the corresponding set. Further we fix a point z and denote $r = |z - \eta|$, $M = \hat{B} \cap \Omega(z, 2r)$, $N = \hat{B} - \Omega(z, 2r)$. Using the triangular inequality and the fundamental properties of the integral, we obtain the estimate

$$v(z) \leqq \int_M \frac{|n(y)\,(y - z)|}{|y - z|^m}\,\mathrm{d}H_{m-1}(y) + \int_N \frac{|n(y)\,(y - \eta)|}{|y - \eta|^m}\,\mathrm{d}H_{m-1}(y) + \tag{1.10}$$
$$+ \int_N \left| \frac{|n(y)\,(y - z)|}{|y - z|^m} - \frac{|n(y)\,(y - \eta)|}{|y - \eta|^m} \right| \mathrm{d}H_{m-1}(y)\,.$$

Now we number the quantities on the right-hand side of this inequality I, II, III respectively. Then we get

$$\mathrm{I} \leqq \frac{H_{m-1}(\Omega(z, 2r) \cap \hat{B})}{(ar)^{m-1}} \leqq \frac{2^{m-1}}{a^{m-1}}\,c\,, \quad \mathrm{II} \leqq v(\eta)\,.$$

To estimate III, we use

$$\int_{R^m} f(x)\,\mathrm{d}\mu(x) = \int_0^\infty \mu(\{x \in R^m; f(x) > t\})\,\mathrm{d}t\,,$$

where μ is a Borel measure and f is a non-negative, μ-integrable function on R^m. The last relation follows from [11] (there only non-negative measures are considered; in the present case we first decompose μ to the difference of the positive and the negative variations). There is $\Theta \in \Gamma$ such that $z = \eta + r\Theta$ so that we obtain

$$\left| \frac{|n(y)\,(y - z)|}{|y - z|^m} - \frac{|n(y)\,(y - \eta)|}{|y - \eta|^m} \right| \leqq \left| \frac{n(y)\,(y - z)}{|y - z|^m} - \frac{n(y)\,(y - \eta)}{|y - \eta|^m} \right| =$$
$$= \left| \frac{|y - \eta|^m - |y - z|^m}{|y - \eta|^m\,|y - z|^m}\,n(y)\,(y - \eta) - r\,n(y)\,\Theta\,\frac{1}{|y - z|^m} \right| \leqq$$
$$\leqq \frac{\big||y - \eta|^m - |y - z|^m\big|}{|y - \eta|^m\,|y - z|^m}\,|n(y)\,(y - \eta)| + r\,\frac{1}{|y - z|^m}\,.$$

Using the substitution $t^{-1/m} = x$ and lemma 1.2; we obtain the following estimate:

$$r \int_N \frac{dH_{m-1}(y)}{|y - z|^m} = r \int_0^\infty H_{m-1}\left(N \cap \left\{x \in R^m; \frac{1}{|x - z|^m} > t\right\}\right) dt =$$

$$= r \int_0^{(2r)^{-m}} H_{m-1}(\hat{B} \cap \Omega(z, t^{-1/m})) \, dt = rm \int_{2r}^\infty \frac{H_{m-1}(\hat{B} \cap \Omega(z, x))}{x^{m+1}} \, dx \leqq$$

$$\leqq crm \int_{2r}^\infty \frac{dx}{x^2} = \frac{c}{2} m \, .$$

Since for $y \in N$

$$|y - \eta| \leqq 2|y - z| \, ,$$

it is also

$$||y - \eta|^m - |y - z|^m| \leqq |y - \eta|^m + |y - z|^m \leqq (1 + 2^m) |y - z|^m \, .$$

Thus we have

$$\int_N \frac{||y - \eta|^m - |y - z|^m|}{|y - z|^m |y - \eta|^m} |n(y)(y - \eta)| \, dH_{m-1}(y) \leqq$$

$$\leqq (1 + 2^m) \int_N \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, dH_{m-1}(y) \leqq (1 + 2^m) \, v(\eta) \, .$$

Finally, we conclude that

$$v(z) \leqq c\left(\frac{2^{m-1}}{a^{m-1}} + \frac{m}{2}\right) + v(\eta)(2 + 2^m) \, .$$

Theorem 1.3 may be converted in this manner:

1.4 Theorem. *Let $\eta \in B$ and suppose that there are linearly independent vectors $\Theta_i \in \Gamma$ $(i = 1, \ldots, m)$ and a number $\delta > 0$ such that*

$$\sup \{v(z); z \in \bigcup_{i=1}^m H(\Theta_i, \eta) \cap \Omega(\eta, \delta)\} = c < \infty \, . \tag{1.11}$$

Then

$$\sup_{r>0} \frac{H_{m-1}(\Omega(\eta, r) \cap \hat{B})}{r^{m-1}} < \infty \, . \tag{1.12}$$

Proof. Assume that $\eta = 0$, $\delta \leqq 1$ and let Θ_i $(i = 1, \ldots, m)$ be linearly independent vectors. Then there is $b > 0$ such that for each $y \in \Omega(\eta, 2b)$ the vectors $(y - \Theta_i)$ are linearly independent. There is $d > 0$ such that

$$\sum_{i=1}^m |u(y - \Theta_i)| \geqq d$$

holds for each $y \in \Omega(\eta, b)$ and each $u \in \Gamma$. Obviously $b \leqq 1$ and thus $|y - \Theta_i| \leqq 2$. Hence

$$\sum_{i=1}^{m} \frac{|u(y - \Theta_i)|}{|y - \Theta_i|^m} \geqq \frac{1}{2^m} d\,.$$

Let now $0 < r < b\delta$ and consider $y \in \Omega(\eta, r) \cap \hat{B}$. Then we have

$$(1.13) \qquad 1 \leqq 2^m d^{-1} \sum_{i=1}^{m} \frac{\left|n(y)\left(\frac{b}{r} y - \Theta_i\right)\right|}{\left|\frac{b}{r} y - \Theta_i\right|^m} =$$

$$= r^{m-1} \,.\, 2^m \frac{1}{db^{m-1}} \sum_{i=1}^{m} \frac{\left|n(y)\left(y - \frac{r}{b}\Theta_i\right)\right|}{\left|y - \frac{r}{b}\Theta_i\right|^m}\,.$$

If we integrate the inequality (1.13) on the set $\hat{B} \cap \Omega(\eta, r)$ with respect to H_{m-1}, we obtain for each r, $0 < r < b\delta$

$$(1.14) \qquad H_{m-1}(\Omega(\eta, r) \cap \hat{B}) \leqq$$

$$\leqq r^{m-1} \,.\, 2^m d^{-1} b^{1-m} \sum_{i=1}^{m} v\left(\frac{r}{b}\Theta_i\right) \leqq r^{m-1} m \,.\, 2^m d^{-1} b^{1-m} c\,.$$

Since $H_{m-1}(\hat{B}) < \infty$, (1.12) follows from (1.14).

1.5 Remark. The assumptions of theorem 1.4 are satisfied for example whenever $\eta \in B$ and there are $\Theta' \in \Gamma$, $\delta > 0$ such that

$$(1.15) \qquad \limsup_{\substack{z \to \eta \\ z \in H(\Theta, \eta)}} v(z) < \infty$$

holds for each $\Theta \in \Gamma$ with $|\Theta - \Theta'| < \delta$. That last assumption is satisfied for example whenever $\operatorname{contg}(\hat{B}, \eta) \neq \mathscr{S}(\eta)$ (or $\operatorname{contg}(G_{1/2}, \eta) \neq \mathscr{S}(\eta)$ or $\operatorname{contg}(B, \eta) \neq \mathscr{S}(\eta)$) and (1.15) holds for each $\Theta \in \Gamma$ with $H(\Theta, \eta) \notin \operatorname{contg}(\hat{B}, \eta)$ (or $H(\Theta, \eta) \notin \operatorname{contg}(G_{1/2}, \eta)$ or $H(\Theta, \eta) \notin \operatorname{contg}(B, \eta)$).

