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EINE BENUTZUNG DER ZAHLENZERLEGUNG ZUR BESTIMMUNG DER ANZAHL UNISOMORPHER ZYKLEN IN ξ-TURNIEREN

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1.

In der ganzen Arbeit verstehen wir unter einer Zahl eine natürliche Zahl.

Bezeichnen wir $p_n(2n + 1, k)$ die Anzahl der Zerlegung der Zahl $2n + 1$ in k Zahlen, aus denen keine grösser als die Zahl n ist.

Weiter leiten wir die Formeln für die Anzahl der Zerlegungen $p_n(2n + 1, k)$ ab, wenn $k = 3, 4$ ist.

Satz 1. *Es sei $p_n(2n + 1,3)$ die Anzahl der Zerlegungen der Zahl $2n + 1$ in 3 Zahlen, die nicht grösser als die Zahl n sind. Dann gilt*

$$p_n(2n + 1, 3) = \sum_{k=0}^{[(n-1)/3]} \left[\frac{n + 1 - 3k}{2}\right]. \tag{1}$$

Beweis. Gestalten wir folgenderweise das Schema 1, in dem die Summe jeder Spalte $2n + 1$ ist:

$n\ n$	$\dots\ n$	$n-1\ \ n-1$	$\dots\ n-1$	$\dots$	$\dots\ h$
$n\ \ n-1$	$\dots\ r_1$	$n-1\ \ n-2$	$\dots\ r_2$	$\dots$	$\dots\ r_p$
$1\ 2$	$\dots\ s_1$	$3\ \ 4$	$\dots\ s_2$	$\dots$	$\dots\ s_p$

Schema 1

Dabei legen wir $n \geqq r_1 \geqq s_1$, $n - 1 \geqq r_2 \geqq s_2$, $h \geqq r_p \geqq s_p$, $r_1 + s_1 = n + 1$, $r_2 + s_2 = n + 2, \ldots$ und

$$r_p + s_p = \begin{cases} \frac{2}{3}(2n + 1), \text{ wenn } 2n + 1 \equiv 0 \pmod 3 \text{ ist}, \\ 2\left[\frac{2n + 1}{3}\right], \text{ wenn } 2n + 1 \equiv 1 \pmod 3 \text{ ist}, \\ 2\left[\frac{2n + 1}{3}\right] + 1, \text{ wenn } 2n + 1 \equiv 2 \pmod 3 \text{ ist}, \end{cases}$$

$$h = \begin{cases} \frac{2n + 1}{3}, \text{ wenn } 2n + 1 \equiv 0 \pmod 3 \text{ ist}, \\ \left[\frac{2n + 1}{3}\right] + 1, \text{ wenn } 2n + 1 \equiv 0 \pmod 3 \text{ ist} \end{cases}$$

legen.

Die Anzahl der Spalten im Schema 1 ist der Zahl $p_n(2n + 1, 3)$ gleich. Wir überzeugen uns leicht aus dem Schema 1, dass für eine beliebige Zahl n

$$p_n(2n + 1, 3) = \left[\frac{n + 1}{2}\right] + \left[\frac{n - 2}{2}\right] + \left[\frac{n - 5}{2}\right] + \ldots + \left[\frac{n + 1 - 3\left[\frac{n - 1}{3}\right]}{2}\right]$$

gilt.

Damit ist der Beweis des Satzes 1 beendet.

Es ist ersichtlich, dass $[a_1] + \ldots + [a_k] \leqq a_1 + \ldots + a_k$ für beliebige Zahlen $a_1, \ldots, a_k$ gilt. Darum ist:

$$p_n(2n + 1, 3) \leqq \frac{n + 1}{2} + \frac{n + 1 - 3 \,.\, 1}{2} + \frac{n + 1 - 3 \,.\, 2}{2} + \frac{n + 1 - 3 \,.\, 3}{2} + \ldots +$$

$$+ \frac{n + 1 - 3 \,.\left(\frac{n - 1}{3}\right)}{2} = \frac{1}{2}\left\{(n + 1)\left(\frac{n - 1}{3} + 1\right) - 3\left(1 + 2 + \ldots + \frac{n - 1}{3}\right)\right\} =$$

$$= \frac{1}{6}\binom{n + 3}{2}.$$

Wenn wir weiter überlegen, dass $m - 1 \leqq [m]$ für jede Zahl m gilt und dass in der Zahlenfolge $n + 1, n - 2, n - 5, \ldots$ gerade und ungerade Zahlen sich gegenseitig abwechseln, dann bekommen wir die untere Grenze für $p_n(2n + 1, 3)$, wenn wir von $\frac{1}{6}\binom{n+3}{2}$ die Zahl $\frac{1}{2}\left(\frac{n-1}{3}+1\right)$ subtrahieren. Es gilt also

$$\frac{1}{6}\binom{n+2}{2} \leqq p_n(2n+1,3) \leqq \frac{1}{6}\binom{n+3}{2}. \tag{2}$$

Satz 2. *Es sei $p_n(2n + 1, 4)$ die Anzahl der Zerlegungen der Zahl $2n + 1$ in 4 Zahlen, die nicht grösser als die Zahl n sind. Dann gilt*

$$p_n(2n+1,4) = \sum_{\substack{i=0 \\ 0 \leqq 3i+j \leqq n-2}} \sum_{j=0} \left[\frac{n-3i-j}{2}\right]. \tag{3}$$

Beweis. Gestalten wir das Schema 2, in dem die Summe der Elemente in jeder Spalte $2n + 1$ ist, folgenderweise:

$n \quad \dots\ n$	$n-1\ \dots\ n-1$	$\dots$	$\dots\ q$
$n-1\ \dots\ x_1$	$n-1\ \dots\ x_2$	$\dots$	$\dots\ x_p$
$1 \quad \dots\ y_1$	$2 \quad \dots\ y_2$	$\dots$	$\dots\ y_p$
$1 \quad \dots\ z_1$	$1 \quad \dots\ z_2$	$\dots$	$\dots\ z_p$

Schema 2

Wir legen dabei $n \geqq x_1 \geqq y_1 \geqq z_1$, $n - 1 \geqq x_2 \geqq y_2 \geqq z_2, \dots, q \geqq x_p \geqq y_p \geqq z_p$, $x_1 + y_1 + z_1 = n + 1$, $x_2 + y_2 + z_2 = n + 2, \dots, q + x_p + y_p + z_p = 2n + 1$.

Wenn wir die Ergebnisse des Schemas 1 für die zweite, dritte und vierte Spalte des Schemas 2 benutzen, bekommen wir

$$p_n(2n+1,4) = \left[\frac{n}{2}\right] + \left[\frac{n-3}{2}\right] + \dots + \left[\frac{n-3\,.\left[\frac{n-2}{3}\right]}{2}\right] +$$

$$+ \left[\frac{n-1}{2}\right] + \left[\frac{n-4}{2}\right] + \dots + \left[\frac{n-1-3\,.\left[\frac{n-3}{3}\right]}{2}\right] + \dots + \left[\frac{d}{2}\right],$$

wobei d eine der Zahlen 2, 3, 4 ist. Damit ist der Satz 2 bewiesen.

Aus (2) und (3) folgt dass

$$\frac{1}{6}\binom{n+1}{2} + \frac{1}{6}\binom{n}{2} + \dots + \frac{1}{6}\binom{3}{2} \leqq$$

$$\leqq p_n(2n+1,4) \leqq \frac{1}{6}\binom{n+2}{2} + \frac{1}{6}\binom{n+1}{2} + \dots + \frac{1}{6}\binom{4}{2},$$

ist, woraus wir nach einer Berechnung

(4) $$\frac{1}{6}\left\{\binom{n+2}{3}-1\right\} \leqq p_n(2n+1,4)=\frac{1}{6}\left\{\binom{n+3}{3}-4\right\},$$

bekommen.

In der Tabelle 1 werden die Zahlen $p_n(2n + 1, 3)$ und $p_n(2n + 1, 4)$ für $n =$ $= 1, 2, \ldots, 10$ angeführt.

Tab. 1

n	1	2	3	4	5	6	7	8	9	10
$p_n(2n + 1, 3)$	1	1	2	3	4	5	7	8	10	12
$p_n(2n + 1, 4)$	0	1	2	4	7	11	16	23	31	41

2.

A. Kotzig behandelt in der Arbeit [1] die Zyklen in Turnieren und führt eine Definition spezieller Turniere an, woher wir die folgende Definition übernehmen:

Definition 1. Es sei G ein Turnier mit m Knotenpunkten. Wir nennen G genau dann ein ξ-Turnier, wenn wir dessen Knotenpunkten so numerieren können, dass für jedes Paar der Indexen $i, j \in \{1, 2, \ldots, m\}$. Folgendes gilt: Wenn G die Kante $\overrightarrow{v_iv_j}$ enthält, dann enthält G auch die Kante $\overrightarrow{v_{i+1}v_{j+1}}$ (wobei wir $v_{m+1} = v_1$ legen).

Es ist ersichtlich, dass jedes ξ-Turnier eine ungerade Anzahl von Knotenpunkten enthält.

Es sei G ein ξ-Turnier mit $2n + 1$ Knotenpunkten. Nehmen wir die Kante $\overrightarrow{v_iv_j}$ des Graphes G. Benennen wir eine gleichzeitige Vergrösserung der beiden Indexe um eins, bei der wir aus der Kante $\overrightarrow{v_iv_j}$ die Kante $\overrightarrow{v_{i+1}v_{j+1}}$ bekommen die Umdrehung der Kante $\overrightarrow{v_iv_j}$ (dabei ist $v_{i+k(2n+1)} = v_i$ für $k = 1, 2, \ldots$). Wir bezeichnen die Kantenlänge $\overrightarrow{v_iv_j}$ ($\overrightarrow{v_jv_i}$) des Graphes G mit d_{ij} und wir definieren:

(6) $$d_{ij} = \min\left(|i-j|, 2n+1-|i-j|\right).$$

Bei so einer Definition der Kantenlänge gibt es nur Kanten der Länge $1, 2, \ldots, n$, wobei es in G genau $2n + 1$ Kanten der Länge i ($i = 1, 2, \ldots, n$) gibt. Es ist ersichtlich, dass die Kantenlänge bei der Umdrehung (auch bei der vielfachen Umdrehung) sich nicht wechselt.

Es sei C ein gegebener Zyklus des Graphes G. Eine Umdrehung des Zyklus C sei die gleichzeitige Umdrehung aller seiner Kanten genannt.

Definition 2. Unter der Länge des Zyklus C verstehen wir die Anzahl seiner Kanten.

Es sei G ein ξ-Turnier mit $2n + 1$ Knotenpunkten. Bezeichnen wir $S(G, k)$ die Summe der Kantenlängen des Zyklus der Länge k des Graphes G. Aus der Definition 2 und aus (6) folgt, dass

$$3 \leqq S(G, k) \leqq \binom{2n+1}{2} \quad \text{ist.}$$

Betrachten wir soein spezielles ξ-Turnier mit den Knotenpunkten $v_1, v_2, \ldots, v_{2n+1}$, das die Kanten

$$\text{(7)} \qquad \overrightarrow{v_1v_2}, \overrightarrow{v_1v_3}, \overrightarrow{v_1v_4}, \ldots, \overrightarrow{v_1v_{n+1}}$$

enthält und bezeichnen wir es mit G_1. Aus (7) und aus der Definition 1 folgt, dass der Graph G_1 genau folgende Kanten enthält:

$$\text{(8)} \qquad \begin{array}{l} \overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \ldots, \overrightarrow{v_{2n}v_{2n+1}}, \overrightarrow{v_{2n+1}v_1}\,, \\ \overrightarrow{v_1v_3}, \overrightarrow{v_2v_4}, \ldots, \overrightarrow{v_{2n}v_1}, \overrightarrow{v_{2n+1}v_2}\,, \\ \overrightarrow{v_1v_4}, \overrightarrow{v_2v_5}, \ldots, \overrightarrow{v_{2n}v_2}, \overrightarrow{v_{2n+1}v_3}\,, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \overrightarrow{v_1v_n}, \overrightarrow{v_2v_{n+1}}, \ldots, \overrightarrow{v_{2n}v_{n-2}}, \overrightarrow{v_{2n+1}v_{n-1}}\,, \\ \overrightarrow{v_1v_{n+1}}, \overrightarrow{v_2v_{n+2}}, \ldots, \overrightarrow{v_{2n}v_{n-1}}, \overrightarrow{v_{2n+1}v_n}\,. \end{array}$$

Aus (6) folgt, dass alle Kanten in der ersten Zeile die Länge 1, der in zweiten Zeile die Länge 2, usw. bis in der n-en Zeile die Länge n haben.

Lemma 1. *Es seinen k, p solche ganzen Zahlen, für die $1 \leqq p \leqq n$, $3 \leqq k \leqq$ $\leqq 2n + 1$ gilt. Es sei G_1 ein ξ-Turnier, das die Kanten (7) enthält. Bezeichnen wir $S(G_1, k)$ die Summe der Kantenlängen von einem beliebigen Zyklus der Länge k des Graphes G_1. Dann gilt*

$$\text{(9)} \qquad S(G_1, k) = p(2n + 1)\,.$$

Der Beweis folgt aus (6) und (8). Ohne Einschränkung der Allgemeinheit genügt es die Summe der Kantenlängen der Zyklen der Länge k zu untersuchen, welche Zyklen durch einen fest ausgewählten Knotenpunkt gehen (z. B. v_1).

Bemerkung 1. Auf dem Grund von (8) überzeugen wir uns leicht, dass $S(G_1, 3) =$ $= 2n + 1$, $S(G_1, 4) = 2n + 1$ ist.

Satz 3. *Es sei G_1 ein ξ-Turnier mit $2n + 1$ Knotenpunkten, welches die Kanten (8) enthält. Es seien $p_n(2n + 1, 3)$ und $p_n(2n + 1, 4)$ die Zahlen aus den Sätzen 1 und 2. Bezeichnen wir $Q(G_1, k)$ die Anzahl der unisomorphen Zyklen des Graphes G_1*

der Länge k. Dann gilt

$$p_n(2n + 1, 3) \leqq Q(G_1, 3) < 2p_n(2n + 1, 3), \tag{10}$$

$$p_n(2n + 1, 4) \leqq Q(G_1, 4) < 6p_n(2n + 1, 4). \tag{11}$$

Beweis. Es sei G_1 so ein ξ-Turnier, welches die Kanten (8) enthält. Es sei $C_3 = (v_{i_1}, v_{i_1}v_{i_2}, v_{i_2}, v_{i_2}v_{i_3}, v_{i_3}, v_{i_3}v_{i_1}, v_{i_1})$ ein Zyklus des Graphes G_1 der Länge 3. Es ist ersichtlich, dass $\{i_1, i_2, i_3\} \subset \{1, 2, \ldots, 2n + 1\}$ ist. Die Kantenlängen des Zyklus C_3 bilden nach (6) die Zahlenfolge $\{d_{i_1 i_2}, d_{i_2 i_3}, d_{i_3 i_1}\}$, für die es gilt, dass deren jedes Glied eine ganze positive Zahl nicht grösser als n ist und dass die Summe aller 3 Glieder $2n + 1$ ist; also, die angeführte Zahlenfolge schafft die Zerlegung der Zahl $2n + 1$ in 3 Zahlen, die nicht grösser als n sind. Darum bekommen wir auf Grund des Eingeführten und der Bemerkung 1 die Anzahl $Q(G_1, k)$ aller unisomorphen Zyklen des Graphen G_1 der Länge 3, wenn wir alle Zerlegungen der Zahl $2n + 1$ in 3 Zahlen nicht grösse als n finden und wenn wir aus diesen Zerlegungen alle verschiedene unzyklische Permutation bilden. Wenn alle drei oder zwei Zahlen in dieser Zerlegung verschiedenen sind, dann bekommen wir eine unzyklische Permutation. In diesem Fall ist $Q(G_1, 3) = p_n(2n + 1, 3)$. Wenn alle drei Zahlen in der angeführten Zerlegung verschieden sind, dann existieren zwei unzyklische Permutationen von diesen Zahlen und es ist $Q(G_1, 3) = 2p_n(2n + 1, 3)$. Für $k = 4$ durchläuft der Beweis analogisch ausser dem Fall, dass $Q(G_1, 4) = p_n(2n + 1, 4)$ für die Zerlegung der Zahl $2n + 1$ in 4 gleiche oder drei gleiche Zahlen ist; in dem Fall von zwei verschiedenen und zwei gleichen Zahlen in derselben Zerlegung ist $Q(G_1, 4) = 3p_n(2n + 1, 4)$. Endlich in dem Fall, wenn alle Zahlen in der Zerlegung verschieden sind, ist $Q(G_1, 4) = 6p_n(2n + 1, 4)$. Damit haben wir den Satz 4 bewiesen.

In der Tabelle 2 sind die Zahlen $Q(G_1, 3)$ und $Q(G_1, 4)$ angeführt, wenn G_1 ein ξ-Turnier ist, welches 3, 5, 7, 11, 13 und 15 Knotenpunkten enthält.

Tab. 2

Anzahl der Knotenpunkten des Graphen G_1	3	5	7	9	11	13	15
$Q(G_1, 3)$	1	1	2	4	5	7	10
$Q(G_1, 4)$	0	1	4	10	20	35	56

Literatur

[1] *Kotzig A.:* Über Zyklen in Turnieren, Beiträge zur Graphentheorie, Leipzig 1968, 85–89.

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NON-TANGENTIAL LIMITS OF THE DOUBLE LAYER POTENTIALS

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INTRODUCTION

We shall first introduce some fundamental notations, notions and theorems that will be used later.