Let us make still a note that theorem 1.3 holds also when we write in its assumptions $\operatorname{contg}(G_{1/2}, \eta)$ or $\operatorname{contg}(B, \eta)$ instead of $\operatorname{contg}(\hat{B}, \eta)$.

Taking into account the preceding remark, proposition 1.1 and theorems 1.3 and 1.4, we obtain immediately the following theorem.

1.6 Theorem. *Let $\eta \in B$. Then there is a finite limit*

$$(1.16) \qquad \lim_{\substack{z \to \eta \\ z \in H(\Theta, \eta)}} W(f, z)$$

for each $f \in C$ *(or each* $f \in \mathscr{B}$ *continuous at the point* η*) and for each half-line* $H(\Theta, \eta) \notin \operatorname{contg}(B, \eta)$*, if and only if* (1.8) *holds (provided* $\operatorname{contg}(B, \eta) \neq \mathscr{S}(\eta)$*). If* $H(\Theta, \eta) \notin \operatorname{contg}(B, \eta)$*, then there exist* $\delta > 0$*,* $i \in \{0, 1\}$ *such that*

$$H(\Theta, \eta) \cap \Omega(\eta, \delta) \subset G_i$$

and whenever (1.8) *holds, then the value of the limit* (1.16) *is given by* (1.5).

In the case $m = 2$ we may change the suppositions of theorem 1.4 as follows.

1.7 Theorem. *Let* $m = 2$ *and* $\eta \in B$*,* $\Theta \in \Gamma$ *such that* $H(\Theta, \eta) \notin \operatorname{contg}(\hat{B}, \eta)$*,* $H(-\Theta, \eta) \notin \operatorname{contg}(\hat{B}, \eta)$*. If there is* $r_0 > 0$ *such that*

(1.17) $$c = \sup \{v(z);\ z \in H(\Theta, \eta) \cap \Omega(\eta, r_0)\} < \infty\,,$$

then also

(1.18) $$\sup_{r>0} \frac{H_1(\hat{B} \cap \Omega(\eta, r))}{r} < \infty\,.$$

Proof. Suppose $\eta = 0$, $\Theta = [1, 0]$, $r_0 \leqq 1$. Choose r, $0 < r < r_0$ and $y \in \hat{B} \cap \Omega(\eta, r)$. Then there is $\beta \in \langle 0, 2\pi \rangle$ for which $y = |y| [\cos \beta, \sin \beta]$. Since neither $H(\Theta, \eta)$ nor $H(-\Theta, \eta)$ belong to $\operatorname{contg}(\hat{B}, \eta)$, we may find r', δ so that $r' > 0$, $0 < \delta < \frac{1}{2}\pi$, and

(1.19) $$\beta \in (\delta, \pi - \delta) \cup (\pi + \delta, 2\pi - \delta)$$

for every $y \in \hat{B}$ with $|y| < r'$, $y = |y| [\cos \beta, \sin \beta]$. Further it may be supposed that $r_0 = r'$. Let $y \in \hat{B}$. Then there is $\alpha \in \langle 0, 2\pi \rangle$ such that

(1.20) $$n(y) = [\cos \alpha, \sin \alpha]\,.$$

The rest of the proof will be divided into the following two parts:

a) $$\alpha \in \langle 0, \tfrac{1}{2}(\pi - \delta)\rangle \cup \langle \tfrac{1}{2}(\pi + \delta), \tfrac{3}{2}(\pi - \delta)\rangle \cup \langle \tfrac{3}{2}(\pi + \delta, 2\pi \rangle\,,$$

b) $$\alpha \in (\tfrac{1}{2}(\pi - \delta), \tfrac{1}{2}(\pi + \delta)) \cup (\tfrac{3}{2}(\pi - \delta), \tfrac{3}{2}(\pi + \delta))\,.$$

Put $z = [r, 0]$. It is easy to establish that

(1.21) $$|n(y)\, y| + |n(y)\,(y - z)| \geqq r|\cos \alpha|\,.$$

In the case *a*) we may write $r \cos \frac{1}{2}(\pi - \delta)$ on the right-hand side of the inequality (1.21).

We have $|n(y)\, y| = |y|\, |\cos (\beta - \alpha)|$. In the case *b*), by (1.19) it is evident that $|n(y)\, y| \geqq |y| \cos \frac{1}{2}(\pi - \delta)$.

Together we obtain that

(1.22) $$\frac{|n(y)\, y|}{|y|^2} + \frac{|n(y)\,(y - z)|}{|y - z|^2} \geqq \frac{\cos \frac{1}{2}(\pi - \delta)}{4r}$$

holds for each r, $0 < r \leqq r_0$, each $y \in \hat{B} \cap \Omega(\eta, r)$ and $z = [r, 0]$. It follows from the lower semicontinuity of $v(x)$ and from the assumption (1.17) that also $v(\eta) \leqq c$. If we integrate the inequality (1.22) on $\hat{B} \cap \Omega(\eta, r)$ (for r such that $0 < r \leqq r_0$) with respect to H_1, we arrive at

$$\frac{H_1(\Omega(\eta, r) \cap \hat{B})}{r} \leqq \frac{8c}{\cos \frac{1}{2}(\pi - \delta)} . \tag{1.23}$$

(1.18) is now a corollary of (1.23) and of $H_1(\hat{B}) < \infty$.

2.

Throughout this paragraph $G \subset R^m$ ($m \geqq 2$) denotes again a Borel set with a compact boundary B and with a finite perimeter. Now we shall deal with double layer potential $W(\mu, z)$ for $\mu \in C^*$.

$D \in R^1$ will be termed the H_{m-1}-derivative on $\hat{B}$ of $\mu \in C^*$ at the point $\eta \in B$ (briefly the derivative at η) if for every $r > 0$

$$H_{m-1}(\hat{B} \cap \Omega(\eta, r)) > 0 \tag{2.1}$$

and if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\left| \frac{\mu(M)}{H_{m-1}(M)} - D \right| < \varepsilon \tag{2.2}$$

holds for each Borel set $M \subset \hat{B} \cap \Omega(\eta, \delta)$ with $H_{m-1}(M) > 0$.

$D \in R^1$ will be termed the symmetric H_{m-1}-derivative on $\hat{B}$ of $\mu \in C^*$ at the point $\eta \in B$ (briefly the symmetric derivative at η) if there exists the limit

$$\lim_{r \to 0+} \frac{\mu(\Omega(\eta, r) \cap \hat{B})}{H_{m-1}(\Omega(\eta, r) \cap \hat{B})} = D . \tag{2.3}$$

(Note that in this definition also the assumption that (2.1) holds for each $r > 0$ is contained. This is valid, by [5], lemma 3.7, for each $\eta \in B$ with $|d_G(\eta) - \frac{1}{2}| < \frac{1}{2}$).

Obviously, if μ has the derivative at η, then there exists also the symmetric derivative of μ at η and their values are equal.