Let G be a fixed Borel set in the Euclidean m-space R^m, $m \geqq 2$, and suppose that the boundary B of G is compact. Let the points of R^m be identified with m-dimensional vectors. For each $x, y \in R^m$ denote by xy the scalar product of the vectors x, y; denote by $|x|$ the Euclidean norm of the vector x. Further define, for any $y \in R^m$ and $r > 0$,

$$\Omega(y, r) = \{x \in R^m;\ |x - y| < r\}\ ;$$

the boundary of $\Omega(0, 1)$ denote by Γ. For a natural number α, $\alpha \leqq m$, denote by H_α the Hausdorff α-dimensional measure. Put

$$d_M(y) = \lim_{r \to 0+} \frac{H_m(\Omega(y, r) \cap M)}{H_m(\Omega(y, r))}$$

for any Borel set $M \subset R^m$ provided the limit exists. $d_M(y)$ is called the m-dimensional density of the set M at the point y. The vector $\Theta \in \Gamma$ is called the exterior normal of G at the point $y \in R^m$ in the sense of Federer provided the symmetric difference of G and the half-space

$$\{x \in R^m;\ (x - y)\,\Theta < 0\}$$

has m-dimensional density 0 at y. Since at every point $y \in R^m$ there exists at most one exterior normal in the sense of Federer, we may define a vector-valued function $n(y)$ in this way: we put $n(y) = \Theta$ if there is the exterior normal Θ at y; otherwise $n(y)$ equals the zero vector. Let $\hat{B}$ stand for the reduced boundary of G, i.e. the set of all $y \in R^m$ with $n(y) \neq 0$ (always $\hat{B} \subset B$). It follows from [3], theorem 4.5 that $n(y)$ is a Baire function; in particular, $\hat{B}$ is a Borel set.

$P(G)$ will denote the perimeter of G defined by

$$P(G) = \sup_v \int_G \operatorname{div} v(x) \, dx ,$$

where v ranges over all m-dimensional infinitely differentiable vector-valued functions with compact supports in R^m, satisfying $|v(x)| \leqq 1$ for each $x \in R^m$. In what follows we shall assume

$$(0.1) \qquad P(G) < \infty .$$

Then (cf. [5]) $H_{m-1}(\hat{B}) < \infty$.

For any $\Theta \in \Gamma$ and $z \in R^m$ put

$$H(\Theta, z) = \{z + r\Theta; r > 0\}, \quad \mathscr{S}(z) = \{H(\Theta, z); \ \Theta \in \Gamma\} .$$

A point $y \in H(\Theta, z)$ is called a hit of $H(\Theta, z)$ on G provided both

$$H(\Theta, z) \cap G \cap \Omega(y, r) \quad \text{and} \quad (H(\Theta, z) - G) \cap \Omega(y, r)$$

have a positive H_1-measure for every $r > 0$. If $n(\Theta, z)$ denotes the total number of all the hits of $H(\Theta, z)$ on G, then according to [5], prop. 1.6 $n(\Theta, z)$ is a non-negative Baire function of the variable $\Theta \in \Gamma$. We may thus define a cyclic variation of G at the point z by

$$v(z) = \int_\Gamma n(\Theta, z) \, dH_{m-1}(\Theta) .$$

By [5], lemma 2.12 and with respect to assumption (0.1) we have

$$(0.2) \qquad v(z) = \int_B \frac{|n(y) (y - z)|}{|y - z|^m} \, dH_{m-1}(y)$$

for every $z \in R^m$. Since $H_{m-1}(\hat{B}) < \infty$ and for any fixed $z \notin B$ the integrand in (0, 2) is a bounded function, it is $v(z) < \infty$ (cf. also [5], lemma 2.9). Notice that $v(z) < \infty$ implies the existence of $d_G(z)$ (cf. [5], lemma 2.7).

Let C be a space of all continuous functions on B equiped with the supremum norm. Denote C^* the space of all linear continuous functionals on C. Elements of C^* may be interpreted as bounded measures with supports in B (cf. [1]). For $\mu \in C^*$ let μ^+, μ^- and $|\mu|$ be positive, negative and total variations of the measure μ respectively (cf. [1]). It is known that $\mu = \mu^+ - \mu^-$, $|\mu| = \mu^+ + \mu^-$ and the norm of μ equals $|\mu| (B)$. We define the integrability and measurability of functions and sets with respect to $\mu \in C^*$ in the same way as in [1].

If φ_M stands for the characteristic function of the set $M \subset R^m$, put, for a Borel set $A \subset B$, $\mu \mid A = \varphi_M \mu$ (for the multiplication of a measure by a function see [1]). For every $\mu \in C^*$ there exists a Borel set $A \subset B$ such that $\mu \mid A = \mu^+$, $\mu \mid (B - A) = \mu^-$. By [1], chap. V, § 5, part 7, corollary of theorem 13 there are actually two disjoint sets $M, N \subset B$ such that μ^+ is concentrated on M and μ^- is concentrated on N. Clearly the set M is μ-measurable (it is μ^+-measurable as $\mu^+(B - M) = 0$ and μ^--measurable as $\mu^-(M) = 0$). Thus there exists a Borel set $A \subset B$ such that $M \subset A$ and $|\mu| (A - M) = 0$. It is evident that A satisfies the above requirements.

Let $\mathscr{B}$ be the system of all bounded Baire functions on B. Assuming

$$v(z) < \infty \,, \tag{0.3}$$

we define the double layer potential for each $f \in \mathscr{B}$, $z \in R^m$ by

$$W(f, z) = \int_B f(y) \frac{n(y)\,(y - z)}{|y - z|^m} \, \mathrm{d}H_{m-1}(y) \tag{0.4}$$

(cf. [5], lemma 2.12). Let $\mu \in C^*$. Then we define the double layer potential $W(\mu, z)$ for all $z \notin B$ and for $z \in B$ such that

$$\int_B \frac{|n(y)\,(y - z)|}{|y - z|^m} \, \mathrm{d}|\mu|\,(y) < \infty \,, \tag{0.5}$$

by

$$W(\mu, z) = \int_B \frac{n(y)\,(y - z)}{|y - z|^m} \, \mathrm{d}\mu(y) \,. \tag{0.6}$$

For $M \subset R^m$, $y \in R^m$ let us call the contingent of M at y and denote by contg (M, y) the system of all half-lines $H(\Theta, y) \in \mathscr{S}(y)$ for which there is a sequence of points $y_n \in M$ $(n = 1, 2, \ldots)$ with $y_n \neq y$, $y_n \to y$ and

$$\lim_{n \to \infty} \frac{y_n - y}{|y_n - y|} = \Theta \,.$$

Obviously, contg $(M, y) \neq \emptyset$ if and only if y is an accumulation point of M.

Now we prove the following statement which will be needed later.

0.1 Proposition. *Let* $M \subset R^m$, $S \subset R^m$, $\eta \in R^m$ *and*

$$\operatorname{contg}(M, \eta) \cap \operatorname{contg}(S, \eta) = \emptyset \,.$$

Then there are $a > 0$, $\delta > 0$ *such that*

$$(M \cap S \cap \Omega(\eta, \delta)) - \{\eta\} = \emptyset \tag{0.7}$$

and if dist (y, M) *denotes the distance of the point* y *from the set* M, *then*

$$\operatorname{dist}(y, M) \geqq a|y - \eta| \tag{0.8}$$

holds for each $y \in S \cap \Omega(\eta, \delta)$.

Proof. The relation (0.7) follows from (0.8). Obviously, the statement is true in the case $y \notin \bar{S} \cap \bar{M}$.

If the statement (0.8) were false, we could find, for any sequence $\{a_n\}_{n=1}^{\infty}$ with $0 < a_n < 1$, $a_n \to 0$, two sequences $\{y_n\}_{n=1}^{\infty}$, $\{z_n\}_{n=1}^{\infty}$ with $y_n \in S \cap \Omega(\eta, a_n) - \{\eta\}$, $z_n \in M$ and

$$|y_n - z_n| < a_n |y_n - \eta| = a_n r_n ,$$

where $|y_n - \eta| = r_n$. Putting $|z_n - \eta| = \bar{r}_n$, we get

$$r_n - a_n r_n \leqq \bar{r}_n \leqq r_n + a_n r_n .$$

Further

$$0 \leqq \left| \frac{z_n - \eta}{|z_n - \eta|} - \frac{y_n - \eta}{|y_n - \eta|} \right| \leqq \frac{|z_n - y_n|}{\bar{r}_n} + \left| \frac{y_n - \eta}{\bar{r}_n} - \frac{y_n - \eta}{r_n} \right| \leqq \tag{0.9}$$

$$\leqq \frac{a_n r_n}{\bar{r}_n} + r_n \frac{|r_n - \bar{r}_n|}{r_n \bar{r}_n} \leqq 2 \frac{a_n}{1 - a_n} \to 0$$

as $n \to \infty$. Since the sequence $\{(z_n - \eta)/|z_n - \eta|\}_{n=1}^{\infty}$ is a sequence of points of the compact set Γ, there is a convergent subsequence; we may assume it to have been already extracted. This implies

$$\lim_{n \to \infty} \frac{z_n - \eta}{|z_n - \eta|} = \Theta \in \Gamma .$$

On the other hand, by (0.9) also

$$\lim_{n \to \infty} \frac{y_n - \eta}{|y_n - \eta|} = \Theta .$$

Hence $H(\Theta, \eta) \in \operatorname{contg}(M, \eta) \cap \operatorname{contg}(S, \eta)$ which is the desired contradiction.

The preceding proposition implies that for $\eta \in B$ with $H(\Theta, \eta) \notin \operatorname{contg}(B, \eta)$ a $\delta > 0$ may be found such that the set

$$S = \{\eta + r\Theta;\ 0 < r < \delta\}$$

is included either in the interior of G or in $R^m - G$. Denoting for $\alpha \in \{0, \frac{1}{2}, 1\}$

$$G_\alpha = \{x \in R^m;\ d_G(x) = \alpha\} ,$$

then obviously $G_{1/2} \subset B$, $G_1 \subset \bar{G}$, $R^m - \bar{G} \subset G_0$. We have $S \subset G_1$ or $S \subset G_0$. Further $\hat{B} \subset G_{1/2}$ and by [5], lemma 3.7

$$H_{m-1}(G_{1/2} - \hat{B}) = 0 .$$

In the end let us make a note that the Hausdorff measure of a set is an invariant of the motion (i.e. a translation and a rotation) in R^m. Then also the quantities $v(x)$, $d_G(x)$, $W(f, x)$ are invariants of the motion, as well as the existence of the exterior normal in the sense of Federer; so for example the reduced boundary of the set after a motion is equal to the reduced boundary of the original set G, subjected to the motion.

1.

Recall that the symbol G denotes a fixed Borel set in R^m, $m \geqq 2$ with a compact boundary B and with a finite perimeter.

Now we shall prove this statement:

1.1 Proposition. *Let $S \subset R^m - B$, $\eta \in \bar{S} \cap B$. Then*

$$\limsup_{\substack{x \to \eta \\ x \in S}} W(f, x) < \infty \tag{1.1}$$

holds for every function $f \in C$ (or for every $f \in \mathscr{B}$) if and only if

$$\limsup_{\substack{x \to \eta \\ x \in S}} v(x) < \infty\,. \tag{1.2}$$

If, moreover, there is $\delta > 0$ such that

$$S \cap \Omega(\eta, \delta) \subset G_i \tag{1.3}$$

holds for $i = 0$ or $i = 1$, then the limit

$$\lim_{\substack{x \to \eta \\ x \in S}} W(f, x) \tag{1.4}$$

exists for each function $f \in C$ (or for each $f \in \mathscr{B}$ continuous at the point η) if and only if (1.2) holds. The value of the limit (1.4) is then given by

$$W(f, \eta) + f(\eta)\, H_{m-1}(\Gamma)\,(i - d_G(\eta))\,. \tag{1.5}$$

Proof. First we shall prove that the condition (1.2) is necessary and sufficient for (1.1) to be true for each $f \in C$. If this were false, we could find $x_k \in S$ $(k = 1, 2, \ldots)$, $x_k \to \eta$, $v(x_k) \to \infty$. The point $x \in R^m$ being fixed, the quantity $W(f, x)$ determines a linear functional on the space C, whose norm is equal to $v(x)$ (cf. [5], relation (2.5)). It follows from (1.1) by Banach-Steinhaus theorem that there are two numbers k_0 and c such that $v(x_k) \leqq c$ for each $k > k_0$. This is the desired contradiction.

Let (1.2) hold. Hence we have $v(\eta) < \infty$ as the function $v(x)$ is lower semicontinuous with respect to $x \in R^m$ according to the statement 2.9 in [5]. Further, this implies that the density $d_G(\eta)$ at the point η exists (cf. [5], lemma 2.7).

. Taking into account (0.2) and (0.4), we get that the condition (1.1) is satisfied for each function $f \in \mathscr{B}$. Now suppose that (1.3) holds and prove the existence of the limit (1.4) for any $f \in \mathscr{B}$ continuous at the point η. According to (1.2) there is δ_1, $0 < \delta_1 < \delta$ such that

$$c = \sup \{v(x);\ x \in S \cap \Omega(\eta, \delta_1)\} < \infty .$$

From the lower semicontinuity of $v(x)$ we obtain

$$c = \sup \{v(x);\ x \in \bar{S} \cap \Omega(\eta, \delta_1)\} .$$

First assume that $f(x) = 1$ for all $x \in B$. This (by [5], lemma 2.5, provided $v(z) < \infty$) implies

$$W(f, z) = H_{m-1}(\Gamma)\, d_G(z)$$

if G is bounded and

$$W(f, z) = H_{m-1}(\Gamma)\,(1 - d_G(z))$$

if G is unbounded. By the assumption (1.3) just one of the following cases occurs: either $d_G(z) = 1$ for each $x \in S \cap \Omega(\eta, \delta)$ or $d_G(z) = 0$ for each $x \in S \cap \Omega(\eta, \delta)$. Moreover, comparing the values $W(f, \eta)$ and $W(f, z)$ for $z \in S \cap \Omega(\eta, \delta)$, we arrive at

$$\lim_{\substack{x \to \eta \\ x \in S}} W(f, x) = W(f, \eta) + H_{m-1}(\Gamma)\,(i - d_G(\eta)) .$$

Now let $f \in \mathscr{B}$, f continuous at the point η and $f(\eta) = 0$. Certainly there exists a function h continuous on R^m such that $0 \leqq h \leqq 1$, $h(x) = 1$ for each $x \in \Omega(0, \frac{1}{2})$ and $h(x) = 0$ for each $x \in R^m - \Omega(0, 1)$. Putting

$$g_r(x) = f(x)\, h\left(\frac{1}{r}(x - \eta)\right), \quad f_r(x) = f(x) - g_r(x)$$

for any $r > 0$, we have $g_r(x) = 0$ on $B - \Omega(\eta, r)$ and

$$\lim_{r \to 0+} \sup \{|g_r(x)|;\ x \in B\} = 0 .$$

Since $f_r(x) = 0$ on $B \cap \Omega(\eta, r/2)$, the function $W(f_r, x)$ is continuous on $\Omega(\eta, r/2)$. To prove

$$\lim_{\substack{x \to \eta \\ x \in S}} W(f, x) = W(f, \eta) ,$$

we shall prove that $W(g_r, x)$ tends to zero uniformly on $\bar{S} \cap \Omega(\eta, \delta_1)$ as $r \to 0+$. This will be sufficient because

$$W(f, x) = W(f_r, x) + W(g_r, x)$$

holds on $\bar{S} \cap \Omega(\eta, \delta_1)$. We have for each $x \in \bar{S} \cap \Omega(\eta, \delta_1)$

$$|W(g_r, x)| = \left| \int_B g_r(y) \frac{n(y)(y - x)}{|y - x|^m} \, \mathrm{d}H_{m-1}(y) \right| \leqq$$
$$\leqq \sup \{|g_r(z)|;\, z \in B\} \int_B \frac{|n(y)(y - x)|}{|y - x|^m} \, \mathrm{d}H_{m-1}(y) \leqq$$
$$\leqq c \sup \{|g_r(z)|;\ z \in B\} \to 0$$

as $r \to 0+$. If now $f \in \mathscr{B}$, f continuous at the point η, we may express this function f in the form of a sum of two functions, a constant function on B and a function lying in $\mathscr{B}$ continuous and vanishing at η. As $W(f, x)$ for a fixed x is linear with respect to f, the proof is complete.

Now we shall establish conditions for the validity of (1.2). Let us prove first the following auxiliary statement.

1.2 Lemma. *Let* $S \subset R^m - B$, $\eta \in \bar{S} \cap B$,

$$\operatorname{contg}(S, \eta) \cap \operatorname{contg}(\hat{B}, \eta) = \emptyset$$

and suppose

$$\sup_{r>0} \frac{H_{m-1}(\Omega(\eta, r) \cap \hat{B})}{r^{m-1}} = k < \infty .$$

Then there are $\delta > 0$, $c < \infty$ *such that for each* $z \in S \cap \Omega(\eta, \delta)$ *and each* $r > 0$

$$\frac{H_{m-1}(\Omega(z, r) \cap \hat{B})}{r^{m-1}} \leqq c . \tag{1.6}$$

Proof. Proposition 0.1 implies that there are $\delta > 0$, $a > 0$ such that for every $z \in S \cap \Omega(\eta, \delta)$

$$\operatorname{dist}(z, \hat{B}) \geqq a|z - \eta| . \tag{1.7}$$

Put $r_1 = |z - \eta|$ and $r = r_1 b$ for $b > 0$. Certainly the relation (1.6) holds for that r for which its corresponding value b satisfies $b < a$ because in that case $\Omega(z, r) \cap \hat{B} = \emptyset$ and thus also $H_{m-1}(\Omega(z, r) \cap \hat{B}) = 0$. For that r for which its corresponding value b satisfies $b \geqq a$ we have the following estimate:

$$\frac{H_{m-1}(\Omega(z, r) \cap \hat{B})}{r^{m-1}} \leqq \frac{H_{m-1}(\Omega(\eta, r_1 + r) \cap \hat{B})}{r^{m-1}} =$$
$$= \frac{H_{m-1}(\Omega(\eta, (1 + b) r_1) \cap \hat{B})}{(r_1(1 + b))^{m-1}} \frac{(1 + b)^{m-1}}{b^{m-1}} \leqq k \frac{(1 + b)^{m-1}}{b^{m-1}} \leqq k \frac{(1 + a)^{m-1}}{a^{m-1}} .$$

Now it is sufficient to put $c = k[(1 + a)^{m-1}/a^{m-1}]$.

1.3 Theorem. *Let* $S \subset R^m - B$, $\eta \in \bar{S} \cap B$ *and*

$$\operatorname{contg}(S, \eta) \cap \operatorname{contg}(\hat{B}, \eta) = \emptyset .$$

Further suppose

$$v(\eta) + \sup_{r>0} \frac{H_{m-1}(\Omega(\eta, r) \cap \hat{B})}{r^{m-1}} < \infty . \tag{1.8}$$

Then

$$\limsup_{\substack{z \to \eta \\ z \in S}} v(z) < \infty . \tag{1.9}$$

Proof. By the statements 0.1 and 1.2 we determine the constants a, δ, c such that (1.7) and (1.6) hold in the corresponding set. Further we fix a point z and denote $r = |z - \eta|$, $M = \hat{B} \cap \Omega(z, 2r)$, $N = \hat{B} - \Omega(z, 2r)$. Using the triangular inequality and the fundamental properties of the integral, we obtain the estimate

$$\begin{aligned} v(z) \leqq \int_M \frac{|n(y)(y - z)|}{|y - z|^m} \, \mathrm{d}H_{m-1}(y) + \int_N \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, \mathrm{d}H_{m-1}(y) + \\ + \int_N \left| \frac{|n(y)(y - z)|}{|y - z|^m} - \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \right| \mathrm{d}H_{m-1}(y) . \end{aligned} \tag{1.10}$$

Now we number the quantities on the right-hand side of this inequality I, II, III respectively. Then we get

$$\mathrm{I} \leqq \frac{H_{m-1}(\Omega(z, 2r) \cap \hat{B})}{(ar)^{m-1}} \leqq \frac{2^{m-1}}{a^{m-1}} \, c , \quad \mathrm{II} \leqq v(\eta) .$$

To estimate III, we use

$$\int_{R^m} f(x) \, \mathrm{d}\mu(x) = \int_0^\infty \mu(\{x \in R^m; f(x) > t\}) \, \mathrm{d}t ,$$

where μ is a Borel measure and f is a non-negative, μ-integrable function on R^m. The last relation follows from [11] (there only non-negative measures are considered; in the present case we first decompose μ to the difference of the positive and the negative variations). There is $\Theta \in \Gamma$ such that $z = \eta + r\Theta$ so that we obtain

$$\begin{aligned} \left| \frac{|n(y)(y - z)|}{|y - z|^m} - \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \right| &\leqq \left| \frac{n(y)(y - z)}{|y - z|^m} - \frac{n(y)(y - \eta)}{|y - \eta|^m} \right| = \\ = \left| \frac{|y - \eta|^m - |y - z|^m}{|y - \eta|^m \, |y - z|^m} \, n(y)(y - \eta) - r \, n(y) \, \Theta \, \frac{1}{|y - z|^m} \right| &\leqq \\ \leqq \frac{\left| |y - \eta|^m - |y - z|^m \right|}{|y - \eta|^m \, |y - z|^m} \, |n(y)(y - \eta)| + r \frac{1}{|y - z|^m} . \end{aligned}$$

Using the substitution $t^{-1/m} = x$ and lemma 1.2; we obtain the following estimate:

$$r\int_N \frac{dH_{m-1}(y)}{|y-z|^m} = r\int_0^\infty H_{m-1}\left(N \cap \left\{x \in R^m; \frac{1}{|x-z|^m} > t\right\}\right) dt =$$

$$= r\int_0^{(2r)^{-m}} H_{m-1}(\hat{B} \cap \Omega(z, t^{-1/m}))\, dt = rm\int_{2r}^\infty \frac{H_{m-1}(\hat{B} \cap \Omega(z, x))}{x^{m+1}}\, dx \leqq$$

$$\leqq crm\int_{2r}^\infty \frac{dx}{x^2} = \frac{c}{2} m .$$

Since for $y \in N$

$$|y - \eta| \leqq 2|y - z| ,$$

it is also

$$||y-\eta|^m - |y-z|^m| \leqq |y-\eta|^m + |y-z|^m \leqq (1 + 2^m)\,|y-z|^m .$$

Thus we have

$$\int_N \frac{||y-\eta|^m - |y-z|^m|}{|y-z|^m\,|y-\eta|^m}\, |n(y)\,(y-\eta)|\, dH_{m-1}(y) \leqq$$

$$\leqq (1 + 2^m)\int_N \frac{|n(y)\,(y-\eta)|}{|y-\eta|^m}\, dH_{m-1}(y) \leqq (1 + 2^m)\, v(\eta) .$$

Finally, we conclude that

$$v(z) \leqq c\left(\frac{2^{m-1}}{a^{m-1}} + \frac{m}{2}\right) + v(\eta)\,(2 + 2^m) .$$

Theorem 1.3 may be converted in this manner:

1.4 Theorem. *Let $\eta \in B$ and suppose that there are linearly independent vectors $\Theta_i \in \Gamma$ $(i = 1, \ldots, m)$ and a number $\delta > 0$ such that*

$$\sup\left\{v(z);\, z \in \bigcup_{i=1}^m H(\Theta_i, \eta) \cap \Omega(\eta, \delta)\right\} = c < \infty . \tag{1.11}$$

Then

$$\sup_{r>0} \frac{H_{m-1}(\Omega(\eta, r) \cap \hat{B})}{r^{m-1}} < \infty . \tag{1.12}$$

Proof. Assume that $\eta = 0$, $\delta \leqq 1$ and let Θ_i $(i = 1, \ldots, m)$ be linearly independent vectors. Then there is $b > 0$ such that for each $y \in \Omega(\eta, 2b)$ the vectors $(y - \Theta_i)$ are linearly independent. There is $d > 0$ such that

$$\sum_{i=1}^m |u(y - \Theta_i)| \geqq d$$

holds for each $y \in \Omega(\eta, b)$ and each $u \in \Gamma$. Obviously $b \leqq 1$ and thus $|y - \Theta_i| \leqq 2$. Hence

$$\sum_{i=1}^{m} \frac{|u(y - \Theta_i)|}{|y - \Theta_i|^m} \geqq \frac{1}{2^m} d .$$

Let now $0 < r < b\delta$ and consider $y \in \Omega(\eta, r) \cap \hat{B}$. Then we have

$$1 \leqq 2^m d^{-1} \sum_{i=1}^{m} \frac{\left| n(y) \left(\frac{b}{r} y - \Theta_i \right) \right|}{\left| \frac{b}{r} y - \Theta_i \right|^m} = \tag{1.13}$$

$$= r^{m-1} \, . \, 2^m \frac{1}{db^{m-1}} \sum_{i=1}^{m} \frac{\left| n(y) \left(y - \frac{r}{b} \Theta_i \right) \right|}{\left| y - \frac{r}{b} \Theta_i \right|^m} .$$

If we integrate the inequality (1.13) on the set $\hat{B} \cap \Omega(\eta, r)$ with respect to H_{m-1}, we obtain for each r, $0 < r < b\delta$

$$H_{m-1}(\Omega(\eta, r) \cap \hat{B}) \leqq \tag{1.14}$$

$$\leqq r^{m-1} \, . \, 2^m d^{-1} b^{1-m} \sum_{i=1}^{m} v\left(\frac{r}{b} \Theta_i\right) \leqq r^{m-1} m \, . \, 2^m d^{-1} b^{1-m} c .$$

Since $H_{m-1}(\hat{B}) < \infty$, (1.12) follows from (1.14).

1.5 Remark. The assumptions of theorem 1.4 are satisfied for example whenever $\eta \in B$ and there are $\Theta' \in \Gamma$, $\delta > 0$ such that

$$\limsup_{\substack{z \to \eta \\ z \in H(\Theta, \eta)}} v(z) < \infty \tag{1.15}$$

holds for each $\Theta \in \Gamma$ with $|\Theta - \Theta'| < \delta$. That last assumption is satisfied for example whenever $\operatorname{contg}(\hat{B}, \eta) \neq \mathscr{S}(\eta)$ (or $\operatorname{contg}(G_{1/2}, \eta) \neq \mathscr{S}(\eta)$ or $\operatorname{contg}(B, \eta) \neq \mathscr{S}(\eta)$) and (1.15) holds for each $\Theta \in \Gamma$ with $H(\Theta, \eta) \notin \operatorname{contg}(\hat{B}, \eta)$ (or $H(\Theta, \eta) \notin \operatorname{contg}(G_{1/2}, \eta)$ or $H(\Theta, \eta) \notin \operatorname{contg}(B, \eta)$).

Let us make still a note that theorem 1.3 holds also when we write in its assumptions $\operatorname{contg}(G_{1/2}, \eta)$ or $\operatorname{contg}(B, \eta)$ instead of $\operatorname{contg}(\hat{B}, \eta)$.

Taking into account the preceding remark, proposition 1.1 and theorems 1.3 and 1.4, we obtain immediately the following theorem.

1.6 Theorem. *Let $\eta \in B$. Then there is a finite limit*

$$\lim_{\substack{z \to \eta \\ z \in H(\Theta, \eta)}} W(f, z) \tag{1.16}$$

for each $f \in C$ (*or each* $f \in \mathscr{B}$ *continuous at the point* η) *and for each half-line* $H(\Theta, \eta) \notin \operatorname{contg}(B, \eta)$, *if and only if* (1.8) *holds* (*provided* $\operatorname{contg}(B, \eta) \neq \mathscr{S}(\eta)$). *If* $H(\Theta, \eta) \notin \operatorname{contg}(B, \eta)$, *then there exist* $\delta > 0$, $i \in \{0, 1\}$ *such that*

$$H(\Theta, \eta) \cap \Omega(\eta, \delta) \subset G_i$$

and whenever (1.8) *holds, then the value of the limit* (1.16) *is given by* (1.5).

In the case $m = 2$ we may change the suppositions of theorem 1.4 as follows.

1.7 Theorem. *Let* $m = 2$ *and* $\eta \in B$, $\Theta \in \Gamma$ *such that* $H(\Theta, \eta) \notin \operatorname{contg}(\hat{B}, \eta)$, $H(-\Theta, \eta) \notin \operatorname{contg}(\hat{B}, \eta)$. *If there is* $r_0 > 0$ *such that*

$$c = \sup \{v(z);\ z \in H(\Theta, \eta) \cap \Omega(\eta, r_0)\} < \infty\,, \tag{1.17}$$

then also

$$\sup_{r > 0} \frac{H_1(\hat{B} \cap \Omega(\eta, r))}{r} < \infty\,. \tag{1.18}$$

Proof. Suppose $\eta = 0$, $\Theta = [1, 0]$, $r_0 \leqq 1$. Choose r, $0 < r < r_0$ and $y \in \hat{B} \cap \Omega(\eta, r)$. Then there is $\beta \in \langle 0, 2\pi\rangle$ for which $y = |y|\,[\cos \beta, \sin \beta]$. Since neither $H(\Theta, \eta)$ nor $H(-\Theta, \eta)$ belong to $\operatorname{contg}(\hat{B}, \eta)$, we may find r', δ so that $r' > 0$, $0 < \delta < \frac{1}{2}\pi$, and

$$\beta \in (\delta, \pi - \delta) \cup (\pi + \delta, 2\pi - \delta) \tag{1.19}$$

for every $y \in \hat{B}$ with $|y| < r'$, $y = |y|\,[\cos \beta, \sin \beta]$. Further it may be supposed that $r_0 = r'$. Let $y \in \hat{B}$. Then there is $\alpha \in \langle 0, 2\pi\rangle$ such that

$$n(y) = [\cos \alpha, \sin \alpha]\,. \tag{1.20}$$

The rest of the proof will be divided into the following two parts:

a) $\quad \alpha \in \langle 0, \frac{1}{2}(\pi - \delta)\rangle \cup \langle \frac{1}{2}(\pi + \delta), \frac{3}{2}(\pi - \delta)\rangle \cup \langle \frac{3}{2}(\pi + \delta, 2\pi\rangle\,,$

b) $\quad \alpha \in (\frac{1}{2}(\pi - \delta), \frac{1}{2}(\pi + \delta)) \cup (\frac{3}{2}(\pi - \delta), \frac{3}{2}(\pi + \delta))\,.$

Put $z = [r, 0]$. It is easy to establish that

$$|n(y)\, y| + |n(y)\,(y - z)| \geqq r|\cos \alpha|\,. \tag{1.21}$$

In the case *a*) we may write $r \cos \frac{1}{2}(\pi - \delta)$ on the right-hand side of the inequality (1.21).

We have $|n(y)\, y| = |y|\,|\cos(\beta - \alpha)|$. In the case *b*), by (1.19) it is evident that $|n(y)\, y| \geqq |y| \cos \frac{1}{2}(\pi - \delta)$.

Together we obtain that

$$\frac{|n(y)\, y|}{|y|^2} + \frac{|n(y)\,(y - z)|}{|y - z|^2} \geqq \frac{\cos \frac{1}{2}(\pi - \delta)}{4r} \tag{1.22}$$

holds for each r, $0 < r \leqq r_0$, each $y \in \hat{B} \cap \Omega(\eta, r)$ and $z = [r, 0]$. It follows from the lower semicontinuity of $v(x)$ and from the assumption (1.17) that also $v(\eta) \leqq c$. If we integrate the inequality (1.22) on $\hat{B} \cap \Omega(\eta, r)$ (for r such that $0 < r \leqq r_0$) with respect to H_1, we arrive at

$$\frac{H_1(\Omega(\eta, r) \cap \hat{B})}{r} \leqq \frac{8c}{\cos \frac{1}{2}(\pi - \delta)}. \tag{1.23}$$

(1.18) is now a corollary of (1.23) and of $H_1(\hat{B}) < \infty$.

2.

Throughout this paragraph $G \subset R^m$ $(m \geqq 2)$ denotes again a Borel set with a compact boundary B and with a finite perimeter. Now we shall deal with double layer potential $W(\mu, z)$ for $\mu \in C^*$.

$D \in R^1$ will be termed the H_{m-1}-derivative on $\hat{B}$ of $\mu \in C^*$ at the point $\eta \in B$ (briefly the derivative at η) if for every $r > 0$

$$H_{m-1}(\hat{B} \cap \Omega(\eta, r)) > 0 \tag{2.1}$$

and if for each $\varepsilon > 0$ there is $\delta > 0$ such that

$$\left| \frac{\mu(M)}{H_{m-1}(M)} - D \right| < \varepsilon \tag{2.2}$$

holds for each Borel set $M \subset \hat{B} \cap \Omega(\eta, \delta)$ with $H_{m-1}(M) > 0$.

$D \in R^1$ will be termed the symmetric H_{m-1}-derivative on $\hat{B}$ of $\mu \in C^*$ at the point $\eta \in B$ (briefly the symmetric derivative at η) if there exists the limit

$$\lim_{r \to 0+} \frac{\mu(\Omega(\eta, r) \cap \hat{B})}{H_{m-1}(\Omega(\eta, r) \cap \hat{B})} = D. \tag{2.3}$$

(Note that in this definition also the assumption that (2.1) holds for each $r > 0$ is contained. This is valid, by [5], lemma 3.7, for each $\eta \in B$ with $|d_G(\eta) - \frac{1}{2}| < \frac{1}{2}$).

Obviously, if μ has the derivative at η, then there exists also the symmetric derivative of μ at η and their values are equal.

2.1 Lemma. *Let $\mu \in C^*$, $\eta \in B$, $S \subset R^m - B$,*

$$\operatorname{contg}(S, \eta) \cap \operatorname{contg}(\hat{B}, \eta) = \emptyset$$

and suppose that μ is a non-negative measure with the symmetric derivative on $\hat{B}$ at η equal to zero. Further suppose that (1.8) holds and that

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, d\mu(y) < \infty. \tag{2.4}$$

Then

$$\lim_{\substack{z \to \eta \\ z \in S}} W(\mu, z) = W(\mu, \eta) . \tag{2.5}$$

Proof. For $R > 0$ put $\lambda = \mu \mid \Omega(\eta, R)$, $\nu = \mu \mid (R^m - \Omega(\eta, R))$. We have $W(\mu, z) = W(\lambda, z) + W(\nu, z)$ for each $z \in R^m$ for which the left-hand side is defined. Analogously to the proof of the proposition 1.1, it is sufficient to prove that there is $\delta > 0$ such that

$$W(\lambda, z) \to 0$$

as $R \to 0+$ uniformly on $\{\eta\} \cup S \cap \Omega(\eta, \delta)$. For $z \in S$ denote $r = |z - \eta|$ and

$$M = \Omega(\eta, R) \cap \hat{B} - \Omega(\eta, 2r) , \quad N = \Omega(\eta, R) \cap \hat{B} \cap \Omega(\eta, 2r) .$$

We have

$$W(\lambda, z) = \int_M \frac{n(y)(y - z)}{|y - z|^m} \mathrm{d}\mu(y) + \int_N \frac{n(y)(y - z)}{|y - z|^m} \mathrm{d}\mu(y) . \tag{2.6}$$

Denote by I, II respectively the absolute values of the integrals on the right-hand side of (2.6). Applying the proposition 0.1 we find $a, \delta > 0$ such that

$$\operatorname{dist}(z, \hat{B}) \geqq a|z - \eta|$$

holds for each $z \in S \cap \Omega(\eta, \delta)$. If now $z \in S \cap \Omega(\eta, \delta)$, $|z - \eta| = r$, we arrive at

$$\mathrm{II} \leqq \frac{\mu(N)}{(ar)^{m-1}} \leqq \frac{2^{m-1}k}{a^{m-1}} \frac{\mu(N)}{H_{m-1}(\Omega(\eta, 2r) \cap \hat{B})} ,$$

where

$$k = \sup_{r > 0} \frac{H_{m-1}(\Omega(\eta, r) \cap \hat{B})}{r^{m-1}} .$$

Since the symmetric derivative of μ vanishes at η, for each $\varepsilon > 0$ there is $\delta_1 > 0$ such that

$$\frac{\mu(\Omega(\eta, \varrho) \cap \hat{B})}{H_{m-1}(\Omega(\eta, \varrho) \cap \hat{B})} \leqq \varepsilon \frac{a^{m-1}}{2^{m-1}k}$$

for any ϱ, $0 < \varrho < \delta_1$. Hence

$$\mathrm{II} \leqq \varepsilon$$

for each R such that $0 < R < \delta_1$, as we have

$$\frac{2^{m-1}k}{a^{m-1}} \frac{\mu(N)}{H_{m-1}(\Omega(\eta, 2r) \cap \hat{B})} = \frac{2^{m-1}k}{a^{m-1}} \frac{\mu(\Omega(\eta, 2r) \cap \hat{B})}{H_{m-1}(\Omega(\eta, 2r) \cap \hat{B})} < \varepsilon$$

if $R \geqq 2r$ (then $0 < 2r < \delta_1$) and

$$\frac{2^{m-1}k}{a^{m-1}} \frac{\mu(N)}{H_{m-1}(\Omega(\eta, 2r) \cap \hat{B})} \leqq \frac{2^{m-1}k}{a^{m-1}} \frac{\mu(\Omega(\eta, R) \cap \hat{B})}{H_{m-1}(\Omega(\eta, R) \cap \hat{B})} < \varepsilon$$

if $R < 2r$.