2.1 Lemma. *Let $\mu \in C^*$, $\eta \in B$, $S \subset R^m - B$,*

$$\operatorname{contg}(S, \eta) \cap \operatorname{contg}(\hat{B}, \eta) = \emptyset$$

and suppose that μ is a non-negative measure with the symmetric derivative on $\hat{B}$ at η equal to zero. Further suppose that (1.8) holds and that

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, d\mu(y) < \infty . \tag{2.4}$$

Then

$$\lim_{\substack{z \to \eta \\ z \in S}} W(\mu, z) = W(\mu, \eta) . \tag{2.5}$$

Proof. For $R > 0$ put $\lambda = \mu \mid \Omega(\eta, R)$, $\nu = \mu \mid (R^m - \Omega(\eta, R))$. We have $W(\mu, z) = W(\lambda, z) + W(\nu, z)$ for each $z \in R^m$ for which the left-hand side is defined. Analogously to the proof of the proposition 1.1, it is sufficient to prove that there is $\delta > 0$ such that

$$W(\lambda, z) \to 0$$

as $R \to 0+$ uniformly on $\{\eta\} \cup S \cap \Omega(\eta, \delta)$. For $z \in S$ denote $r = |z - \eta|$ and

$$M = \Omega(\eta, R) \cap \hat{B} - \Omega(\eta, 2r) , \quad N = \Omega(\eta, R) \cap \hat{B} \cap \Omega(\eta, 2r) .$$

We have

$$W(\lambda, z) = \int_M \frac{n(y)(y - z)}{|y - z|^m} \,\mathrm{d}\mu(y) + \int_N \frac{n(y)(y - z)}{|y - z|^m} \,\mathrm{d}\mu(y) . \tag{2.6}$$

Denote by I, II respectively the absolute values of the integrals on the right-hand side of (2.6). Applying the proposition 0.1 we find $a, \delta > 0$ such that

$$\operatorname{dist}(z, \hat{B}) \geqq a|z - \eta|$$

holds for each $z \in S \cap \Omega(\eta, \delta)$. If now $z \in S \cap \Omega(\eta, \delta)$, $|z - \eta| = r$, we arrive at

$$\mathrm{II} \leqq \frac{\mu(N)}{(ar)^{m-1}} \leqq \frac{2^{m-1}k}{a^{m-1}} \frac{\mu(N)}{H_{m-1}(\Omega(\eta, 2r) \cap \hat{B})} ,$$

where

$$k = \sup_{r > 0} \frac{H_{m-1}(\Omega(\eta, r) \cap \hat{B})}{r^{m-1}} .$$

Since the symmetric derivative of μ vanishes at η, for each $\varepsilon > 0$ there is $\delta_1 > 0$ such that

$$\frac{\mu(\Omega(\eta, \varrho) \cap \hat{B})}{H_{m-1}(\Omega(\eta, \varrho) \cap \hat{B})} \leqq \varepsilon \frac{a^{m-1}}{2^{m-1}k}$$

for any ϱ, $0 < \varrho < \delta_1$. Hence

$$\mathrm{II} \leqq \varepsilon$$

for each R such that $0 < R < \delta_1$, as we have

$$\frac{2^{m-1}k}{a^{m-1}} \frac{\mu(N)}{H_{m-1}(\Omega(\eta, 2r) \cap \hat{B})} = \frac{2^{m-1}k}{a^{m-1}} \frac{\mu(\Omega(\eta, 2r) \cap \hat{B})}{H_{m-1}(\Omega(\eta, 2r) \cap \hat{B})} < \varepsilon$$

if $R \geqq 2r$ (then $0 < 2r < \delta_1$) and

$$\frac{2^{m-1}k}{a^{m-1}} \frac{\mu(N)}{H_{m-1}(\Omega(\eta, 2r) \cap \hat{B})} \leqq \frac{2^{m-1}k}{a^{m-1}} \frac{\mu(\Omega(\eta, R) \cap \hat{B})}{H_{m-1}(\Omega(\eta, R) \cap \hat{B})} < \varepsilon$$

if $R < 2r$.

This estimate is independent of $z \in S \cap \Omega(\eta, \delta)$.

Now estimate the expression I. We may consider only $z \in S \cap \Omega(\eta, \delta)$ with $2r < R$ (for a fixed R) because in the opposite case $M = \emptyset$ and thus $\mathrm{I} = 0$. Since

$$\int_{\Omega(\eta,\varrho)\cap B} \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, d\mu(y) \to 0$$

as $\varrho \to 0+$, it is sufficient to prove that

$$V(z) = \left| \int_M \left(\frac{|n(y)(y - z)|}{|y - z|^m} - \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \right) d\mu(y) \right| \to 0 \tag{2.7}$$

as $R \to 0+$ uniformly with respect to z on the set $S \cap \Omega(\eta, \delta)$. We have

$$V(z) \leqq (1 + 2^m) \int_M \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, d\mu(y) + \int_M \frac{r}{|y - z|^m} \, d\mu(y) \tag{2.8}$$

(cf. an analogous estimate in the proof of theorem 1.3). Further

$$(1 + 2^m) \int_M \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, d\mu(y) \leqq (1 + 2^m) \int_{\Omega(\eta,R)\cap B} \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, d\mu(y) \to 0$$

as $R \to 0+$, where the last expression is independent of $z \in S \cap \Omega(\eta, \delta)$. Now estimate the expression II. Taking into account $|y - z| \geqq \frac{1}{2}|y - \eta|$ for $y \in M$, we arrive at

$$r \int_M \frac{d\mu(y)}{|y - z|^m} \leqq 2^m r \int_M \frac{d\mu(y)}{|y - \eta|^m} . \tag{2.9}$$

According to the proof of theorem 1.3, one obtains

$$r \int_M \frac{d\mu(y)}{|y - \eta|^m} = r \int_0^\infty \mu\left(\left\{x \in M; \frac{1}{|y - \eta|^m} > u\right\}\right) du . \tag{2.10}$$

However,

$$\left\{x \in M; \frac{1}{|y - \eta|^m} > u\right\} = (\Omega(\eta, R) \cap \hat{B} - \Omega(\eta, 2r)) \cap \Omega(\eta, u^{-1/m}) .$$

For $u \geqq (2r)^{-m}$ this set is empty and thus for these u the integrand on the right-hand side of (2.10) equals zero. For u such that $0 < u < R^{-m}$ this set is equal to M and thus for these u the integrand on the right-hand side of (2.10) equals $\mu(M)$. Now it is evident that

$$r \int_M \frac{d\mu(y)}{|y - \eta|^m} = r \frac{\mu(M)}{R^m} + r \int_{R^{-m}}^{(2r)^{-m}} \mu(M \cap \Omega(\eta, u^{-1/m})) \, du . \tag{2.11}$$

The first term on the right-hand side of (2.11) may be estimated by

$$r\frac{\mu(M)}{R^m} \leqq \frac{k}{2}\frac{\mu(\Omega(\eta, R) \cap \hat{B})}{H_{m-1}(\Omega(\eta, R) \cap \hat{B})}. \tag{2.12}$$

By the substitution $t = u^{-1/m}$ in the second term on the right-hand side of (2.11) we obtain

$$r\int_{R^{-m}}^{(2r)^{-m}} \mu(M \cap \Omega(\eta, u^{-1/m}))\, du = \tag{2.13}$$

$$= mr\int_{2r}^{R} \frac{\mu((\hat{B} - \Omega(\eta, 2r)) \cap \Omega(\eta, t))}{t^{m+1}}\, dt \leqq mrk\int_{2r}^{R} \frac{\mu(\Omega(\eta, t) \cap \hat{B})}{H_{m-1}(\Omega(\eta, t) \cap \hat{B})}\frac{dt}{t^2} \leqq$$

$$\leqq mrk \sup_{x\in(0,R)} \frac{\mu(\Omega(\eta, x) \cap \hat{B})}{H_{m-1}(\Omega(\eta, x) \cap \hat{B})}\int_{2r}^{R}\frac{dt}{t^2} \leqq \frac{mk}{2}\sup_{x\in(0,R)} \frac{\mu(\Omega(\eta, x) \cap \hat{B})}{H_{m-1}(\Omega(\eta, x) \cap \hat{B})}.$$

It follows from (2.13), (2.12), (2.11) and (2.9) that

$$r\int_M \frac{d\mu(y)}{|y - z|^m} \leqq 2^{m-1}k(m + 1) \sup_{x\in(0,R)} \frac{\mu(\Omega(\eta, x) \cap \hat{B})}{H_{m-1}(\Omega(\eta, x) \cap \hat{B})} \to 0 \tag{2.14}$$

as $R \to 0+$. The quantity on the right-hand side of the last inequality is independent of $z \in S \cap \Omega(\eta, \delta)$. Now it is evident that $V(z)$ tends to zero uniformly on $S \cap \Omega(\eta, \delta)$ as $R \to 0+$. Hence, in fact, $W(\lambda, z) \to 0$ as $R \to 0+$ uniformly on $\{\eta\} \cup S \cap \Omega(\eta, \delta)$, which completes the proof.