This estimate is independent of $z \in S \cap \Omega(\eta, \delta)$.

Now estimate the expression I. We may consider only $z \in S \cap \Omega(\eta, \delta)$ with $2r < R$ (for a fixed R) because in the opposite case $M = \emptyset$ and thus $\mathrm{I} = 0$. Since

$$\int_{\Omega(\eta,\varrho)\cap B} \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \,\mathrm{d}\mu(y) \to 0$$

as $\varrho \to 0+$, it is sufficient to prove that

$$V(z) = \left|\int_M \left(\frac{|n(y)(y - z)|}{|y - z|^m} - \frac{|n(y)(y - \eta)|}{|y - \eta|^m}\right) \mathrm{d}\mu(y)\right| \to 0 \tag{2.7}$$

as $R \to 0+$ uniformly with respect to z on the set $S \cap \Omega(\eta, \delta)$. We have

$$V(z) \leqq (1 + 2^m) \int_M \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \,\mathrm{d}\mu(y) + \int_M \frac{r}{|y - z|^m} \,\mathrm{d}\mu(y) \tag{2.8}$$

(cf. an analogous estimate in the proof of theorem 1.3). Further

$$(1 + 2^m) \int_M \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \,\mathrm{d}\mu(y) \leqq (1 + 2^m) \int_{\Omega(\eta,R)\cap B} \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \,\mathrm{d}\mu(y) \to 0$$

as $R \to 0+$, where the last expression is independent of $z \in S \cap \Omega(\eta, \delta)$. Now estimate the expression II. Taking into account $|y - z| \geqq \frac{1}{2}|y - \eta|$ for $y \in M$, we arrive at

$$r \int_M \frac{\mathrm{d}\mu(y)}{|y - z|^m} \leqq 2^m r \int_M \frac{\mathrm{d}\mu(y)}{|y - \eta|^m}\,. \tag{2.9}$$

According to the proof of theorem 1.3, one obtains

$$r \int_M \frac{\mathrm{d}\mu(y)}{|y - \eta|^m} = r \int_0^\infty \mu\left(\left\{x \in M;\ \frac{1}{|y - \eta|^m} > u\right\}\right) \mathrm{d}u\,. \tag{2.10}$$

However,

$$\left\{x \in M;\ \frac{1}{|y - \eta|^m} > u\right\} = (\Omega(\eta, R) \cap \hat{B} - \Omega(\eta, 2r)) \cap \Omega(\eta, u^{-1/m})\,.$$

For $u \geqq (2r)^{-m}$ this set is empty and thus for these u the integrand on the right-hand side of (2.10) equals zero. For u such that $0 < u < R^{-m}$ this set is equal to M and thus for these u the integrand on the right-hand side of (2.10) equals $\mu(M)$. Now it is evident that

$$r \int_M \frac{\mathrm{d}\mu(y)}{|y - \eta|^m} = r \frac{\mu(M)}{R^m} + r \int_{R^{-m}}^{(2r)^{-m}} \mu(M \cap \Omega(\eta, u^{-1/m}))\,\mathrm{d}u\,. \tag{2.11}$$

The first term on the right-hand side of (2.11) may be estimated by

$$r\frac{\mu(M)}{R^m} \leqq \frac{k}{2}\frac{\mu(\Omega(\eta, R) \cap \hat{B})}{H_{m-1}(\Omega(\eta, R) \cap \hat{B})}. \tag{2.12}$$

By the substitution $t = u^{-1/m}$ in the second term on the right-hand side of (2.11) we obtain

$$r\int_{R^{-m}}^{(2r)^{-m}} \mu(M \cap \Omega(\eta, u^{-1/m}))\,\mathrm{d}u = \tag{2.13}$$

$$= mr\int_{2r}^{R} \frac{\mu((\hat{B} - \Omega(\eta, 2r)) \cap \Omega(\eta, t))}{t^{m+1}}\,\mathrm{d}t \leqq mrk\int_{2r}^{R} \frac{\mu(\Omega(\eta, t) \cap \hat{B})}{H_{m-1}(\Omega(\eta, t) \cap \hat{B})}\frac{\mathrm{d}t}{t^2} \leqq$$

$$\leqq mrk \sup_{x\in(0,R)} \frac{\mu(\Omega(\eta, x) \cap \hat{B})}{H_{m-1}(\Omega(\eta, x) \cap \hat{B})}\int_{2r}^{R}\frac{\mathrm{d}t}{t^2} \leqq \frac{mk}{2}\sup_{x\in(0,R)} \frac{\mu(\Omega(\eta, x) \cap \hat{B})}{H_{m-1}(\Omega(\eta, x) \cap \hat{B})}.$$

It follows from (2.13), (2.12), (2.11) and (2.9) that

$$r\int_M \frac{\mathrm{d}\mu(y)}{|y - z|^m} \leqq 2^{m-1}k(m + 1) \sup_{x\in(0,R)} \frac{\mu(\Omega(\eta, x) \cap \hat{B})}{H_{m-1}(\Omega(\eta, x) \cap \hat{B})} \to 0 \tag{2.14}$$

as $R \to 0+$. The quantity on the right-hand side of the last inequality is independent of $z \in S \cap \Omega(\eta, \delta)$. Now it is evident that $V(z)$ tends to zero uniformly on $S \cap \Omega(\eta, \delta)$ as $R \to 0+$. Hence, in fact, $W(\lambda, z) \to 0$ as $R \to 0+$ uniformly on $\{\eta\} \cup S \cap \Omega(\eta, \delta)$, which completes the proof.

2.2 Lemma. *Let $\eta \in B$ such that $v(\eta) < \infty$ and $H_{m-1}(\hat{B} \cap \Omega(\eta, r)) > 0$ for every $r > 0$. Let $\mu \in C^*$ and suppose that there are $\delta > 0$ and $k < \infty$ such that*

$$\left|\frac{\mu(M)}{H_{m-1}(M)}\right| \leqq k \tag{2.15}$$

for any Borel set $M \subset \hat{B} \cap \Omega(\eta, \delta)$ with $H_{m-1}(M) > 0$. Then

$$\int_B \frac{|n(y)\,(y - \eta)|}{|y - \eta|^m}\,\mathrm{d}|\mu|\,(y) < \infty. \tag{2.16}$$

Proof. There exists a Borel set $A \subset B$ with $\mu^+ = \mu \mid A$, $\mu^- = \mu \mid (B - A)$. Putting $\lambda = \mu \mid (\hat{B} \cap \Omega(\eta, \delta))$, we obtain $\lambda^+ = \lambda \mid A$, $\lambda^- = \lambda \mid (B - A)$ and

$$\int_B \frac{|n(y)\,(y - \eta)|}{|y - \eta|^m}\,\mathrm{d}|\mu|\,(y) = \tag{2.17}$$

$$= \int_{B-\Omega(\eta,\delta)} \frac{|n(y)\,(y - \eta)|}{|y - \eta|^m}\,\mathrm{d}|\mu|\,(y) + \int_B \frac{|n(y)\,(y - \eta)|}{|y - \eta|^m}\,\mathrm{d}|\lambda|\,(y).$$

The first integral on the right-hand side of (2.17) is finite because the integrand is bounded on $\hat{B} - \Omega(\eta, \delta)$ and $|\mu|(\hat{B}) < \infty$. It can be easily seen that

$$\lambda^+(M) \leqq kH_{m-1}(M), \quad \lambda^-(M) \leqq kH_{m-1}(M) \tag{2.18}$$

for any Borel set $M \subset \hat{B}$. Since λ^+ and λ^- are concentrated on two disjoint subsets of $\hat{B} \cap \Omega(\eta, \delta)$, it follows from Radon-Nikodym theorem that there is $\varphi \in \mathscr{B}$ with $|\varphi(x)| \leqq k$ for each $x \in B$, $\varphi(x) = 0$ for each $x \in B - (\Omega(\eta, \delta) \cap \hat{B})$ and $\lambda = \varphi(H_{m-1} \mid \hat{B})$. For such function φ we have

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, \mathrm{d}|\lambda|(y) = \int_B |\varphi(y)| \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, \mathrm{d}H_{m-1}(y) \leqq k\, v(\eta)$$

so that (2.16) is true.

2.3 Lemma. *Let $\eta \in B$ and let $\mu \in C^*$ has the derivative D at η. Then there exist derivatives of μ^+, μ^- and $|\mu|$ at η and they are equal to*

$$\frac{D + |D|}{2}, \quad \frac{-D + |D|}{2}, \quad |D|$$

respectively.

Proof. There is a Borel set $A \subset B$ for which $\mu^+ = \mu \mid A$, $\mu^- = \mu \mid (B - A)$. Further there is $\delta > 0$ such that

$$\left|\frac{\mu(M)}{H_{m-1}(M)}\right| \leqq |D| + 1$$

for any Borel set $M \subset \hat{B} \cap \Omega(\eta, \delta)$ with $H_{m-1}(M) > 0$. Now the proof will be divided into two parts:

a) Let $D = 0$.

The following two cases may occur: either

$$H_{m-1}(A \cap \hat{B} \cap \Omega(\eta, r)) > 0$$

for every $r > 0$ or

$$H_{m-1}((\hat{B} - A) \cap \Omega(\eta, r)) > 0$$

for every $r > 0$. Consider the first case. Let $M \subset \hat{B} \cap \Omega(\eta, \delta)$ be a Borel set with $H_{m-1}(M) > 0$. If $H_{m-1}(A \cap M) = 0$, then also $\mu^+(M) = 0$; if $H_{m-1}(A \cap M) > 0$, then

$$\frac{\mu^+(M)}{H_{m-1}(M)} \leqq \frac{\mu(A \cap M)}{H_{m-1}(A \cap M)}.$$

Therefore, since the derivative of μ vanishes at η, we obtain that μ^+ has the derivative vanishing at η. From the relations $\mu^- = \mu^+ - \mu$ and $|\mu| = \mu^+ + \mu^-$ we now conclude that μ^- and $|\mu|$ have also derivatives which vanish at η. In the second case we can proceed analogously.

b) Let $D \neq 0$.

Assume $D > 0$. There is δ_1, $0 < \delta_1 < \delta$ such that

$$\left|\frac{\mu(M)}{H_{m-1}(M)} - D\right| < \frac{D}{2} \tag{2.19}$$

holds for each Borel set $M \subset \hat{B} \cap \Omega(\eta, \delta_1)$ with $H_{m-1}(M) > 0$. Then necessarily

$$H_{m-1}((\hat{B} - A) \cap \Omega(\eta, \delta_1)) = 0 .$$

Indeed, if this is not the case, the inequality (2.19) with $(\hat{B} - A) \cap \Omega(\eta, \delta_1)$ written there instead of M is false. Hence

$$\mu^-(\hat{B} \cap \Omega(\eta, \delta_1)) = 0 .$$

This means that μ^- has the derivative which vanishes at η, μ^+ and $|\mu|$ have derivatives at η equal to D.

The case $D < 0$ is analogous.

2.4 Theorem. *Let $S \subset R^m - B$, $\eta \in \bar{S} \cap B$,*

$$\operatorname{contg}(S, \eta) \cap \operatorname{contg}(\hat{B}, \eta) = \emptyset ,$$

suppose that (1.8) holds and there is $\delta > 0$ such that (1.3) holds. Let $\mu \in C^*$, $\mu = \lambda + \nu$, $\lambda, \nu \in C^*$ such that λ has the derivative D at η, $|\nu|$ has the symmetric derivative which vanishes at η. Further suppose

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^m} \, \mathrm{d}|\nu|(y) < \infty .$$

Then there exists the limit

$$\lim_{\substack{z \to \eta \\ z \in S}} W(\mu, z) = W(\mu, \eta) + D H_{m-1}(\Gamma)(i - d_G(\eta)) . \tag{2.20}$$

Proof. We have

$$W(\mu, z) = W(\lambda, z) + W(\nu^+, z) - W(\nu^-, z)$$

for those $z \in R^m$ for which both sides of this equality are defined. It follows from lemma 2.1 that

$$\lim_{\substack{z \to \eta \\ z \in S}} W(\nu, z) = W(\nu, \eta) .$$

It is sufficient to prove that (2.20) holds if we write there λ instead of μ. Put $\gamma = \lambda - D(H_{m-1} \mid \hat{B})$. Since λ has the derivative D at η and $D(H_{m-1} \mid \hat{B})$ has the derivative D at η γ has the derivative vanishing at η. According to lemma 2.3, γ^+ and γ^- have also derivatives vanishing at η. If $f \in C$ is a function equal to unity on B, we have

$$W(\lambda, z) = D\, W(f, z) + W(\gamma^+, z) - W(\gamma^-, z)$$

for those $z \in R^m$ for which the left-hand side is defined. It is known (cf. the proof of the proposition 1.1) that there exists the limit

$$\lim_{\substack{z \to \eta \\ z \in S}} W(f, z) = W(f, \eta) + H_{m-1}(\Gamma)\,(i - d_G(\eta))\,.$$

According to lemma 2.1 the limit

$$\lim_{\substack{z \to \eta \\ z \in S}} W(\gamma, z) = W(\gamma, \eta)$$

also exists (to verify the assumptions one uses lemma 2.2). This implies the statement of the present theorem.

2.5 Remark. It is not possible to replace the requirement (2.15) in the lemma 2.2 by the "symmetric requirement", i.e. by

$$\limsup_{r \to 0+} \left| \frac{\mu(\Omega)(\eta, r) \cap \hat{B})}{H_{m-1}(\Omega(\eta, r) \cap \hat{B})} \right| < \infty\,.$$

Moreover, we shall introduce an example proving that it is not sufficient to suppose that μ is a non-negative measure with the symmetric derivative vanishing at η.

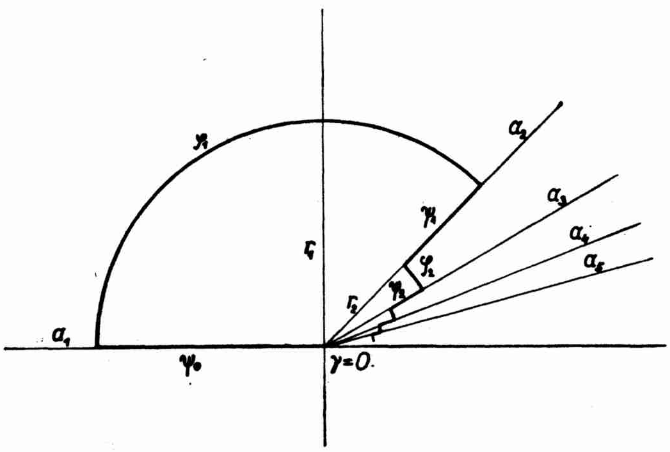

Fig. 1

Let $m = 2$. Denote by $[x, y]$ $(x, y \in R^1)$ the points of R^2. We construct in R^2 the curve φ consisting of the curves φ_i and ψ_j as in fig. 1 – the reader certainly can describe this curve precisely. Here we put $r_k = 1/k$ $(k = 1, 2, \ldots)$, $\alpha_k = \pi/4k$ $(k = 2, 3, \ldots)$, $\alpha_1 = \frac{1}{2}\pi$, r_k denotes the radius of the arc φ_k, α_k the angle. For the curve φ we may easily find a rectification, for example by an arc length – but we shall not need it here. The curve φ is a Jordan curve (i.e. simple closed curve) and thus we may consider the domain $G = \operatorname{Int} \varphi$. It is evident that $P(G) < \infty$, $B = \langle\varphi\rangle$ and $B - \hat{B}$ is a denumerable set. Let $\eta = [0, 0]$. We have $v(\eta) < \infty$. Now we define a function f on B as follows:

$$f(z) = \frac{4}{\pi} \frac{k + 1}{\log k}$$

for all z on the open arc φ_k, $k = 2, 3, \ldots$,

$$f(z) = 0$$

for all other $z \in B$. Putting $\mu = f H_1 \mid B$, we have that $\mu \in C^*$ and μ is a non-negative measure. Let

$$q_k = \mu((\varphi_k)) = r_k(\alpha_k - \alpha_{k+1}) \frac{4}{\pi} \frac{k + 1}{\log k} = \frac{1}{k^2 \log k}$$

for $k = 2, 3, \ldots$. We shall prove that μ has the symmetric derivative which vanishes at η. Given r, $0 < r < 1$, there is a natural number k such that $r \in (r_{k+1}, r_k\rangle$. Then

$$\mu(\Omega(\eta, r)) \cap \hat{B}) = \sum_{n=k+1}^{\infty} q_n = \sum_{n=k+1}^{\infty} \frac{1}{n^2 \log n} \leqq$$

$$\leqq \frac{1}{\log(k + 1)} \sum_{n=k+1}^{\infty} \frac{1}{n^2} \leqq \frac{1}{\log(k + 1)} \int_k^{\infty} \frac{dt}{t^2} = \frac{1}{k \log(k + 1)} .$$

Taking into account

$$H_1(\hat{B} \cap \Omega(\eta, r)) \geqq 2r \left(> 2r_{k+1} = \frac{2}{k + 1} \right),$$

we see that μ has the symmetric derivative vanishing at η.

For $y \in (\varphi_k)$ we have $n(y) = y/|y|$ and therefore

$$\int_B \frac{|n(y)(y - \eta)|}{|y - \eta|^2} \, d\mu(y) = \int_B f(y) \frac{|n(y)(y - \eta)|}{|y - \eta|^2} \, dH_1(y) =$$

$$= \sum_{k=2}^{\infty} \frac{q_k}{r_k} = \sum_{k=2}^{\infty} \frac{1}{k \log k} = \infty .$$

The measure μ satisfies a desired requirements. Let us remark that in the preceding example one may require φ to be a smooth curve.

3.

Throughout this paragraph we always assume that $m = 2$. Where necessary, we identify R^2 with the set of all complex numbers. Introduce the following notation:

If $\alpha \in R^1$, $z \in R^2$, write $H(\alpha, z) = H(\Theta, z) = \{z + r\Theta;\ r > 0\}$, where $\Theta = [\cos\alpha, \sin\alpha]$. $\mathscr{D}$ stands for the set of all infinitely differentiable functions with compact supports in R^2. For $z \in R^2$ put

$$\mathscr{D}(z) = \{\varphi \in \mathscr{D};\ z \notin \operatorname{supp} \varphi\},$$

where supp φ denotes the support of the function φ.

Now we shall prove two simple auxiliary assertions (which could be pronounced in a more general form).

3.1 Lemma. *Let φ be a Jordan curve in R^2 defined on $\langle a, b\rangle$ and ϑ a function with a finite variation on $\langle a, b\rangle$. Further suppose that the function ϑ is either continuous from the right on $\langle a, b)$ or continuous from the left on $(a, b\rangle$. Then*

$$\operatorname{var}[\vartheta; \langle a, b\rangle] = \sup\left\{\int_a^b f(\varphi(t))\, d\vartheta(t);\ f \in \mathscr{D},\ |f| \leqq 1\right\} \tag{3.1}$$

(the integrals in (3.1) are meant in the sense of Stieltjes).

Proof. If $\operatorname{var}[\vartheta; \langle a, b\rangle] = 0$, then the statement is obvious. Suppose that $\operatorname{var}[\vartheta; \langle a, b\rangle] > 0$. It is known that

$$\operatorname{var}[\vartheta; \langle a, b\rangle] = \sup\left\{\int_a^b f(t)\, d\vartheta(t);\ f \in C(\langle a, b\rangle),\ |f| \leqq 1\right\}$$

(integrals are always meant in the sense of Stieltjes).

Given $\varepsilon > 0$, there is $f_1 \in C(\langle a, b\rangle)$, $|f_1| \leqq 1$ such that

$$\int_a^b f_1(t)\, d\vartheta(t) > \operatorname{var}[\vartheta; \langle a, b\rangle] - \frac{\varepsilon}{3}. \tag{3.2}$$

Assume conversely that ϑ is continuous from the right on $\langle a, b)$. Then the function

$$s(t) = \operatorname{var}[\vartheta; \langle a, b\rangle]$$

is continuous from the right at the point a and thus there is $t_0 \in (a, b)$ such that for each $t \in \langle a, t_0\rangle$

$$s(t) < \frac{\varepsilon}{6}.$$

Further there exists $f_2 \in C(\langle\varphi\rangle)$ with $|f_2| \leqq 1$, $f_2(\varphi(t)) = f_1(t)$ for each $t \in \langle t_0, b\rangle$. Then

$$\int_a^b f_2(\varphi(t))\,\mathrm{d}\vartheta(t) = \int_{t_0}^b f_1(t)\,\mathrm{d}\vartheta(t) + \int_a^{t_0} f_2(\varphi(t))\,\mathrm{d}\vartheta(t) \geqq$$

$$\geqq \int_a^b f_1(t)\,\mathrm{d}\vartheta(t) - \left|\int_a^{t_0} (f_2(\varphi(t)) - f_1(t))\,\mathrm{d}\vartheta(t)\right| > \operatorname{var}[\vartheta; \langle a, b\rangle] - \tfrac{2}{3}\varepsilon\,.$$

Since $\langle\varphi\rangle$ is a compact set and $f_2 \in C(\langle\varphi\rangle)$, there is $f \in \mathscr{D}$, $|f| \leqq 1$ such that

$$|f(z) - f_2(z)| \leqq \frac{\varepsilon}{3\operatorname{var}[\vartheta; \langle a, b\rangle]}$$

holds for each $z \in \langle\varphi\rangle$. Then

$$\int_a^b f(\varphi(t))\,\mathrm{d}\vartheta(t) \geqq \int_a^b f_2(\varphi(t))\,\mathrm{d}\vartheta(t) -$$

$$- \left|\int_a^b (f(\varphi(t)) - f_2(\varphi(t)))\,\mathrm{d}\vartheta(t)\right| > \operatorname{var}[\vartheta; \langle a, b\rangle] - \varepsilon\,.$$

In the case of ϑ continuous from the left on $(a, b\rangle$ we may proceed completely analogously.

3.2 Lemma. *Let φ be a Jordan curve in R^2 defined on $\langle a, b\rangle$, let $t_0 \in \langle a, b\rangle$, $I_1 = \langle a, t_0)$, $I_2 = (t_0, b\rangle$ (of course, if $t_0 = a$, then $I_1 = \emptyset$, if $t_0 = b$, then $I_2 = \emptyset$), let ϑ_j $(j = 1, 2)$ be a continuous function with a locally finite variation on I_j. Then*

$$\sum_{j=1}^{2} \operatorname{var}[\vartheta_j, I_j] = \sup\left\{\sum_{j=1}^{2} \int_{I_j} f(\varphi(t))\,\mathrm{d}\vartheta_j(t);\ f \in \mathscr{D}(\varphi(t_0)),\ |f| \leqq 1\right\} \tag{3.3}$$

(it is obvious how (3.3) reduces in the case $t_0 = a$ or $t_0 = b$).

Proof. a) Let $\sum_{j=1}^{2} \operatorname{var}[\vartheta_j; I_j] < \infty$. Suppose $t_0 \in (a, b)$. Define a function ϑ on $\langle a, b\rangle$ by

$$\vartheta(t) = \vartheta_1(t) \quad \text{for} \quad t \in \langle a, t_0)\,,$$

$$\vartheta(t) = \vartheta_2(t) - \lim_{z \to t_0^+} \vartheta_2(z) + \lim_{z \to t_0^-} \vartheta_1(z) \quad \text{for} \quad t \in (t_0, b\rangle\,,$$

$$\vartheta(t_0) = \lim_{z \to t_0^-} \vartheta_1(z).$$

Obviously, ϑ is a continuous function on $\langle a, b\rangle$ with a finite variation

$$\operatorname{var}[\vartheta; \langle a, b\rangle] = \sum_{j=1}^{2} \operatorname{var}[\vartheta_j; I_j]\,.$$

For $f \in C(\langle a, b \rangle)$ we have

$$\sum_{j=1}^{2} \int_{I_j} f(t) \, \mathrm{d}\vartheta_j(t) = \int_a^b f(t) \, \mathrm{d}\vartheta(t) \, .$$

Given $\varepsilon > 0$, then according to lemma 3.1 we may find $f_1 \in \mathscr{D}$, $|f_1| \leqq 1$ such that

$$\int_a^b f_1(\varphi(t)) \, \mathrm{d}\vartheta(t) > \operatorname{var}[\vartheta; \langle a, b \rangle] - \frac{\varepsilon}{2} \, .$$

Further there is δ, $0 < \delta < \min\{t_0 - a, b - t_0\}$ such that

$$\operatorname{var}[\vartheta; \langle t_0 - \delta, t_0 + \delta \rangle] < \frac{\varepsilon}{4} \, .$$

Since $\varphi(t_0)$ is not contained in the compact set

$$\varphi(\langle a, t_0 - \delta \rangle \cup \langle t_0 + \delta, b \rangle) \, ,$$

there is $f \in \mathscr{D}(\varphi(t_0))$ such that $|f| \leqq 1$ and $f(z) = f_1(z)$ for each

$$z \in \varphi(\langle a, t_0 - \delta \rangle \cup \langle t_0 + \delta, b \rangle) \, .$$

From the choice of f_1 and δ it follows

$$\int_a^b f(\varphi(t)) \, \mathrm{d}\vartheta(t) > \operatorname{var}[\vartheta; \langle a, b \rangle] - \varepsilon \, .$$

Analogously in the cases $t_0 = a$ or $t_0 = b$.

b) Suppose, conversely, $\operatorname{var}[\vartheta_1; \langle a, t_0)] = \infty$.

Let $t_0 \in (a, b)$. Given $k > 0$, there is $t_1 \in (a, t_0)$ such that $\operatorname{var}[\vartheta_1; \langle a, t_1 \rangle] > k + 2$ and thus there is $f_1 \in \mathscr{D}$ with $|f_1| \leqq 1$ and

$$\int_a^{t_1} f_1(\varphi(t)) \, \mathrm{d}\vartheta_1(t) > k + 1 \, .$$

There is $\delta_1 > 0$ such that $\Omega(\varphi(t_0), 2\delta_1) \cap \varphi(\langle a, t_1 \rangle) = \emptyset$. Further there is $t_2 \in (t_1, t_0)$ such that

$$\operatorname{var}[\vartheta_1; \langle t_1, t_2 \rangle] < \tfrac{1}{3}$$

(since ϑ_1 is continuous). We may find $\delta_2 > 0$, $2\delta_2 < t_1 - a$ such that

$$\operatorname{var}[\vartheta_1; \langle a, a + 2\delta_2 \rangle] < \tfrac{1}{3} \, .$$

Then $\varphi(\langle a + 2\delta_2, t_1 \rangle)$ and $\varphi(\langle a, a + \delta_2 \rangle \cup \langle t_2, b \rangle) \cup \overline{\Omega(\varphi(t_0), \delta_1)}$ are two disjoint compact sets and thus there is $f \in \mathscr{D}$ with $|f| \leqq 1$, $f(z) = f_1(z)$ on the former of both described sets and $f(z) = 0$ on the latter. Therefore, moreover, $f \in \mathscr{D}(\varphi(t_0))$. We arrive at

$$\sum_{j=1}^{2} \int_{I_j} f(\varphi(t))\, \mathrm{d}\vartheta_j(t) = \int_{a+\delta_2}^{t_2} f(\varphi(t))\, \mathrm{d}\vartheta_1(t) =$$

$$= \int_a^{t_1} f_1 * \varphi \mathrm{d}\vartheta_1 - \int_a^{a+2\delta_2} f_1 * \varphi\, \mathrm{d}\vartheta_1 + \int_{t_1}^{t_2} f * \varphi\, \mathrm{d}\vartheta_1 + \int_{a+\delta_2}^{a+2\delta_2} f * \varphi\, \mathrm{d}\vartheta_1 > k\,.$$

Analogously for $t_0 = b$.

The case var $[\vartheta_2; I_2] = \infty$ may be solved in the same way.

Throughout the rest of this paragraph ψ stands for a Jordan curve in R^2 defined on a compact interval $\langle \alpha, \beta \rangle$ $(\alpha < \beta)$. Further suppose that ψ is a positively oriented curve with a finite length. Denote $G = \operatorname{Int} \psi$ and, according to the preceding notation, $B = \langle \psi \rangle$, $\hat{B}$ being the reduced boundary of the set G. From [12], part 8, we get var $[\psi; \langle \alpha, \beta \rangle] = P(G)$ and so

$$P(G) < \infty\,. \tag{3.4}$$

For $z \in R^2$, $\alpha \in \langle 0, 2\pi \rangle$ let $N(\alpha, z)$ be the number of all points of the set $\langle \psi \rangle \cap$ $\cap\, H(\alpha, z)$. The function $N(\alpha, z)$ is a measurable function with respect to $\alpha \in \langle 0, 2\pi \rangle$ (and non-negative), thus we may define

$$V(z) = \int_0^{2\pi} N(\alpha, z)\, \mathrm{d}\alpha$$

(cf., for example, [6], lemma 2.1). If $\Theta = [\cos \alpha, \sin \alpha]$, then $n(\Theta, z) \leqq N(\alpha, z)$ (where $n(\Theta, z)$ has the same meaning as in the introduction). Hence

$$v(z) \leqq V(z)\,. \tag{3.5}$$

For $z \in R^2$ let $\mathfrak{A}$ be the system of all components of the set $\langle \alpha, \beta \rangle - \psi^{-1}(z)$ (in the present case $\mathfrak{A}$ has at most two elements) and for $I \in \mathfrak{A}$ let ϑ_z^I be a single-valued continuous argument of $\psi(t) - z$ on I. Define, for $z \in R^2$ and $f \in C$,

$$W^*(f, z) = \sum_{I \in \mathfrak{A}} \int_I f(\psi(t))\, \mathrm{d}\vartheta_z^I(t) \tag{3.6}$$

provided the integrals on the right-hand side exist and their sum is defined.

Prove that if $\varphi \in \mathscr{D}(z)$, then

$$W^*(\varphi, z) = W(\varphi, z)\,. \tag{3.7}$$

Hence we obtain by passing to the limit that if $V(z) < \infty$, then $W^*(f, z) = W(f, z)$ for each $f \in C$ — as regards this, see the equality (3.10) in the following.

If $\varphi \in \mathscr{D}(z)$, then (cf. [5])

$$W(\varphi, z) = \int_G \operatorname{grad} \varphi(x) \frac{x - z}{|x - z|^2}\, \mathrm{d}x\,.$$

The proposition 2.3 in [8] implies

$$W^*(\varphi, z) = -\int_\alpha^\beta \varphi(\psi(t)) \frac{\psi_2(t) - y}{|\psi(t) - z|^2} \, d\psi_1(t) + \int_\alpha^\beta \varphi(\psi(t)) \frac{\psi_1(t) - x}{|\psi(t) - z|^2} \, d\psi_2(t),$$

where $z = [x, y]$, $\psi = [\psi_1, \psi_2]$. For ψ and the function

$$w(\zeta) = \left[-\varphi(\zeta) \frac{\eta - y}{|\zeta - z|^2}, \varphi(\zeta) \frac{\xi - x}{|\zeta - z|^2} \right]$$

(where $\zeta = [\xi, \eta]$) the requirements of Green theorem are satisfied (cf. [4], theorem 8.49) and thus we conclude

$$W^*(\varphi, z) = \int_\psi w_1 \, d\xi + w_2 \, d\eta = \int_G \operatorname{rot} w = \int_G \operatorname{grad} \varphi(u) \frac{u - z}{|u - z|^2} \, du = W(\varphi, z).$$

3.3 Theorem. *If $z \in R^2$, then*

(3.8) $$V(z) = v(z).$$

Proof. Since by [5], assertion 1.6

$$v(z) = \sup \{ W(\varphi, z); \varphi \in \mathscr{D}(z), |\varphi| \leqq 1 \},$$

it is sufficient to prove, with respect to (3.7), that

(3.9) $$V(z) = \sup \{ W^*(\varphi, z); \varphi \in \mathscr{D}(z), |\varphi| \leqq 1 \}.$$

Let $\mathfrak{A}$, ϑ_z^I have the same meaning as in the definition of $W^*(f, z)$. It follows from (6) in [8] that

(3.10) $$V(z) = \sum_{I \in \mathfrak{A}} \operatorname{var} [\vartheta_z^I; I].$$

If $\alpha \leqq a < b \leqq \beta$, $z \notin \psi(\langle a, b \rangle)$ and ϑ is some single-valued argument of $\psi(t) - z$ on $\langle a, b \rangle$, then (by 1.12 from [7])

$$\operatorname{var} [\vartheta; \langle a, b \rangle] \leqq \operatorname{dist} (z; \psi(\langle a, b \rangle)) \operatorname{var} [\psi; \langle a, b \rangle].$$

This implies that ϑ_z^I has a locally finite variation on $I \in \mathfrak{A}$. If now $z \in B$, we may use lemma 3.2, therefore we see that (3.9) holds. If $z \notin B$, then (3.9) follows from lemma 3.1.

3.4 Remark. Since $n(\Theta, z) \leqq N(\alpha, z)$ (where $\Theta = [\cos \alpha, \sin \alpha]$), it follows from theorem 3.3 that for each fixed $z \in R^2$, $n(\Theta, z) = N(\alpha, z)$ for almost all $\alpha \in \langle 0, 2\pi \rangle$.

In the same way as in [8] we define for $t_0 \in (\alpha, \beta)$

(3.11) $$\tau_\psi^+(t_0) = \lim_{t \to t_0+} \frac{\psi(t) - \psi(t_0)}{|\psi(t) - \psi(t_0)|} = e^{i\alpha+}, \quad \tau_\psi^-(t_0) = \lim_{t \to t_0-} \frac{\psi(t) - \psi(t_0)}{|\psi(t) - \psi(t_0)|} = e^{i\alpha-}$$

provided the limits exist. We may suppose that $\alpha_+ \leqq \alpha_- < \alpha_+ + 2\pi$. If $\tau_\psi^+(t_0) = -\tau_\psi^-(t_0)$, then we put

$$\tau_\psi(t_0) = \tau_\psi^+(t_0)\,. \tag{3.12}$$

3.5 Lemma. *Let $t \in (\alpha, \beta)$. If there exist $\tau_\psi^+(t)$ and $\tau_\psi^-(t)$, then there exists the density $d_G(z)$ for $z = \psi(t)$. If moreover $\alpha_+ \neq \alpha_-$, then*

$$d_G(z) = \frac{1}{2\pi}(\alpha_- - \alpha_+)\,; \tag{3.13}$$

if $\alpha_+ = \alpha_-$, then either $d_G(z) = 0$ or $d_G(z) = 1$.