2.2 Lemma. *Let $\eta \in B$ such that $v(\eta) < \infty$ and $H_{m-1}(\hat{B} \cap \Omega(\eta, r)) > 0$ for every $r > 0$. Let $\mu \in C^*$ and suppose that there are $\delta > 0$ and $k < \infty$ such that*

$$\left|\frac{\mu(M)}{H_{m-1}(M)}\right| \leqq k \tag{2.15}$$

for any Borel set $M \subset \hat{B} \cap \Omega(\eta, \delta)$ with $H_{m-1}(M) > 0$. Then

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^m}\, d|\mu|(y) < \infty. \tag{2.16}$$

Proof. There exists a Borel set $A \subset B$ with $\mu^+ = \mu \mid A$, $\mu^- = \mu \mid (B - A)$. Putting $\lambda = \mu \mid (\hat{B} \cap \Omega(\eta, \delta))$, we obtain $\lambda^+ = \lambda \mid A$, $\lambda^- = \lambda \mid (B - A)$ and

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^m}\, d|\mu|(y) = \tag{2.17}$$

$$= \int_{B-\Omega(\eta,\delta)} \frac{|n(y)(y - \eta)|}{|y - \eta|^m}\, d|\mu|(y) + \int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^m}\, d|\lambda|(y).$$

The first integral on the right-hand side of (2.17) is finite because the integrand is bounded on $\hat{B} - \Omega(\eta, \delta)$ and $|\mu| (\hat{B}) < \infty$. It can be easily seen that

$$\lambda^+(M) \leqq kH_{m-1}(M)\,, \quad \lambda^-(M) \leqq kH_{m-1}(M) \tag{2.18}$$

for any Borel set $M \subset \hat{B}$. Since λ^+ and λ^- are concentrated on two disjoint subsets of $\hat{B} \cap \Omega(\eta, \delta)$, it follows from Radon-Nikodym theorem that there is $\varphi \in \mathscr{B}$ with $|\varphi(x)| \leqq k$ for each $x \in B$, $\varphi(x) = 0$ for each $x \in B - (\Omega(\eta, \delta) \cap \hat{B})$ and $\lambda = \varphi(H_{m-1} \mid \hat{B})$. For such function φ we have

$$\int_B \frac{|n(y)\,(y - \eta)|}{|y - \eta|^m} \,\mathrm{d}|\lambda|\,(y) = \int_B |\varphi(y)| \,\frac{|n(y)\,(y - \eta)|}{|y - \eta|^m} \,\mathrm{d}H_{m-1}(y) \leqq k\, v(\eta)$$

so that (2.16) is true.

2.3 Lemma. *Let $\eta \in B$ and let $\mu \in C^*$ has the derivative D at η. Then there exist derivatives of μ^+, μ^- and $|\mu|$ at η and they are equal to*

$$\frac{D + |D|}{2}\,, \quad \frac{-D + |D|}{2}\,, \quad |D|$$

respectively.

Proof. There is a Borel set $A \subset B$ for which $\mu^+ = \mu \mid A$, $\mu^- = \mu \mid (B - A)$. Further there is $\delta > 0$ such that

$$\left|\frac{\mu(M)}{H_{m-1}(M)}\right| \leqq |D| + 1$$

for any Borel set $M \subset \hat{B} \cap \Omega(\eta, \delta)$ with $H_{m-1}(M) > 0$. Now the proof will be divided into two parts:

a) Let $D = 0$.

The following two cases may occur: either

$$H_{m-1}(A \cap \hat{B} \cap \Omega(\eta, r)) > 0$$

for every $r > 0$ or

$$H_{m-1}((\hat{B} - A) \cap \Omega(\eta, r)) > 0$$

for every $r > 0$. Consider the first case. Let $M \subset \hat{B} \cap \Omega(\eta, \delta)$ be a Borel set with $H_{m-1}(M) > 0$. If $H_{m-1}(A \cap M) = 0$, then also $\mu^+(M) = 0$; if $H_{m-1}(A \cap M) > 0$, then

$$\frac{\mu^+(M)}{H_{m-1}(M)} \leqq \frac{\mu(A \cap M)}{H_{m-1}(A \cap M)}\,.$$

Therefore, since the derivative of μ vanishes at η, we obtain that μ^+ has the derivative vanishing at η. From the relations $\mu^- = \mu^+ - \mu$ and $|\mu| = \mu^+ + \mu^-$ we now conclude that μ^- and $|\mu|$ have also derivatives which vanish at η. In the second case we can proceed analogously.

b) Let $D \neq 0$.

Assume $D > 0$. There is δ_1, $0 < \delta_1 < \delta$ such that

$$\left|\frac{\mu(M)}{H_{m-1}(M)} - D\right| < \frac{D}{2} \tag{2.19}$$

holds for each Borel set $M \subset \hat{B} \cap \Omega(\eta, \delta_1)$ with $H_{m-1}(M) > 0$. Then necessarily

$$H_{m-1}((\hat{B} - A) \cap \Omega(\eta, \delta_1)) = 0 .$$

Indeed, if this is not the case, the inequality (2.19) with $(\hat{B} - A) \cap \Omega(\eta, \delta_1)$ written there instead of M is false. Hence

$$\mu^-(\hat{B} \cap \Omega(\eta, \delta_1)) = 0 .$$

This means that μ^- has the derivative which vanishes at η, μ^+ and $|\mu|$ have derivatives at η equal to D.

The case $D < 0$ is analogous.

2.4 Theorem. *Let $S \subset R^m - B$, $\eta \in \bar{S} \cap B$,*

$$\operatorname{contg}(S, \eta) \cap \operatorname{contg}(\hat{B}, \eta) = \emptyset ,$$

suppose that (1.8) holds and there is $\delta > 0$ such that (1.3) holds. Let $\mu \in C^*$, $\mu = \lambda + \nu$, $\lambda, \nu \in C^*$ such that λ has the derivative D at η, $|\nu|$ has the symmetric derivative which vanishes at η. Further suppose

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, \mathrm{d}|\nu|(y) < \infty .$$

Then there exists the limit

$$\lim_{\substack{z \to \eta \\ z \in S}} W(\mu, z) = W(\mu, \eta) + DH_{m-1}(\Gamma)(i - d_G(\eta)) . \tag{2.20}$$

Proof. We have

$$W(\mu, z) = W(\lambda, z) + W(\nu^+, z) - W(\nu^-, z)$$

for those $z \in R^m$ for which both sides of this equality are defined. It follows from lemma 2.1 that

$$\lim_{\substack{z \to \eta \\ z \in S}} W(\nu, z) = W(\nu, \eta) .$$

It is sufficient to prove that (2.20) holds if we write there λ instead of μ. Put $\gamma = \lambda - D(H_{m-1} \mid \hat{B})$. Since λ has the derivative D at η and $D(H_{m-1} \mid \hat{B})$ has the derivative D at η γ has the derivative vanishing at η. According to lemma 2.3, γ^+ and γ^- have also derivatives vanishing at η. If $f \in C$ is a function equal to unity on B, we have

$$W(\lambda, z) = D\, W(f, z) + W(\gamma^+, z) - W(\gamma^-, z)$$

for those $z \in R^m$ for which the left-hand side is defined. It is known (cf. the proof of the proposition 1.1) that there exists the limit

$$\lim_{\substack{z \to \eta \\ z \in S}} W(f, z) = W(f, \eta) + H_{m-1}(\Gamma)\,(i - d_G(\eta))\,.$$

According to lemma 2.1 the limit

$$\lim_{\substack{z \to \eta \\ z \in S}} W(\gamma, z) = W(\gamma, \eta)$$

also exists (to verify the assumptions one uses lemma 2.2). This implies the statement of the present theorem.