If, besides that, there exists $\tau_\psi(t)$, then there exists the exterior normal of G in the sense of Federer

$$n(z) = -i\tau_\psi(z)\,.$$

Proof. Suppose that $\psi(t) = 0$, $\alpha_+ \neq \alpha_-$ and that there is $\gamma \in (0, \pi)$ such that

$$\alpha_+ = -\gamma\,, \quad \alpha_- = \gamma\,.$$

Given ε, $0 < \varepsilon < \gamma$, then by the definition of τ_ψ^+ and τ_ψ^- there is $\delta > 0$, $\delta < \min\{t - \alpha, \beta - t\}$ such that

$$[u \in (t, t + \delta), \psi(u) - \psi(t) = e^{i\beta_1}|\psi(u) - \psi(t)|, \beta_1 \in \langle -\pi - \gamma, \pi - \gamma\rangle] \Rightarrow \Rightarrow |\beta_1 + \gamma| < \varepsilon\,, \tag{3.14}$$

$$[u \in (t - \delta, t), \psi(u) - \psi(t) = e^{i\beta_2}|\psi(u) - \psi(t)|, \beta_2 \in \langle \gamma - \pi, \gamma + \pi\rangle] \Rightarrow \Rightarrow |\beta_2 - \gamma| < \varepsilon\,.$$

There is $r_0 > 0$ such that $\Omega(0, r_0) \cap \psi(\langle \alpha, \beta\rangle - (t - \delta, t + \delta)) = \emptyset$. Prove that for each r such that $0 < r < r_0$

$$\Omega(0, r) \cap \{z = |z|\, e^{i\eta};\ z \neq 0,\ \eta \in \langle \varepsilon - \gamma, \gamma - \varepsilon\rangle\} \subset \Omega(0, r) \cap G \subset \subset \Omega(0, r) \cap \{z = |z|\, e^{i\eta};\ \eta \in \langle -\varepsilon - \gamma, \varepsilon + \gamma\rangle\}\,. \tag{3.15}$$

The sets

$$\Omega(0, r) \cap \{z = |z|\, e^{i\eta};\ z \neq 0,\ \eta \in \langle \varepsilon - \gamma, \gamma - \varepsilon\rangle\}\,, \tag{3.16}$$

$$\Omega(0, r) \cap \{z = |z|\, e^{i\eta};\ z \neq 0,\ \eta \in \langle \gamma + \varepsilon, 2\pi - \gamma - \varepsilon\rangle\} \tag{3.17}$$

are connected. To prove that (3.16) is contained in $\operatorname{Int}\psi$ and (3.17) is contained in $\operatorname{Ext}\psi$ (which implies (3.15)), it is sufficient to prove that there is a point z_1 in (3.16) with $\operatorname{ind}_\psi(z_1) = 1$ and a point z_2 in (3.17) with $\operatorname{ind}_\psi(z_2) = 0$. Put $z_1 = \frac{1}{2}r$, $z_2 = -\frac{1}{2}r$ (z_1, z_2 are considered in the terms of complex numbers). Since there exist $\tau_\psi^+(t)$, $\tau_\psi^-(t)$ and $\tau_\psi^+(t) = e^{-i\gamma}$, $\tau_\psi^-(t) = e^{i\gamma}$ where $\gamma \in (0, \pi)$, it is clear that the function $\operatorname{Im}\psi$ is decreasing at the point t. By Mařík theorem (cf. [2], theorem 126) we have

$$\operatorname{ind}_\psi(z_2) = \operatorname{ind}_\psi(z_1) - 1\,.$$

Since ψ is a positively oriented curve, this equation yields necessarily $\text{ind}_\psi(z_1) = 1$, $\text{ind}_\psi(z_2) = 0$. The relation (3.15) implies

$$(\gamma - \varepsilon)\, r^2 \leqq H_2(\Omega(0, r) \cap G) \leqq (\gamma + \varepsilon)\, r^2$$

and thus, in fact, $d_G(z) = \gamma/\pi\ (=(\alpha_- - \alpha_+)/2\pi)$. The rest of the proof, i.e. $d_G(z) = 0$ or $d_G(z) = 1$ if $\alpha_+ = \alpha_-$ and the existence of the exterior normal in the sense of Federer if $\tau_\psi(t)$ exists is analogous.

Let $z \in R^2$, $t > 0$ and let $M(t, z)$ stand for the number of all points of the set $\psi^{-1}(\{x;\ |x - z| = t\})$. Then $M(t, z)$ is a measurable function with respect to $t \in (0, \infty)$ (cf. ,e.g., [6], lemma 2.5) and we may thus define, for each $r > 0$,

$$u(z, r) = \int_0^r M(t, z)\, dt\,. \tag{3.18}$$

3.6 Theorem. *If $\eta \in R^2$ with $v(\eta) < \infty$, then*

$$\sup_{r>0} \frac{u(\eta, r)}{r} < \infty$$

holds if and only if

$$\sup_{r>0} \frac{H_1(\Omega(\eta, r) \cap \hat{B})}{r} < \infty\,.$$

Proof. If $\eta \notin B$ is the case the statement is obvious, because $n(z, \infty) \leqq \leqq \text{var}\,[\psi; \langle\alpha, \beta\rangle]$ for each $z \in R^2$ (cf. (7) in [8]) and $H_1(\hat{B}) < \infty$.

Let $\eta \in B$. Therefore by [8], theorem 3.9

$$u(\eta, r) \leqq \text{var}\,[\psi; K_r] \leqq r\, v(\eta) + u(\eta, r)\,, \tag{3.19}$$

where $K_r = \psi^{-1}(\{z;\ |z - \eta| \leqq r\})$. Now it is sufficient to prove that

$$\text{var}\,[\psi; K_r] = H_1(\hat{B} \cap \Omega(\eta, r))\,. \tag{3.20}$$

According to [13], theorem 1.1 we have

$$\text{var}\,[\psi; K_r] = H_1(\psi(K_r)) = H_1(B \cap \Omega(\eta, r))$$

(in the present case $N_\psi(z; K_r)$ from theorem 1.1 in [13] is equal to unity on $\psi(K_r)$ except at most at one point). Further we have $\hat{B} \subset B$. Prove $H_1(B - \hat{B}) = 0$. Taking into account theorem 1.17 from [13] we obtain that there exists $\tau_\psi(t)$ for var_ψ-almost all $t \in \langle\alpha, \beta\rangle$. By [13], theorem 1.4, $\text{var}\,[\psi; M] = 0$ for any $M \subset \langle\alpha, \beta\rangle$ if and only if $H_1(\psi(M)) = 0$. By lemma 3.5, $\hat{B}$ contains the set of all $z \in B$ for which there exists τ_ψ in $\psi^{-1}(z)$.

3.7 Remark. As (3.20) holds, it is

$$\sup_{r>0} \frac{H_1(\Omega(\eta, r) \cap \hat{B})}{r} < \infty \Rightarrow \sup_{r>0} \frac{u(\eta, r)}{r} < \infty \,.$$

If $v(\eta) < \infty$, then the converse of this implication holds by theorem 3.6. If $v(\eta) = \infty$, then the converse of this implication need not hold. This will be proved by the following example.

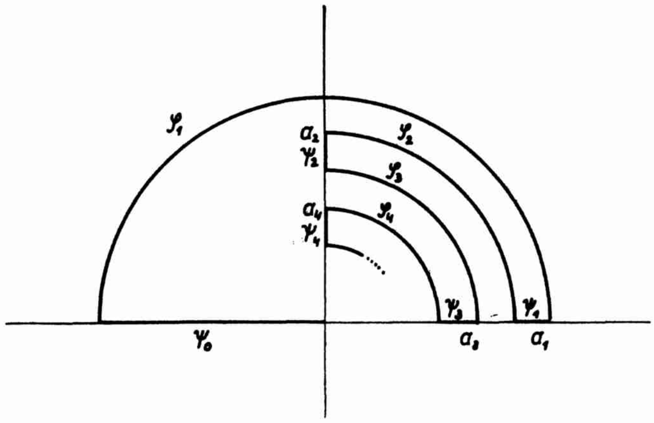

Fig. 2

Analogously to the remark 2.5 we construct a positively oriented Jordan curve φ as in fig. 2. (The figure is only a sketch.) Here we put $a_k = 1/k^2$ $(k = 1, 2, \ldots)$. The curve φ has a finite length and if $\eta = [0, 0]$ then $v(\eta) = \infty$. For $t > 1$ we have $M(t, \eta) = 0$ and for t with $0 < t < 1$, $t \neq a_k$, we have $M(t, \eta) = 2$, therefore

$$\sup_{r>0} \frac{u(\eta, r)}{r} = 2 \,.$$

Further

$$H_1(\Omega(\eta, a_k) \cap \hat{B}) \geqq \frac{\pi}{2} \sum_{n=k+1}^{\infty} a_n \geqq \frac{\pi}{2} \int_{k+2}^{\infty} \frac{\mathrm{d}x}{x^2} = \frac{\pi}{2} \frac{1}{k+2} \,.$$

Hence

$$\frac{H_1(\Omega(\eta, a_k) \cap \hat{B})}{a_k} \geqq \frac{\pi}{2} \frac{k^2}{k+2} \to \infty$$

as $k \to \infty$.

3.8 Remark. In [8] (cf. also [4]) it is proved that if $\eta \in B$, then the limit

(3.21) $$\lim_{\substack{z \to \eta \\ z \in H(\theta, \eta)}} W(f, z)$$

exists for any function $f \in C$ and any half-line $H(\Theta, \eta) \notin \operatorname{contg}(\hat{B}, \eta)$ if and only if

$$v(\eta) + \sup_{r>0} \frac{u(\eta, r)}{r} < \infty .$$

Here this assertion follows immediately from theorems 1.6 and 3.6. If we compare the value of the limit (3.21) introduced in [8] (or [4]) with the value of that introduced in theorem 1.6, then lemma 3.5 certifies that these values are equal.

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[1]) The analogous problems are studied from a little different point of view in the article Einige Eigenschaften von k-dimensionalen λ-Potentialen der einfachen und der doppelten Belegung by S. Dümmel (Atti della Accademia Nazionale dei Lincei, Memorie, ser. VIII, vol. VII, 173—201, 1965).

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OPTIMAL UNIVERSAL APPROXIMATIONS OF FOURIER COEFFICIENTS IN SPACES OF CONTINUOUS PERIODIC FUNCTIONS

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1. INTRODUCTION

The computation of the Fourier coefficients, i.e. the integrals

$$I_p(f) = \frac{1}{2\pi} \int_0^{2\pi} f(x)\, e^{-ipx}\, dx \tag{1.1}$$

is a problem occurring frequently in practice. As a rule, the problem is numerically solved by successive calculation of the coefficients needed using some quadrature formula (see e.g. [4]). The effort to lower the amount of calculations in the case of the computation of a large number of the integrals (1.1) has resulted in methods based on the approximation of $I_p(f)$ for a given function f by an expression of the type

$$\sum_{k=1}^{n} a_k(f)\, g_k(p) \tag{1.2}$$

and on the successive substitution of the values of p [7]. The point of the procedure is that the number of the terms of this expression (and also of the functionals $a_k(f)$) is lower than that of the Fourier coefficients computed. In this way, the number of the evaluated functionals is reduced.

The question is which method is the most suitable one for solving the problem described. Recently, great attention has been paid to the optimal approximations of linear functionals (see e.g. [3], [5]). Especially, BABUŠKA [1], [2] has dealt with the optimal quadrature formulae for the computation of the Fourier coefficients. It is necessary to keep in mind that the question of optimality is a rather ambiguous one. Namely, the optimality problem is always studied relatively, i.e. with respect to a given functional space. The optimal approximation and its error depend on this space naturally and, as shown in [2], this dependence can be very strong. The available information on $f(x)$ does not allow us usually to determine a definite space where we could choose from a given class of approximations the optimal one. This implies the importance of finding so called universal approximations. These are the approximations the error of which does not differ "too much" (in a precisely defined manner) from that of the optimal approximation in a wide class of spaces. The universal approximation need not be optimal in any space from the class given, but it provides us with the desirable independence of the choice of the numerical method on the choice of the space.

In this paper, we shall treat the universal approximations of the integrals (1.1) supposing that we want to calculate their values for a given function f and a set P of subscripts p. We shall assume the function f to belong to a Babuška-introduced space of continuous 2π-periodic functions. We shall give our attention to the methods of the type (1.2) where we shall study the problem of the choice of the functions $g_k(p)$, which is of decisive importance for the universality of these methods. For our considerations, the choice of the functionals $a_k = a_k(f)$ is not relevant though we shall touch it in some discussions.

The paper is divided into six sections, the introduction being the first one.

The properties of Hilbert spaces of periodic functions the Fourier coefficients of which are to be computed are summarized in the second section. Sec. 3, which has rather auxiliary character, describes some ways of the approximation of the integrals (1.1) by the expressions of the type (1.2). Some of the given approximations are, as shown in the following sections, of practical importance; others are used as examples and as counterexamples in the proofs of the theorems in the following sections.

The theoretical basis of the paper is the fourth section where the criterion of the optimality for our problem is formulated and a lower bound on the error of the optimal approximation is derived. This bound is used frequently later. The concept of the universal approximation is introduced here, too. The study of the universality is the subject of Sec. 5. The classes of spaces in which universal approximations do not exist are given. An approximation universal with respect to a wide class of the spaces of periodic functions is constructed. Further, the necessary conditions which the functions $g_k(p)$ must satisfy for (1.2) to be a universal approximation are derived and the optimal approximations are studied in the class of universal approximations.

The sixth section surveys the practical aspects of the results proved in the paper.

2. THE SPACES OF PERIODIC FUNCTIONS

In this section, we describe to the necessary extent the Hilbert spaces of continuous 2π-periodic functions used in our considerations. A function is meant as a complex-valued function of one real argument in this paper unless otherwise stated. The properties of the periodic spaces will not be proved, the theorems of this section being particular cases of the theorems contained in [6].

First, let us define a general periodic space.

Definition 2.1. A Hilbert space H the elements of which are continuous 2π-periodic functions is called *periodic* if the following conditions are satisfied:

(a) For all $f \in H$,

$$\|f\|_C \leqq B(H) \,.\, \|f\|$$

where $\|.\|_C$ is the usual norm in the space $C_{2\pi}$ of continuous 2π-periodic functions and the constant $B(H)$ does not depend on f.

(b) Let $f \in H$; then $g(x) = f(x + c) \in H$ for every real number c and $\|f\| = \|g\|$.

A trivial example of a periodic space is the linear space of all the functions of the form

$$ae^{ikx}$$

where a is an arbitrary complex number and k is a fixed integer. The scalar product is defined as

$$(a_1 e^{ikx}, a_2 e^{ikx}) = a_1 \bar{a}_2 \,.$$

Thus the functions e^{ikx}, k integer, can be elements of the periodic spaces. Therefore, we introduce a set A_H of integers for every periodic space H describing those functions e^{ikx} which are in H.

Definition 2.2. Let H be a periodic space. An integer k is said to belong to the set A_H if $e^{ikx} \in H$.

The basic structure of periodic spaces is characterized by

Theorem 2.1. *Let H be a periodic space which does not consist only of the zero function. Then $A_H \neq \emptyset$ and the system $\{e^{ikx}\}$, $k \in A_H$ forms an orthogonal basis of the space H. Furthermore,*

$$\sum_{k \in A_H} \|e^{ikx}\|^{-2} < \infty \,. \tag{2.1}$$

It will prove useful to introduce a convenient notation by the following

Definition 2.3. Let H be a periodic space, $k \in A_H$. We denote

$$\eta_k = \|e^{ikx}\| \,.$$

We shall make use also of the following theorem justifying the rearrangements of Fourier series and the interchanges of limits.

Theorem 2.2. *Let H be a periodic space, $f \in H$. Then the Fourier expansion of the function $f(x)$ converges absolutely and uniformly with respect to x.*

Theorem 2.1 asserts that each non-trivial periodic space determines a sequence $\eta = \{\eta_k\}_{k \in A_H}$ of positive numbers satisfying (2.1). It may be shown that there exists even a one-to-one correspondence between the class of periodic spaces and the class of such sequences.

Theorem 2.3. *Let $A \neq \emptyset$ be a set of integers. Let $\eta_k > 0$, $k \in A$ be real numbers satisfying the condition*

$$\sum_{k \in A} \eta_k^{-2} < \infty . \tag{2.2}$$

Then there exists a periodic space H such that $e^{ikx} \in H$, $k \in A$ and $\|e^{ikx}\| = \eta_k$. This space is the completion of the linear hull generated by e^{ikx}, $k \in A$ with respect to the scalar product

$$\begin{aligned} (e^{ikx}, e^{isx}) &= 0 , \quad k, s \in A , \quad k \neq s , \\ (e^{ikx}, e^{ikx}) &= \eta_k^2 , \quad k \in A . \end{aligned} \tag{2.3}$$

The mutual relation of the norm of an element $f \in H$ where H is a periodic space, and its Fourier coefficients is given by Parseval's identity having the form

$$\|f\|^2 = \sum_{k \in A_H} |I_k(f)|^2 \, \eta_k^2 \tag{2.4}$$

in our notation.

In Sec. 5, we shall prove that for the class of periodic spaces no universal approximation exists in the set of approximating functionals that will be under consideration. Therefore, we introduce the subclass of strongly periodic spaces the properties of which are sufficient for the universal approximation to exist. Our definition of the strongly periodic space differs somewhat from that of Babuška [2], [6]. We omit namely one condition regarding the character of the increase of η_k as $k \to \infty$ which is not necessary for the considerations of this paper.

Definition 2.4. A periodic space H is said to be *strongly periodic* if the following conditions are satisfied:

(c) $e^{ikx} \in H$ for all integers k and $\eta_k = \eta_{-k}$.
(d) $\eta_k \geqq \eta_j$ for $|k| \geqq |j|$, k, j integers.

When referring to the conditions (a)–(d) from Definitions 2.1, 2.4 in what follows we shall denote them only by the corresponding letters. Note that Definition 2.4 implies that A_H is the set of all integers for strongly periodic spaces.

Examples of strongly periodic spaces are the periodic spaces H_γ [2] with the scalar products

$$(g, h)_\gamma = \sum_{j=0}^{\infty} \gamma_j \int_0^{2\pi} g^{(j)}(x) \, \overline{h^{(j)}(x)} \, dx$$

where $\gamma_0, \gamma_1, \ldots$ are real numbers satisfying

(1) $\gamma_j \geqq 0$ for all integers j, $\gamma_0 > 0$;
(2) there exists $j_0 > 0$ such that $\gamma_{j_0} \neq 0$;
(3) $\lim_{j \to \infty} \gamma_j^{1/j} = 0$.

3. THE APPROXIMATIONS OF FOURIER COEFFICIENTS IN PERIODIC SPACES

Let H be a periodic space, $f \in H$. Let us assume that we want to calculate the values of $I_p(f)$ for all p from a given set P of integers. Let P consist of r distinct elements. (This assumption refers to the whole paper and will not be repeated afterwards.)

We shall study r-tuples of the approximating functionals of the form

$$\{G_p\}_{p \in P}\,, \quad G_p(f) = \sum_{k=1}^{n} a_k(f)\, g_k(p)\,, \quad n \geqq 1\,. \tag{3.1}$$

We suppose $g_k(p)$, $k = 1, 2, \ldots, n$ to be complex-valued functions of an integer argument p defined on P. For given n and P we construct the set of all $\{G_p\}$ such that $a_k(f)$ are bounded complex functionals defined on H. This set will be denoted by $M_n(P)$. The elements of the set $M_n(P)$ will be called "approximations" shortly.

Remark 3.1. A *bounded functional* G means here a functional with the property that there exists a constant K such that

$$|G(f)| \leqq K\|f\|$$

is valid for all $f \in H$. The smallest constant K of this kind will be called the *norm* of the functional G and designated by $\|G\|$. For linear functionals, this definition coincides with the usual definition of the norm. If we need to emphasize that the norm of f is considered just in the space H we use the notation $\|f\|_H$. Similarly, in order to emphasize that the norm of a functional G is taken just over the space H we use the symbol $\|G\|_H$.

In the numerical methods, $a_k(f)$ are usually linear combinations of the values of f (and possibly also of its derivatives) in certain points of the interval $[0, 2\pi]$. Therefore, we shall pay attention to the asymptotic properties of the sequences of approximations

$$\{G_p^{(l)}\}\,, \quad p \in P\,, \quad l = 1, 2, \ldots\,, \quad \text{where} \quad G_p^{(l)}(f) = \sum_{k=1}^{n} a_k^{(l)}(f)\, g_k(p) \tag{3.2}$$

and define the set $\tilde{M}_n(P)$ of approximations analogously to $M_n(P)$, but with

$$a_k^{(l)}(f) = \sum_{s=1}^{j(l)} \alpha_{s,j}^{(k)} f(x_{s,j}^{(k)}) \tag{3.3}$$

where $x_{s,j}^{(k)} \in [0, 2\pi]$ and $\alpha_{s,j}^{(k)}$ are complex numbers. The results obtained for the approximations from M_n are related in a very simple manner to the corresponding asymptotic statements for the sequences of approximations from $\tilde{M}_n$. Moreover, the fact that we have confined ourselves in (3.3) only to the linear combinations of the values of f is, as the reader will find, not substantial for the considerations of this paper. Our considerations will be concerned primarily with the properties of the functions $g_k(p)$. The distinction between M_n and $\tilde{M}_n$ is made rather for a better insight and for the practical applications of the results of the paper.

The functionals $a_k^{(l)}(f)$ defined by (3.3) are linear. Their additivity and homogeneity is obvious. The boundedness follows from the boundedness of the functionals in question on $C_{2\pi}$ and from the continuous imbedding of H into $C_{2\pi}$ stated in (a). The boundedness of the above $a_k^{(l)}(f)$ implies $\tilde{M}_n(P) \subset M_n(P)$.

Each approximation from $M_n(P)$ is assigned an r-tuple of the error functionals (also "the error of the approximation" in what follows) defined as

$$J_p(f) = I_p(f) - G_p(f) \tag{3.4}$$

where $f \in H$ and $p \in P$.

It will be necessary for the further study to know the errors of some simple approximations. Thus, in this section we shall prove some statements that will be used mostly as examples and counterexamples. Their proofs are based on the Riesz representation theorem.

Theorem 3.1. *Let H be a periodic space. The functionals I_p, p integer, are linear on H. If $p \notin A_H$ then $I_p(f) = 0$ for all $f \in H$ (the null functional).*

Proof. Obviously, I_p is additive and homogeneous. For $p \in A_H$, (2.4) implies

$$|I_p(f)| \leqq \frac{1}{\eta_p} \|f\|$$

for all $f \in H$, which proves the boundedness in this case. For $p \notin A_H$, the statement of the theorem is obtained through a simple calculation employing Theorem 2.2.

Now we may calculate the norm of the functional I_p.

Theorem 3.2. *Let H be a periodic space, p integer. Then*

$$\|I_p\| = \begin{cases} \dfrac{1}{\eta_p} & \text{for} \quad p \in A_H\,, \\ 0 & \text{for} \quad p \notin A_H\,. \end{cases}$$

Proof. Theorem 3.1 justifies using the Riesz representation theorem according to which for $f \in H$

$$I_p(f) = (f, F_p)\,, \quad \|I_p\| = \|F_p\|\,. \tag{3.5}$$

It is easy to verify that

$$F_p(x) = \begin{cases} \dfrac{1}{\eta_p^2}\, e^{ipx} & \text{for} \quad p \in A_H\,, \\ 0 & \text{for} \quad p \notin A_H\,. \end{cases} \tag{3.6}$$

The statement of the theorem follows immediately through calculating $\|F_p\|$ from the relation

$$\|F_p\|^2 = (F_p, F_p)\,.$$

In the following sections we shall also use the functionals

$$K_s = g(s)\,(\alpha I_p + \beta I_q) \tag{3.7}$$

where α, β are real numbers, s, p, q are integers, $p \neq q$ and $g(s)$ is a real function of an integer argument s. We shall take them for approximations of I_s. The point of this way of approximation is roughly as follows: We know (for a given $f \in H$) the value of some linear combination of two Fourier coefficients. We approximate the Fourier coefficients of f by this fixed linear combination in such a way that for the coefficient I_s we multiply the value of the above combination by the number $g(s)$, which depends on s but not on f. Obviously $\{K_s\} \in M_1$.

When studying the approximations of I_s we shall limit ourselves to the case where $s \in A_H$ (for $s \notin A_H$, I_s is the null functional). The norm of the error of the functional K_s is given by

Theorem 3.3. *Let H be a periodic space, $s, p, q \in A_H$, $p \neq q$. Denote*

$$\gamma(s) = g^2(s)\left(\frac{\alpha^2}{\eta_p^2} + \frac{\beta^2}{\eta_q^2}\right) + \frac{1}{\eta_s^2}\,.$$

Then

$$\|I_s - K_s\|^2 = \begin{cases} \gamma(s) & \text{for} \quad s \neq p, q\,, \\ \gamma(p) - 2g(p)\,\dfrac{\alpha}{\eta_p^2} & \text{for} \quad s = p\,, \\ \gamma(q) - 2g(q)\,\dfrac{\beta}{\eta_q^2} & \text{for} \quad s = q\,. \end{cases}$$

Proof. Clearly, K_s is linear and we can use the Riesz representation theorem. Using (3.5) and (3.6), we get for $s, p, q \in A_H$ and $f \in H$

$$K_s(f) = (f, E_s) \tag{3.8}$$

where

$$E_s(x) = g(s)\left(\frac{\alpha}{\eta_p^2}\,e^{ipx} + \frac{\beta}{\eta_q^2}\,e^{iqx}\right). \tag{3.9}$$

Similarly

$$I_s(f) - K_s(f) = (f, \Phi_s)$$

and using (3.5), (3.6) and (3.8) we get

$$\Phi_s(x) = \frac{e^{isx}}{\eta_s^2} - E_s(x) .$$

The norm of $I_s - K_s$ is obtained again from

$$\|I_s - K_s\|^2 = (\Phi_s, \Phi_s) .$$

The theorem is proved.

It is possible to prove analogous statements regarding the trapezoidal rule with j equally spaced abscissae, i.e. the functional

$$L_p^{(j)}(f) = \frac{1}{j} \sum_{k=1}^{j} e^{-i(2\pi/j)kp} f\left(\frac{2\pi}{j} k\right). \tag{3.10}$$

Before so doing, however, we introduce a convenient notation.

Definition 3.1. Let H be a periodic space, $p \in A_H$. Let j be a positive integer. We denote

$$C(j, p, \boldsymbol{\eta}) = \left(\eta_p^2 \sum_{\substack{t=-\infty \\ p-tj \in A_H}}^{+\infty} \eta_{p-tj}^{-2}\right). \tag{3.11}$$

The series in the above definition converges by virtue of Theorem 2.1. The quantity $C(j, p, \boldsymbol{\eta})$ appears in the expressions for the errors of the approximations using the trapezoidal rule. First we shall investigate its asymptotic behaviour as $j \to \infty$.

Lemma 3.1. *Let H be a periodic space, $p \in A_H$. Then*

$$\lim_{j \to \infty} \sum_{\substack{t; p-tj \in A_H \\ t \neq 0}} \eta_{p-tj}^{-2} = 0 \tag{3.12}$$

and

$$\lim_{j \to \infty} C(j, p, \boldsymbol{\eta}) = 1 . \tag{3.13}$$

Proof. Denote the sum occurring on the left-hand side of (3.12) by Σ'. Let $p \in A_H$ be given. We wish to prove that for every $\varepsilon > 0$ there exists an integer j_0 such that

$$\Sigma' \frac{1}{\eta_{p-tj}^2} < \varepsilon$$

is valid for all $j \geqq j_0$.

For arbitrary $\varepsilon > 0$, Theorem 2.1 implies the existence of K such that

$$\sum_{\substack{|k| > K \\ k \in A_H}} \frac{1}{\eta_k^2} < \varepsilon . \tag{3.14}$$

Clearly, for this K we may find j_0 such that

$$|p - tj| > K$$

is valid for all $j \geqq j_0$ and all $t \neq 0$. Thus for $j \geqq j_0$

$$\sum{}' \frac{1}{\eta_{p-tj}^2} = \sum_{\substack{t;p-tj \in A_H \\ |p-tj|>K}} \frac{1}{\eta_{p-tj}^2} \leqq \sum_{\substack{|k|>K \\ k \in A_H}} \frac{1}{\eta_k^2}.$$

Using (3.14) now, we get the first statement of the lemma. The second one follows immediately from (3.12) if we write

$$C(j, p, \boldsymbol{\eta}) = \left(1 + \eta_p^2 \sum{}' \frac{1}{\eta_{p-tj}^2}\right)^{-1}.$$

The lemma is proved.

Given p and j, we find now the norm of the error functional of the trapezoidal rule.

Theorem 3.4. *Let H be a periodic space, $p \in A_H$. Let j be a positive integer. Then*

$$\|I_p - L_p^{(j)}\|^2 = \eta_p^{-2} \frac{1 - C(j, p, \boldsymbol{\eta})}{C(j, p, \boldsymbol{\eta})}. \tag{3.15}$$

Thus, as $j \to \infty$, $L_p^{(j)}$ converges to I_p in the norm.

Proof is exactly parallel to that of Theorem 3.2 in [2] where the author assumes A_H to be the set of all integers. According to the Riesz representation theorem, for $p \in A_H$ and all $f \in H$

$$I_p(f) - L_p^{(j)}(f) = (f, \Phi_p^{(j)}),$$

where

$$\Phi_p^{(j)}(x) = \frac{e^{ipx}}{\eta_p^2} - \sum_{\substack{t=-\infty \\ p-tj \in A_H}}^{+\infty} \frac{e^{i(p-tj)x}}{\eta_{p-tj}^2}.$$

The norm of the error functional is now obtained from

$$\|I_p - L_p^{(j)}\|^2 = (\Phi_p^{(j)}, \Phi_p^{(j)}).$$

The convergence of $L_p^{(j)}$ follows immediately from (3.15) and Lemma 3.1, and the theorem is proved.

If we take care of the trapezoidal rule, the analogue of the functional (3.7) is the functional

$$N_s^{(j)} = g(s)\left(\alpha L_p^{(j)} + \beta L_q^{(j)}\right) \tag{3.16}$$

where α, β are real numbers, s, p, q are integers, j is a positive integer and $g(s)$ is a real function of an integer argument. Clearly $\{N_s^{(j)}\} \in \tilde{M}_1$. Since we shall be concerned only with the asymptotic properties of the functionals (3.16) as $j \to \infty$, our interest here is not in deriving the formula for the error in case of a given j, but rather in finding the limit of the norm of the error functional.

Theorem 3.5. *Let H be a periodic space, $p, q, s \in A_H$, $p \neq q$. Let there be given real numbers α, β, a function $g(s)$ and the functionals K_s and $N_s^{(j)}$ according to* (3.7) *and* (3.16). *Then*

$$\lim_{j\to\infty} \|I_s - N_s^{(j)}\| = \|I_s - K_s\| .$$

Proof follows immediately from (3.7), (3.16) and Theorem 3.4.

In the remainder of this section, we shall consider another approximation connected with the following problem. Suppose we know the values of $I_p(f)$ for $p \in P_1$ where P_1 is a subset of P. Now, the question is how to approximate I_p for $p \in P - P_1$. The reader will see that the approximation by zero functionals for $p \in P - P_1$ will play an important role in the study of universality.

We give a precise formulation restricting ourselves to the cases where $P \subset A_H$. For each positive integer n, $n < r$ we define the approximation $\{B_p\} \in M_n$ as follows: We divide P into two disjoint sets, $P = P_1 \cup P_2$, such that P_1 has n elements. We write $P_1 = \{p_1, p_2, \ldots, p_n\}$, $P_2 = \{p_{n+1}, p_{n+2}, \ldots, p_n\}$. Setting

$$(3.17)\qquad \begin{aligned} a_k &= I_{p_k}, \quad k = 1, 2, \ldots, n ;\\ g_k(p_j) &= 1 \quad \text{for} \quad k = j ,\\ &\ 0 \quad \text{for} \quad k \neq j , \quad j = 1, 2, \ldots, r \end{aligned}$$

in (3.1), we have clearly

$$\begin{aligned} B_{p_j} &= I_{p_j}, \quad j = 1, 2, \ldots, n ;\\ &\ O , \quad j = n + 1, n + 2, \ldots, r \end{aligned}$$

where O denotes the null functional, and

$$(3.18)\qquad \max_{p\in P} \|I_p - B_p\| = \max_{p\in P_2} \|I_p - B_p\| = \max_{p\in P_2} \frac{1}{\eta_p} .$$

Setting

$$(3.19)\qquad a_k = L_{p_k}^{(j)}, \quad k = 1, 2, \ldots, n ,$$

we obtain approximations $\{B_p^{(j)}\} \in \tilde{M}_n$, which form an analogue of $\{B_p\}$ important in practice.

Theorem 3.6. *Let H be a periodic space, $P \subset A_H$. Let n be a positive integer, $n > r$. Then there exists an integer j_0 such that*

$$\max_{p \in P} \|I_p - B_p^{(j)}\| = \max_{p \in P} \|I_p - B_p\| = \max_{p \in P_2} \frac{1}{\eta_p} \tag{3.20}$$

for every $j \geqq j_0$.

Proof follows immediately from (3.18) using Theorem 3.4 to find the number j_0 such that

$$\max_{p \in P_1} \|I_p - B_p^{(j)}\| = \max_{p \in P_1} \|I_p - L_p^{(j)}\| \leqq \max_{p \in P_2} \frac{1}{\eta_p}$$

for every $j \geqq j_0$.

Theorems 3.1 to 3.6 provide us with the necessary auxiliary material. Now we can pass to discussing the questions regarding the choice of the method.

4. THE OPTIMAL APPROXIMATIONS IN PERIODIC SPACES

Let there be given a periodic space H and the set P. Let $\{G_p\} \in M_n$ for a positive integer n. As a criterion for judging the quality of this approximation we shall use the quantity

$$\omega_H(P, G_p) = \max_{p \in P} \|J_p\|_H \tag{4.1}$$

where J_p, $p \in P$ are the error functionals (3.2). (The boundedness of J_p follows from that of I_p and a_k, $k = 1, 2, \ldots, n$.) Now let $M \subset M_n$ for some n. Then the best possible approximation from M is, for the given H and P, characterized by the quantity

$$\Omega_H(M, P) = \inf_M \omega_H(P, G_p) . \tag{4.2}$$

Further, we denote

$$\Omega_H(n, P) \equiv \Omega_H(M_n, P) , \tag{4.3}$$

$$\tilde{\Omega}_H(n, P) \equiv \Omega_H(\tilde{M}_n, P) .$$

Obviously, $\Omega_H(M, P) \geqq \Omega_H(n, P)$ for any $M \subset M_n$.

Definition 4.1. Let n be a positive integer, $M \subset M_n$. The approximation $\{G_p\}_{p \in P} \in M$ is said to be an *optimal approximation* from the set M and for given P and H if

$$\omega_H(P, G_p) = \Omega_H(M, P) .$$

For the sequences of approximations from $\tilde{M}_n$ the notion of asymptotical optimality is defined. We point out that we treat only such sequences that $g_k(p)$, $k = 1, 2, \ldots, n$ are the same for all the elements of a given sequence (cf. (3.2)). Clearly, this corresponds to the practical meaning of our considerations.

Definition 4.2. Let n be a positive integer, $M \subset \tilde{M}_n$. The sequence of approximations $\{G_p^{(j)}\}_{p \in P} \in M$, $j = 1, 2, \ldots$ is said to be *asymptotically optimal* if

$$\lim_{j \to \infty} \omega_H(P, G_p^{(j)}) = \Omega_H(M, P) .$$

Since for every other sequence $\{F_p^{(j)}\}_{p \in P} \in M$, $j = 1, 2, \ldots$

$$\liminf_{j \to \infty} \omega_H(P, F_p^{(j)}) \geqq \Omega_H(M, P) ,$$

$\{G_p^{(j)}\}$, $j = 1, 2, \ldots$ will be also called an asymptotically optimal sequence in the class of the sequences of approximations from M.

Denote the number of the elements of the set $P \cap A_H$ by m and suppose $m > 1$ in what follows. As indicated in the introduction we confine the considerations of the paper to the approximations from M_n with $n < m$.

Remark 4.1. For $n \geqq m$, the whole problem has diverse character, since in this case also $\{I_p\}$ belongs to M_n. As far as the set $\tilde{M}_n$ is concerned the problem of finding the optimal approximation for a fixed j may be transferred to that solved in [2]. Moreover, the results of [2] may be applied for the study of further aspects (asymptotical optimality, universality).

Now, for the reader's convenience, we sum up the assumptions made on the space H and the set P and regarding the remainder of the paper:

(4.4) (1) H is a periodic space (which does not consist only of the zero function).
(2) The elements of P are distinct; their number is denoted by r.
(3) $P \cap A_H$ has m elements, $m > 1$.
(4) $0 < n < m$, n integer.

Under these assumptions we show in Theorem 4.1 that $\Omega_H(n, P) > 0$. Therefore, for each approximation from $M \subset M_n$ we can form the quotient

$$Q_H(M, P, G_p) = \frac{\omega_H(P, G_p)}{\Omega_H(M, P)} \tag{4.5}$$

and say that the approximation $\{G_p\}$ is the "better" in a given space and with respect to the set M, the smaller $Q_H(M, P, G_p)$ is. Clearly $Q_H \geqq 1$. Analogously to (4.3) we shall use the notation $Q_H(n, P, G_p)$ and $\tilde{Q}_H(n, P, G_p)$. The subscript H will be used only where it is essential for understanding the text.