2.5 Remark. It is not possible to replace the requirement (2.15) in the lemma 2.2 by the "symmetric requirement", i.e. by

$$\limsup_{r \to 0+} \left| \frac{\mu(\Omega)(\eta, r) \cap \hat{B})}{H_{m-1}(\Omega(\eta, r) \cap \hat{B})} \right| < \infty\,.$$

Moreover, we shall introduce an example proving that it is not sufficient to suppose that μ is a non-negative measure with the symmetric derivative vanishing at η.

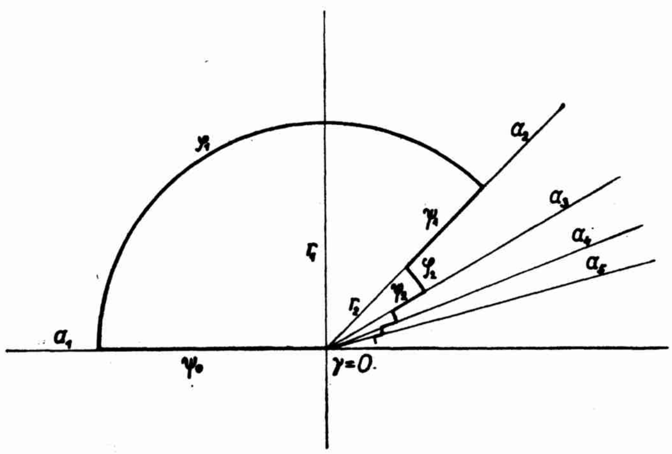

Fig. 1

Let $m = 2$. Denote by $[x, y]$ $(x, y \in R^1)$ the points of R^2. We construct in R^2 the curve φ consisting of the curves φ_i and ψ_j as in fig. 1 – the reader certainly can describe this curve precisely. Here we put $r_k = 1/k$ $(k = 1, 2, \ldots)$, $\alpha_k = \pi/4k$ $(k = 2, 3, \ldots)$, $\alpha_1 = \frac{1}{2}\pi$, r_k denotes the radius of the arc φ_k, α_k the angle. For the curve φ we may easily find a rectification, for example by an arc length – but we shall not need it here. The curve φ is a Jordan curve (i.e. simple closed curve) and thus we may consider the domain $G = \operatorname{Int} \varphi$. It is evident that $P(G) < \infty$, $B = \langle \varphi \rangle$ and $B - \hat{B}$ is a denumerable set. Let $\eta = [0, 0]$. We have $v(\eta) < \infty$. Now we define a function f on B as follows:

$$f(z) = \frac{4}{\pi} \frac{k+1}{\log k}$$

for all z on the open arc φ_k, $k = 2, 3, \ldots$,

$$f(z) = 0$$

for all other $z \in B$. Putting $\mu = f H_1 \mid B$, we have that $\mu \in C^*$ and μ is a non-negative measure. Let

$$q_k = \mu((\varphi_k)) = r_k(\alpha_k - \alpha_{k+1}) \frac{4}{\pi} \frac{k+1}{\log k} = \frac{1}{k^2 \log k}$$

for $k = 2, 3, \ldots$. We shall prove that μ has the symmetric derivative which vanishes at η. Given r, $0 < r < 1$, there is a natural number k such that $r \in (r_{k+1}, r_k\rangle$. Then

$$\mu(\Omega(\eta, r)) \cap \hat{B}) = \sum_{n=k+1}^{\infty} q_n = \sum_{n=k+1}^{\infty} \frac{1}{n^2 \log n} \leqq$$

$$\leqq \frac{1}{\log(k+1)} \sum_{n=k+1}^{\infty} \frac{1}{n^2} \leqq \frac{1}{\log(k+1)} \int_k^{\infty} \frac{dt}{t^2} = \frac{1}{k \log(k+1)} .$$

Taking into account

$$H_1(\hat{B} \cap \Omega(\eta, r)) \geqq 2r \left(> 2r_{k+1} = \frac{2}{k+1} \right),$$

we see that μ has the symmetric derivative vanishing at η.

For $y \in (\varphi_k)$ we have $n(y) = y/|y|$ and therefore

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^2} \, d\mu(y) = \int_B f(y) \frac{|n(y)(y - \eta)|}{|y - \eta|^2} \, dH_1(y) =$$

$$= \sum_{k=2}^{\infty} \frac{q_k}{r_k} = \sum_{k=2}^{\infty} \frac{1}{k \log k} = \infty .$$

The measure μ satisfies a desired requirements. Let us remark that in the preceding example one may require φ to be a smooth curve.

3.

Throughout this paragraph we always assume that $m = 2$. Where necessary, we identify R^2 with the set of all complex numbers. Introduce the following notation:

If $\alpha \in R^1$, $z \in R^2$, write $H(\alpha, z) = H(\Theta, z) = \{z + r\Theta;\ r > 0\}$, where $\Theta = [\cos \alpha, \sin \alpha]$. $\mathscr{D}$ stands for the set of all infinitely differentiable functions with compact supports in R^2. For $z \in R^2$ put

$$\mathscr{D}(z) = \{\varphi \in \mathscr{D};\ z \notin \operatorname{supp} \varphi\},$$

where supp φ denotes the support of the function φ.

Now we shall prove two simple auxiliary assertions (which could be pronounced in a more general form).

3.1 Lemma. *Let φ be a Jordan curve in R^2 defined on $\langle a, b\rangle$ and ϑ a function with a finite variation on $\langle a, b\rangle$. Further suppose that the function ϑ is either continuous from the right on $\langle a, b)$ or continuous from the left on $(a, b\rangle$. Then*

$$\text{(3.1)} \qquad \operatorname{var}[\vartheta; \langle a, b\rangle] = \sup\left\{\int_a^b f(\varphi(t))\, \mathrm{d}\vartheta(t);\ f \in \mathscr{D},\ |f| \leqq 1\right\}$$

(the integrals in (3.1) are meant in the sense of Stieltjes).

Proof. If $\operatorname{var}[\vartheta; \langle a, b\rangle] = 0$, then the statement is obvious. Suppose that $\operatorname{var}[\vartheta; \langle a, b\rangle] > 0$. It is known that

$$\operatorname{var}[\vartheta; \langle a, b\rangle] = \sup\left\{\int_a^b f(t)\, \mathrm{d}\vartheta(t);\ f \in C(\langle a, b\rangle),\ |f| \leqq 1\right\}$$

(integrals are always meant in the sense of Stieltjes).

Given $\varepsilon > 0$, there is $f_1 \in C(\langle a, b\rangle)$, $|f_1| \leqq 1$ such that

$$\text{(3.2)} \qquad \int_a^b f_1(t)\, \mathrm{d}\vartheta(t) > \operatorname{var}[\vartheta; \langle a, b\rangle] - \frac{\varepsilon}{3}.$$

Assume conversely that ϑ is continuous from the right on $\langle a, b)$. Then the function

$$s(t) = \operatorname{var}[\vartheta; \langle a, b\rangle]$$

is continuous from the right at the point a and thus there is $t_0 \in (a, b)$ such that for each $t \in \langle a, t_0\rangle$

$$s(t) < \frac{\varepsilon}{6}.$$

Further there exists $f_2 \in C(\langle\varphi\rangle)$ with $|f_2| \leqq 1$, $f_2(\varphi(t)) = f_1(t)$ for each $t \in \langle t_0, b\rangle$. Then

$$\int_a^b f_2(\varphi(t))\,\mathrm{d}\vartheta(t) = \int_{t_0}^b f_1(t)\,\mathrm{d}\vartheta(t) + \int_a^{t_0} f_2(\varphi(t))\,\mathrm{d}\vartheta(t) \geqq$$

$$\geqq \int_a^b f_1(t)\,\mathrm{d}\vartheta(t) - \left|\int_a^{t_0} (f_2(\varphi(t)) - f_1(t))\,\mathrm{d}\vartheta(t)\right| > \operatorname{var}[\vartheta; \langle a, b\rangle] - \tfrac{2}{3}\varepsilon\,.$$

Since $\langle\varphi\rangle$ is a compact set and $f_2 \in C(\langle\varphi\rangle)$, there is $f \in \mathscr{D}$, $|f| \leqq 1$ such that

$$|f(z) - f_2(z)| \leqq \frac{\varepsilon}{3\operatorname{var}[\vartheta; \langle a, b\rangle]}$$

holds for each $z \in \langle\varphi\rangle$. Then

$$\int_a^b f(\varphi(t))\,\mathrm{d}\vartheta(t) \geqq \int_a^b f_2(\varphi(t))\,\mathrm{d}\vartheta(t) -$$

$$-\left|\int_a^b (f(\varphi(t)) - f_2(\varphi(t)))\,\mathrm{d}\vartheta(t)\right| > \operatorname{var}[\vartheta; \langle a, b\rangle] - \varepsilon\,.$$

In the case of ϑ continuous from the left on $(a, b\rangle$ we may proceed completely analogously.

3.2 Lemma. *Let φ be a Jordan curve in R^2 defined on $\langle a, b\rangle$, let $t_0 \in \langle a, b\rangle$, $I_1 = \langle a, t_0)$, $I_2 = (t_0, b\rangle$ (of course, if $t_0 = a$, then $I_1 = \emptyset$, if $t_0 = b$, then $I_2 = \emptyset$), let ϑ_j $(j = 1, 2)$ be a continuous function with a locally finite variation on I_j. Then*

$$\sum_{j=1}^{2} \operatorname{var}[\vartheta_j, I_j] = \sup\left\{\sum_{j=1}^{2}\int_{I_j} f(\varphi(t))\,\mathrm{d}\vartheta_j(t);\ f \in \mathscr{D}(\varphi(t_0)),\ |f| \leqq 1\right\} \tag{3.3}$$

(it is obvious how (3.3) reduces in the case $t_0 = a$ or $t_0 = b$).

Proof. a) Let $\sum_{j=1}^{2} \operatorname{var}[\vartheta_j; I_j] < \infty$. Suppose $t_0 \in (a, b)$. Define a function ϑ on $\langle a, b\rangle$ by

$$\vartheta(t) = \vartheta_1(t) \quad \text{for} \quad t \in \langle a, t_0)\,,$$

$$\vartheta(t) = \vartheta_2(t) - \lim_{z\to t_0^+} \vartheta_2(z) + \lim_{z\to t_0^-} \vartheta_1(z) \quad \text{for} \quad t \in (t_0, b\rangle\,,$$

$$\vartheta(t_0) = \lim_{z\to t_0^-} \vartheta_1(z).$$

Obviously, ϑ is a continuous function on $\langle a, b\rangle$ with a finite variation

$$\operatorname{var}[\vartheta; \langle a, b\rangle] = \sum_{j=1}^{2} \operatorname{var}[\vartheta_j; I_j]\,.$$

For $f \in C(\langle a, b \rangle)$ we have

$$\sum_{j=1}^{2} \int_{I_j} f(t) \, d\vartheta_j(t) = \int_a^b f(t) \, d\vartheta(t) .$$

Given $\varepsilon > 0$, then according to lemma 3.1 we may find $f_1 \in \mathscr{D}$, $|f_1| \leqq 1$ such that

$$\int_a^b f_1(\varphi(t)) \, d\vartheta(t) > \operatorname{var} [\vartheta; \langle a, b \rangle] - \frac{\varepsilon}{2} .$$

Further there is δ, $0 < \delta < \min \{t_0 - a, b - t_0\}$ such that

$$\operatorname{var} [\vartheta; \langle t_0 - \delta, t_0 + \delta \rangle] < \frac{\varepsilon}{4} .$$

Since $\varphi(t_0)$ is not contained in the compact set

$$\varphi(\langle a, t_0 - \delta \rangle \cup \langle t_0 + \delta, b \rangle) ,$$

there is $f \in \mathscr{D}(\varphi(t_0))$ such that $|f| \leqq 1$ and $f(z) = f_1(z)$ for each

$$z \in \varphi(\langle a, t_0 - \delta \rangle \cup \langle t_0 + \delta, b \rangle) .$$

From the choice of f_1 and δ it follows

$$\int_a^b f(\varphi(t)) \, d\vartheta(t) > \operatorname{var} [\vartheta; \langle a, b \rangle] - \varepsilon .$$

Analogously in the cases $t_0 = a$ or $t_0 = b$.

$b)$ Suppose, conversely, $\operatorname{var} [\vartheta_1; \langle a, t_0)] = \infty$.

Let $t_0 \in (a, b)$. Given $k > 0$, there is $t_1 \in (a, t_0)$ such that $\operatorname{var} [\vartheta_1; \langle a, t_1 \rangle] > k + 2$ and thus there is $f_1 \in \mathscr{D}$ with $|f_1| \leqq 1$ and

$$\int_a^{t_1} f_1(\varphi(t)) \, d\vartheta_1(t) > k + 1 .$$

There is $\delta_1 > 0$ such that $\Omega(\varphi(t_0), 2\delta_1) \cap \varphi(\langle a, t_1 \rangle) = \emptyset$. Further there is $t_2 \in (t_1, t_0)$ such that

$$\operatorname{var} [\vartheta_1; \langle t_1, t_2 \rangle] < \tfrac{1}{3}$$

(since ϑ_1 is continuous). We may find $\delta_2 > 0$, $2\delta_2 < t_1 - a$ such that

$$\operatorname{var} [\vartheta_1; \langle a, a + 2\delta_2 \rangle] < \tfrac{1}{3} .$$

Then $\varphi(\langle a + 2\delta_2, t_1 \rangle)$ and $\varphi(\langle a, a + \delta_2 \rangle \cup \langle t_2, b \rangle) \cup \overline{\Omega(\varphi(t_0), \delta_1)}$ are two disjoint compact sets and thus there is $f \in \mathscr{D}$ with $|f| \leqq 1$, $f(z) = f_1(z)$ on the former of both described sets and $f(z) = 0$ on the latter. Therefore, moreover, $f \in \mathscr{D}(\varphi(t_0))$. We arrive at

$$\sum_{j=1}^{2} \int_{I_j} f(\varphi(t))\, \mathrm{d}\vartheta_j(t) = \int_{a+\delta_2}^{t_2} f(\varphi(t))\, \mathrm{d}\vartheta_1(t) =$$

$$= \int_a^{t_1} f_1 * \varphi \mathrm{d}\vartheta_1 - \int_a^{a+2\delta_2} f_1 * \varphi\, \mathrm{d}\vartheta_1 + \int_{t_1}^{t_2} f * \varphi\, \mathrm{d}\vartheta_1 + \int_{a+\delta_2}^{a+2\delta_2} f * \varphi\, \mathrm{d}\vartheta_1 > k\,.$$

Analogously for $t_0 = b$.

The case var $[\vartheta_2; I_2] = \infty$ may be solved in the same way.

Throughout the rest of this paragraph ψ stands for a Jordan curve in R^2 defined on a compact interval $\langle\alpha, \beta\rangle$ $(\alpha < \beta)$. Further suppose that ψ is a positively oriented curve with a finite length. Denote $G = \operatorname{Int} \psi$ and, according to the preceding notation, $B = \langle\psi\rangle$, $\hat{B}$ being the reduced boundary of the set G. From [12], part 8, we get var $[\psi; \langle\alpha, \beta\rangle] = P(G)$ and so

$$P(G) < \infty\,. \tag{3.4}$$

For $z \in R^2$, $\alpha \in \langle 0, 2\pi\rangle$ let $N(\alpha, z)$ be the number of all points of the set $\langle\psi\rangle \cap \cap H(\alpha, z)$. The function $N(\alpha, z)$ is a measurable function with respect to $\alpha \in \langle 0, 2\pi\rangle$ (and non-negative), thus we may define

$$V(z) = \int_0^{2\pi} N(\alpha, z)\, \mathrm{d}\alpha$$

(cf., for example, [6], lemma 2.1). If $\Theta = [\cos\alpha, \sin\alpha]$, then $n(\Theta, z) \leqq N(\alpha, z)$ (where $n(\Theta, z)$ has the same meaning as in the introduction). Hence

$$v(z) \leqq V(z)\,. \tag{3.5}$$

For $z \in R^2$ let $\mathfrak{A}$ be the system of all components of the set $\langle\alpha, \beta\rangle - \psi^{-1}(z)$ (in the present case $\mathfrak{A}$ has at most two elements) and for $I \in \mathfrak{A}$ let ϑ_z^I be a single-valued continuous argument of $\psi(t) - z$ on I. Define, for $z \in R^2$ and $f \in C$,

$$W^*(f, z) = \sum_{I \in \mathfrak{A}} \int_I f(\psi(t))\, \mathrm{d}\vartheta_z^I(t) \tag{3.6}$$

provided the integrals on the right-hand side exist and their sum is defined.

Prove that if $\varphi \in \mathscr{D}(z)$, then

$$W^*(\varphi, z) = W(\varphi, z)\,. \tag{3.7}$$

Hence we obtain by passing to the limit that if $V(z) < \infty$, then $W^*(f, z) = W(f, z)$ for each $f \in C$ – as regards this, see the equality (3.10) in the following.

If $\varphi \in \mathscr{D}(z)$, then (cf. [5])

$$W(\varphi, z) = \int_G \operatorname{grad} \varphi(x) \frac{x - z}{|x - z|^2}\, \mathrm{d}x\,.$$

The proposition 2.3 in [8] implies

$$W^*(\varphi, z) = -\int_\alpha^\beta \varphi(\psi(t)) \frac{\psi_2(t) - y}{|\psi(t) - z|^2} \, d\psi_1(t) + \int_\alpha^\beta \varphi(\psi(t)) \frac{\psi_1(t) - x}{|\psi(t) - z|^2} \, d\psi_2(t),$$

where $z = [x, y]$, $\psi = [\psi_1, \psi_2]$. For ψ and the function

$$w(\zeta) = \left[-\varphi(\zeta) \frac{\eta - y}{|\zeta - z|^2}, \varphi(\zeta) \frac{\xi - x}{|\zeta - z|^2} \right]$$

(where $\zeta = [\xi, \eta]$) the requirements of Green theorem are satisfied (cf. [4], theorem 8.49) and thus we conclude

$$W^*(\varphi, z) = \int_\psi w_1 \, d\xi + w_2 \, d\eta = \int_G \text{rot } w = \int_G \text{grad } \varphi(u) \frac{u - z}{|u - z|^2} \, du = W(\varphi, z).$$

3.3 Theorem. *If $z \in R^2$, then*

$$V(z) = v(z). \tag{3.8}$$

Proof. Since by [5], assertion 1.6

$$v(z) = \sup \{W(\varphi, z); \varphi \in \mathscr{D}(z), |\varphi| \leqq 1\},$$

it is sufficient to prove, with respect to (3.7), that

$$V(z) = \sup \{W^*(\varphi, z); \varphi \in \mathscr{D}(z), |\varphi| \leqq 1\}. \tag{3.9}$$

Let $\mathfrak{A}$, ϑ_z^I have the same meaning as in the definition of $W^*(f, z)$. It follows from (6) in [8] that

$$V(z) = \sum_{I \in \mathfrak{A}} \text{var } [\vartheta_z^I; I]. \tag{3.10}$$

If $\alpha \leqq a < b \leqq \beta$, $z \notin \psi(\langle a, b \rangle)$ and ϑ is some single-valued argument of $\psi(t) - z$ on $\langle a, b \rangle$, then (by 1.12 from [7])

$$\text{var } [\vartheta; \langle a, b \rangle] \leqq \text{dist } (z; \psi(\langle a, b \rangle)) \text{ var } [\psi; \langle a, b \rangle].$$

This implies that ϑ_z^I has a locally finite variation on $I \in \mathfrak{A}$. If now $z \in B$, we may use lemma 3.2, therefore we see that (3.9) holds. If $z \notin B$, then (3.9) follows from lemma 3.1.

3.4 Remark. Since $n(\Theta, z) \leqq N(\alpha, z)$ (where $\Theta = [\cos \alpha, \sin \alpha]$), it follows from theorem 3.3 that for each fixed $z \in R^2$, $n(\Theta, z) = N(\alpha, z)$ for almost all $\alpha \in \langle 0, 2\pi \rangle$.

In the same way as in [8] we define for $t_0 \in (\alpha, \beta)$

$$\tau_\psi^+(t_0) = \lim_{t \to t_0+} \frac{\psi(t) - \psi(t_0)}{|\psi(t) - \psi(t_0)|} = e^{i\alpha+}, \quad \tau_\psi^-(t_0) = \lim_{t \to t_0-} \frac{\psi(t) - \psi(t_0)}{|\psi(t) - \psi(t_0)|} = e^{i\alpha-} \tag{3.11}$$

provided the limits exist. We may suppose that $\alpha_+ \leqq \alpha_- < \alpha_+ + 2\pi$. If $\tau_\psi^+(t_0) = -\tau_\psi^-(t_0)$, then we put

$$\tau_\psi(t_0) = \tau_\psi^+(t_0)\,. \tag{3.12}$$

3.5 Lemma. *Let $t \in (\alpha, \beta)$. If there exist $\tau_\psi^+(t)$ and $\tau_\psi^-(t)$, then there exists the density $d_G(z)$ for $z = \psi(t)$. If moreover $\alpha_+ \neq \alpha_-$, then*

$$d_G(z) = \frac{1}{2\pi}(\alpha_- - \alpha_+)\,; \tag{3.13}$$

if $\alpha_+ = \alpha_-$, then either $d_G(z) = 0$ or $d_G(z) = 1$.

If, besides that, there exists $\tau_\psi(t)$, then there exists the exterior normal of G in the sense of Federer

$$n(z) = -i\tau_\psi(z)\,.$$

Proof. Suppose that $\psi(t) = 0$, $\alpha_+ \neq \alpha_-$ and that there is $\gamma \in (0, \pi)$ such that

$$\alpha_+ = -\gamma\,, \quad \alpha_- = \gamma\,.$$

Given ε, $0 < \varepsilon < \gamma$, then by the definition of τ_ψ^+ and τ_ψ^- there is $\delta > 0$, $\delta < \min\{t - \alpha, \beta - t\}$ such that

$$\begin{gathered}[u \in (t, t + \delta), \psi(u) - \psi(t) = e^{i\beta_1}|\psi(u) - \psi(t)|, \beta_1 \in \langle -\pi - \gamma, \pi - \gamma\rangle] \Rightarrow \\ \Rightarrow |\beta_1 + \gamma| < \varepsilon\,, \\ [u \in (t - \delta, t), \psi(u) - \psi(t) = e^{i\beta_2}|\psi(u) - \psi(t)|, \beta_2 \in \langle \gamma - \pi, \gamma + \pi\rangle] \Rightarrow \\ \Rightarrow |\beta_2 - \gamma| < \varepsilon\,.\end{gathered} \tag{3.14}$$

There is $r_0 > 0$ such that $\Omega(0, r_0) \cap \psi(\langle \alpha, \beta\rangle - (t - \delta, t + \delta)) = \emptyset$. Prove that for each r such that $0 < r < r_0$

$$\begin{gathered}\Omega(0, r) \cap \{z = |z|\, e^{i\eta};\ z \neq 0,\ \eta \in \langle \varepsilon - \gamma, \gamma - \varepsilon\rangle\} \subset \Omega(0, r) \cap G \subset \\ \subset \Omega(0, r) \cap \{z = |z|\, e^{i\eta};\ \eta \in \langle -\varepsilon - \gamma, \varepsilon + \gamma\rangle\}\,.\end{gathered} \tag{3.15}$$

The sets

$$\Omega(0, r) \cap \{z = |z|\, e^{i\eta};\ z \neq 0,\ \eta \in \langle \varepsilon - \gamma, \gamma - \varepsilon\rangle\}\,, \tag{3.16}$$

$$\Omega(0, r) \cap \{z = |z|\, e^{i\eta};\ z \neq 0,\ \eta \in \langle \gamma + \varepsilon, 2\pi - \gamma - \varepsilon\rangle\} \tag{3.17}$$

are connected. To prove that (3.16) is contained in Int ψ and (3.17) is contained in Ext ψ (which implies (3.15)), it is sufficient to prove that there is a point z_1 in (3.16) with $\text{ind}_\psi(z_1) = 1$ and a point z_2 in (3.17) with $\text{ind}_\psi(z_2) = 0$. Put $z_1 = \frac{1}{2}r$, $z_2 = -\frac{1}{2}r$ (z_1, z_2 are considered in the terms of complex numbers). Since there exist $\tau_\psi^+(t)$, $\tau_\psi^-(t)$ and $\tau_\psi^+(t) = e^{-i\gamma}$, $\tau_\psi^-(t) = e^{i\gamma}$ where $\gamma \in (0, \pi)$, it is clear that the function Im ψ is decreasing at the point t. By Mařík theorem (cf. [2], theorem 126) we have

$$\text{ind}_\psi(z_2) = \text{ind}_\psi(z_1) - 1\,.$$

Since ψ is a positively oriented curve, this equation yields necessarily $\text{ind}_\psi(z_1) = 1$, $\text{ind}_\psi(z_2) = 0$. The relation (3.15) implies

$$(\gamma - \varepsilon)\, r^2 \leqq H_2(\Omega(0, r) \cap G) \leqq (\gamma + \varepsilon)\, r^2$$

and thus, in fact, $d_G(z) = \gamma/\pi\ (=(\alpha_- - \alpha_+)/2\pi)$. The rest of the proof, i.e. $d_G(z) = 0$ or $d_G(z) = 1$ if $\alpha_+ = \alpha_-$ and the existence of the exterior normal in the sense of Federer if $\tau_\psi(t)$ exists is analogous.

Let $z \in R^2$, $t > 0$ and let $M(t, z)$ stand for the number of all points of the set $\psi^{-1}(\{x;\ |x - z| = t\})$. Then $M(t, z)$ is a measurable function with respect to $t \in (0, \infty)$ (cf. ,e.g., [6], lemma 2.5) and we may thus define, for each $r > 0$,

(3.18) $$u(z, r) = \int_0^r M(t, z)\, dt\,.$$

3.6 Theorem. *If $\eta \in R^2$ with $v(\eta) < \infty$, then*

$$\sup_{r>0} \frac{u(\eta, r)}{r} < \infty$$

holds if and only if

$$\sup_{r>0} \frac{H_1(\Omega(\eta, r) \cap \hat{B})}{r} < \infty\,.$$

Proof. If $\eta \notin B$ is the case the statement is obvious, because $n(z, \infty) \leqq \leqq \text{var}\,[\psi; \langle\alpha, \beta\rangle]$ for each $z \in R^2$ (cf. (7) in [8]) and $H_1(\hat{B}) < \infty$.

Let $\eta \in B$. Therefore by [8], theorem 3.9

(3.19) $$u(\eta, r) \leqq \text{var}\,[\psi; K_r] \leqq r\, v(\eta) + u(\eta, r)\,,$$

where $K_r = \psi^{-1}(\{z;\ |z - \eta| \leqq r\})$. Now it is sufficient to prove that

(3.20) $$\text{var}\,[\psi; K_r] = H_1(\hat{B} \cap \Omega(\eta, r))\,.$$

According to [13], theorem 1.1 we have

$$\text{var}\,[\psi; K_r] = H_1(\psi(K_r)) = H_1(B \cap \Omega(\eta, r))$$

(in the present case $N_\psi(z; K_r)$ from theorem 1.1 in [13] is equal to unity on $\psi(K_r)$ except at most at one point). Further we have $\hat{B} \subset B$. Prove $H_1(B - \hat{B}) = 0$. Taking into account theorem 1.17 from [13] we obtain that there exists $\tau_\psi(t)$ for var_ψ-almost all $t \in \langle\alpha, \beta\rangle$. By [13], theorem 1.4, $\text{var}\,[\psi; M] = 0$ for any $M \subset \langle\alpha, \beta\rangle$ if and only if $H_1(\psi(M)) = 0$. By lemma 3.5, $\hat{B}$ contains the set of all $z \in B$ for which there exists τ_ψ in $\psi^{-1}(z)$.

3.7 Remark. As (3.20) holds, it is

$$\sup_{r>0} \frac{H_1(\Omega(\eta, r) \cap \hat{B})}{r} < \infty \Rightarrow \sup_{r>0} \frac{u(\eta, r)}{r} < \infty .$$

If $v(\eta) < \infty$, then the converse of this implication holds by theorem 3.6. If $v(\eta) = \infty$, then the converse of this implication need not hold. This will be proved by the following example.

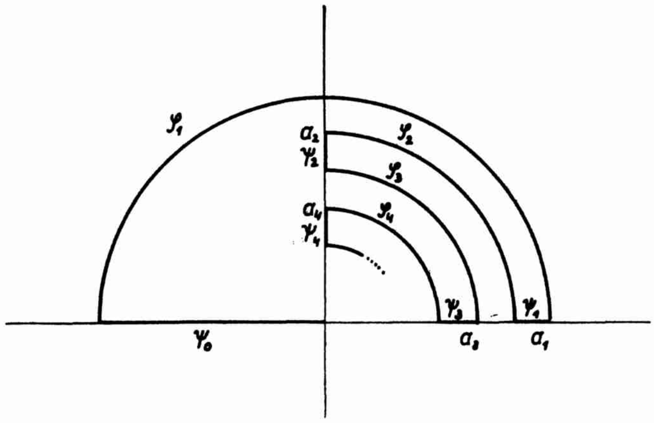

Fig. 2

Analogously to the remark 2.5 we construct a positively oriented Jordan curve φ as in fig. 2. (The figure is only a sketch.) Here we put $a_k = 1/k^2$ $(k = 1, 2, \ldots)$. The curve φ has a finite length and if $\eta = [0, 0]$ then $v(\eta) = \infty$. For $t > 1$ we have $M(t, \eta) = 0$ and for t with $0 < t < 1$, $t \neq a_k$, we have $M(t, \eta) = 2$, therefore

$$\sup_{r>0} \frac{u(\eta, r)}{r} = 2 .$$

Further

$$H_1(\Omega(\eta, a_k) \cap \hat{B}) \geqq \frac{\pi}{2} \sum_{n=k+1}^{\infty} a_n \geqq \frac{\pi}{2} \int_{k+2}^{\infty} \frac{\mathrm{d}x}{x^2} = \frac{\pi}{2} \frac{1}{k+2} .$$

Hence

$$\frac{H_1(\Omega(\eta, a_k) \cap \hat{B})}{a_k} \geqq \frac{\pi}{2} \frac{k^2}{k+2} \to \infty$$

as $k \to \infty$.

3.8 Remark. In [8] (cf. also [4]) it is proved that if $\eta \in B$, then the limit

(3.21) $$\lim_{\substack{z \to \eta \\ z \in H(\theta, \eta)}} W(f, z)$$