In deriving the upper bounds for the quantity Q we shall need a lower bound on Ω. To find the latter we shall use

Lemma 4.1. *Let H be a periodic space with the zero element Θ. Let the set P and a positive integer n be given. If for all $f \in H$ and all approximations $\{G_p\} \in M_n$*

$$\inf_{a_k} \max_{P} |I_p(f) - G_p(f)| \geqq C_H(f, g_1, g_2, \ldots, g_n) \tag{4.6}$$

is valid, then

$$\Omega_H(n, P) \geqq \inf_{g_k} \sup_{\substack{f \in H \\ f \neq \Theta}} \left(\frac{C_H(f, g_1, g_2, \ldots, g_n)}{\|f\|} \right). \tag{4.7}$$

Proof. The inequality

$$\|J_p\| \geqq \frac{|J_p(f)|}{\|f\|},$$

where J_p, $p \in P$ are the error functionals, holds clearly for all approximations from the set M_n, all $p \in P$ and all functions $f \in H, f \neq \Theta$.

Therefore,

$$\max_P \|J_p\| \geqq \|f\|^{-1} \max_P |J_p(f)|$$

for all $f \in H, f \neq \Theta$ and all $\{G_p\} \in M_n$. Moreover, obviously

$$\max_P \|J_p\| \geqq \|f\|^{-1} \inf_{a_k} \max_P |J_p(f)|.$$

Using (4.6) we get now

$$\max_P \|J_p\| \geqq \|f\|^{-1} C_H(f, g_1, g_2, \ldots, g_n)$$

for all $f \in H, f \neq \Theta$ and all $\{G_p\} \in M_n$, where the right-hand side is independent of the functionals a_k.

Further, from this inequality we obtain

$$\max_P \|J_p\| \geqq \sup_{f \in H, f \neq \Theta} (\|f\|^{-1} C_H(f, g_1, g_2, \ldots, g_n)) \tag{4.8}$$

for all $\{G_p\} \in M_n$ and finally

$$\begin{aligned} \inf_{M_n} \max_P \|J_p\| &\geqq \inf_{M_n} \sup_{f \in H, f \neq \Theta} (\|f\|^{-1} C_H(f, g_1, \ldots, g_n)) = \\ &= \inf_{g_k} \sup_{f \in H, f \neq \Theta} (\|f\|^{-1} C_H(f, g_1, \ldots, g_n)), \end{aligned}$$

which completes the proof.

Now we are in position to derive the lower bound for $\Omega(n, P)$.

Theorem 4.1. *Let H be a periodic space. Given the set P, denote $R = P \cap A_H$ and let $\{p_1, p_2, \ldots, p_{n+1}\}$ be an $(n + 1)$-tuple of integers $p_j \in R$. Then*

$$\Omega(n, P) \geqq \Big(\sum_{j=1}^{n+1} \eta_{p_j}^2\Big)^{-1/2} . \tag{4.9}$$

Proof. Our basic tool here will be Lemma 4.1. Therefore, we need a lower bound on

$$\inf_{a_k} \max_{P} |J_p(f)| = \inf_{a_k} \max_{P} \Big|I_p(f) - \sum_{k=1}^{n} a_k(f)\, g_k(p)\Big|$$

satisfying the hypothesis of the above lemma.

We choose some $(n + 1)$-tuple $\{p_1, p_2, \ldots, p_{n+1}\}$ of numbers $p_j \in R$ and denote

$$g_k(p_j) = g_{kj}\,, \quad I_{p_j}(f) = \hat{I}_j(f)\,,$$

and

$$J_{p_j}(f) = \hat{J}_j(f)\,,$$

$k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n + 1$. We note that

$$\max_{p \in P} |J_p(f)| \geqq \max_{p \in R} |J_p(f)| \geqq \max_{j=1,2,\ldots,n+1} |\hat{J}_j(f)|\,. \tag{4.10}$$

Now, let us pay attention to n-dimensional vectors

$$[g_{1j}, g_{2j}, \ldots, g_{nj}]\,, \quad j = 1, 2, \ldots, n + 1\,.$$

Since there is $(n + 1)$ of them, they are linearly dependent and thus we can find numbers λ_j, $j = 1, 2, \ldots, n + 1$ such that

$$\sum_{j=1}^{n+1} \lambda_j g_{kj} = 0\,, \quad k = 1, 2, \ldots, n \tag{4.11}$$

and

$$\sum_{j=1}^{n+1} |\lambda_j| = 1\,. \tag{4.12}$$

For these λ_j's we calculate

$$\sum_{j=1}^{n+1} \lambda_j \hat{J}_j(f) = \sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f) - \sum_{j=1}^{n+1} \lambda_j \Big(\sum_{k=1}^{n} a_k(f)\, g_{kj}\Big) =$$
$$= \sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f) - \sum_{k=1}^{n} a_k(f) \Big(\sum_{j=1}^{n+1} \lambda_j g_{kj}\Big)\,.$$

By (4.11), the second term on the right-hand side is zero and we have

$$\sum_{j=1}^{n+1} \lambda_j \hat{J}_j(f) = \sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f)$$

for every $f \in H$.

For the absolute values we obtain

$$\left|\sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f)\right| \leqq \sum_{j=1}^{n+1} |\lambda_j|\, |\hat{J}_j(f)| \leqq \sum_{j=1}^{n+1} |\lambda_j| \max_{j=1,2,\ldots,n+1} |\hat{J}_j(f)| \,.$$

Substituting from (4.12), we have

$$\max_{j=1,2,\ldots,n+1} |\hat{J}_j(f)| \geqq \left|\sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f)\right| .$$

The λ_j's depend only on $g_k(p_j)$. Therefore, the right-hand side of the above inequality is independent of $a_k(f)$ and we get, using (4.10),

$$\inf_{a_k} \max_{P} |J_p(f)| \geqq \left|\sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f))\right| .$$

This inequality holds for all $f \in H$ and all approximations $\{G_p\} \in M_n$ and satisfies the hypothesis of Lemma 4.1.

Using (4.7) we shall need to know

$$S \equiv \sup_{f \in H, f \neq \theta} \frac{\left|\sum_{j=1}^{n+1} \lambda_j \hat{I}_j(f)\right|}{\|f\|} .$$

According to (3.5) and (3.6)

$$S = \sup_{f \in H, f \neq \theta} \frac{\left|\sum_{j=1}^{n+1} \lambda_j (f, \hat{F}_j)\right|}{\|f\|}$$

and

$$\hat{F}_j(x) = \frac{1}{\hat{\eta}_j^2} e^{ip_j x}$$

where we have denoted $\hat{\eta}_j \equiv \eta_{p_j}$ for $p_j \in R$. We may easily find

$$S = \sup_{f \in H, f \neq \theta} \frac{\left|(f, \sum_{j=1}^{n+1} \bar{\lambda}_j \hat{F}_j)\right|}{\|f\|} = \left\|\sum_{j=1}^{n+1} \bar{\lambda}_j \hat{F}_j\right\|$$

where the bar denotes complex conjugation. From (2.3),

$$(\hat{F}_j, \hat{F}_k) = 0 \quad \text{for} \quad k \neq j \,; \quad k, j = 1, 2, \ldots, n+1 \,,$$

$$(\hat{F}_j, \hat{F}_j) = \frac{1}{\hat{\eta}_j^2} \,, \quad j = 1, 2, \ldots, n+1$$

and we may easily compute

(4.13) $$S = \sqrt{\left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\hat{\eta}_j^2}\right)} .$$

Now, it remains only to find

$$\inf_{g_k} \sqrt{\left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\eta_j^2}\right)}.$$

The expression the exact lower bound of which we are looking for depends only on the values of $g_k(p)$, $k = 1, 2, \ldots, n$ in the points $p = p_j$, $j = 1, 2, \ldots, n + 1$. As stated at the beginning of the proof, each matrix $[g_{kj}]$, $k = 1, 2, \ldots, n$; $j = 1, 2, \ldots, n + 1$ determines some numbers λ_j, $j = 1, 2, \ldots, n + 1$ satisfying (4.12). And, conversely, every $(n + 1)$-tuple of numbers λ_j satisfying (4.12) corresponds to some matrices $[g_{kj}]$. Thus, the problem becomes that of finding

$$\inf_{\substack{[\lambda_j], j=1,2,\ldots,n+1; \\ \sum_{j=1}^{n+1} |\lambda_j| = 1}} \left(\sqrt{\left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\eta_j^2}\right)}\right).$$

As we shall now show using the Cauchy inequality, the expression under consideration attains on the set of the vectors $[\lambda_j]$, $j = 1, 2, \ldots, n + 1$ satisfying (4.12) its exact lower bound, and that for

$$\lambda_j = \frac{\eta_j^2}{\sum_{k=1}^{n+1} \eta_k^2}, \quad j = 1, 2, \ldots, n + 1.$$

We have namely

$$\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\eta_j^2} \cdot \sum_{j=1}^{n+1} \eta_j^2 \geqq \left(\sum_{j=1}^{n+1} |\lambda_j|\right)^2 = 1$$

and therefore

$$\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\eta_j^2} \geqq \left(\sum_{j=1}^{n+1} \eta_j^2\right)^{-1}$$

for all $[\lambda_j]$'s from the set mentioned above. That the lower bound is attained may be verified by substituting for λ_j.

This completes the proof of the theorem.

We now give three remarks regarding Theorem 4.1 and its proof.

Remark 4.2. One may see from Theorem 4.1 that for $m > n + 1$ we can obtain different bounds on Ω choosing different $(n + 1)$-tuples of numbers from R. The best choice yielding the most realistic bound depends on the space H.

Remark 4.3. To find a lower bound on Ω it was not inevitably necessary to use the second inequality in (4.10). If we had bounded, however, $\max_{p \in P} |J_p(f)|$ in the same way, we should have obtained a worse bound on Ω as a consequence of our further procedure.

Remark 4.4. Let there be given functions $g_1(p), g_2(p), \ldots, g_n(p)$. Denote the set of all approximations from M_n employing these $g_k(p)$'s by M_n^g. Then, on the assumptions of Theorem 4.1, (4.8) and (4.13) yield

$$\Omega(M_n^g, P) \geqq \sqrt{\left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\eta_{p_j}^2}\right)} \tag{4.14}$$

where the λ_j's are given by (4.11), (4.12).

A reasonable question to ask at this point is whether the result of Theorem 4.1 with its rather general formulation can be improved on substantially. The answer is negative. First we shall show that the equality sign stands in (4.9) in a special case.

Theorem 4.2. *Let H be a periodic space, $P^* = \{p, q\}$, $P^* \subset A_H$. Then*

$$\Omega(1, P^*) = (\eta_p^2 + \eta_q^2)^{-1/2}.$$

Proof. We have $n = 1$, $r = m = 2$ and the assumptions of Theorem 4.1 are satisfied. We shall find an optimal approximation in $M_1(P^*)$, i.e. a couple of functionals $\{K_p, K_q\}$ such that

$$\max_{s=p,q} \|I_s - K_s\| = (\eta_p^2 + \eta_q^2)^{-1/2}.$$

If K_s is given by (3.7) the couple $\{K_p, K_q\}$ belongs to $M_1(P^*)$ evidently. We denote by $\{K_p^*, K_q^*\}$ the approximation obtained by setting

$$\text{a) } \alpha = \beta = A, \quad A \text{ real}, \quad A \neq 0; \tag{4.15}$$

$$\text{b) } g(p) = \frac{\eta_q^2}{A(\eta_p^2 + \eta_q^2)}, \quad g(q) = \frac{\eta_p^2}{A(\eta_p^2 + \eta_q^2)}$$

in (3.7). The error of this approximation is given by Theorem 3.3. After substitution for α, β and $g(s)$ we get

$$\|I_p - K_p^*\|^2 = \|I_q - K_q^*\|^2 = \frac{1}{\eta_p^2 + \eta_q^2}. \tag{4.16}$$

Thus, $\{K_p^*, K_q^*\}$ is indeed an optimal approximation from $M_1(P^*)$ for given P^* and H and we have

$$\Omega^2(1, P^*) = \frac{1}{\eta_p^2 + \eta_q^2},$$

as required.

Clearly, Theorem 4.1 holds also in the case of the set $\tilde{M}_n$ and the quantity $\tilde{\Omega}$. Even in this case it yields an acceptable bound, for we can show the lower bound in (4.9) to be attainable asymptotically as $j \to \infty$.

Theorem 4.3. *Let H be a periodic space, $P^* = \{p, q\}$, $P^* \subset A_H$. Then*

$$\tilde{\Omega}(1, P^*) = (\eta_p^2 + \eta_q^2)^{-1/2} .$$

Proof is by contradiction. We have $n = 1$, $r = m = 2$ and the assumptions of Theorem 4.1 are satisfied. Since $\tilde{\Omega}(1, P^*) \geqq \Omega(1, P^*)$, (4.9) is true also for $\tilde{\Omega}(1, P^*)$.

Let

$$\tilde{\Omega}(1, P^*) = (\eta_p^2 + \eta_q^2)^{-1/2} + \varepsilon$$

where $\varepsilon > 0$. Denote by $N_s^{*(j)}$, $s = p, q$ the functionals given by (3.16) where α, β and $g(s)$ are the same as those in the proof of Theorem 4.2. The approximations $\{N_p^{*(j)}, N_q^{*(j)}\}$ belong to $\tilde{M}_1(P^*)$ obviously for all positive integers j.

By Theorem 3.5 and with respect to (4.16), there exists a positive integer j_0 such that

$$\max_{s=p,q} \|I_s - N_s^{*(j_0)}\| < (\eta_p^2 + \eta_q^2)^{-1/2} + \varepsilon ,$$

which contradicts the definition of $\tilde{\Omega}(1, P^*)$. The theorem is proved.

We may easily prove an assertion supplementing Theorem 4.2 in a sense.

Theorem 4.4. *Let the set P have r elements, $r > 1$. Given an integer n, $0 < n < r$, for every $\varepsilon > 0$ there exists a periodic space H_ε such that $A_{H_\varepsilon} \supset P$ and that for each $(n+1)$-tuple $\{p_1, p_2, \ldots, p_{n+1}\}$ of numbers $p_j \in P$*

$$\Omega_{H_\varepsilon}(n, P) < \left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{-1/2} + \varepsilon$$

is valid.

Proof. Given an approximation $\{G_p\} \in M_n$,

$$\Omega(n, P) \leqq \omega(P, G_p)$$

holds in every periodic space. Let us consider the class of all periodic spaces with the property that $A_H \supset P$ and take for $\{G_p\}$ especially $\{B_p\}$ from Sec. 3 dividing $P = P_1 \cup P_2$ arbitrarily.

We get for each space from the class considered

$$\Omega(n, P) \leqq \max_{p \in P_2} \frac{1}{\eta_p} . \tag{4.17}$$

We look for the space where

$$\max_{p \in P_2} \frac{1}{\eta_p} < \left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{-1/2} + \varepsilon . \tag{4.18}$$

Such a space does exist, for it is sufficient to require

$$\max_{p \in P_2} \frac{1}{\eta_p} < \varepsilon$$

and use Theorem 2.3. The theorem is proved.

An analogous statement regarding $\tilde{\Omega}(n, P)$ might be proved using $\{B_p^{(j)}\}$ from Sec. 3.

Proving Theorems 4.2 and 4.3 we constructed an optimal approximation and an asymptotically optimal sequence of approximations, respectively. Constructing the optimal functionals we utilized the information on the space we worked in. Therefore, we may expect the properties of the optimal approximation constructed to depend strongly on the space in which we work. That is, the Q of our approximation, which is 1 in a given space, may be expected to be large in other spaces. In fact, this *is* the case. More precisely, the whole situation is illustrated by the following

Theorem 4.5. *Let H be a periodic space, $P^* = \{p, q\}$, $P^* \subset A_H$. Construct the optimal approximation $\{K_s^*\}_{s=p,q}$ from $M_1(P^*)$ in the given space (cf. (4.15)). Then for every $D > 0$ there exists a periodic space H' such that*

$$Q_{H'}(1, P^*, K_s^*) > D\,.$$

Proof. It is sufficient to find, for a given $D > 0$, a periodic space H' such that

$$\frac{\|J_p^*\|_{H'}^2}{\Omega_{H'}^2(1, P^*)} > D^2$$

where $J_p^* = I_p - K_p^*$ is the error functional. Denote

$$\|e^{ikx}\|_H = \eta_k\,,$$
$$\|e^{ikx}\|_{H'} = \varepsilon_k$$

for $k \in A_H$, $k \in A_{H'}$ respectively. We shall take into consideration only the spaces H' with the property $P^* \subset A_{H'}$.

By Theorem 3.3 we get, substituting for α, β, $g(s)$ from (4.15),

$$\|J_p^*\|_{H'}^2 = \frac{\eta_q^4}{(\eta_p^2 + \eta_q^2)^2} \cdot \frac{\varepsilon_p^2 + \varepsilon_q^2}{\varepsilon_p^2 \varepsilon_q^2} + \frac{1}{\varepsilon_p^2}\left[1 - \frac{2\eta_q^2}{\eta_p^2 + \eta_q^2}\right].$$

Introducing a convenient notation, we may write

$$\|J_p^*\|_{H'}^2 = \frac{1}{\varepsilon_p^2}\left[C_1\left(1 + \frac{\varepsilon_p^2}{\varepsilon_q^2}\right) + C_2\right],$$

where C_1, C_2 are constants given by the space H, $C_1 > 0$, $C_1 + C_2 > 0$. In virtue of Theorem 4.2 we have

$$\Omega^2_{H'}(1, P^*) = \frac{1}{\varepsilon_p^2 + \varepsilon_q^2}.$$

Thus

$$\frac{\|J_p^*\|^2_{H'}}{\Omega^2_{H'}(1, P^*)} \geqq \left(1 + \frac{\varepsilon_q^2}{\varepsilon_p^2}\right)(C_1 + C_2).$$

From this we can see that H' may be an arbitrary periodic space such that $A_{H'} \supset P^*$ and

$$\frac{\varepsilon_q^2}{\varepsilon_p^2} > \frac{D^2}{C_1 + C_2} - 1.$$

According to Theorem 2.3, such a space does exist, and the theorem is proved.

Evidently, if $|p| \neq |q|$ the theorem holds even though we confine ourselves to strongly periodic spaces H'. (If $|p| = |q|$ then $g(p) = g(q) = 1/(2A)$ in the class of strogly periodic spaces independently of the space, and $\{K_s^*\}$, $s \in P^*$ is optimal in every strongly periodic space.)

Analogous statements regarding the functionals $N_s^{*(j)}$ from Theorem 4.3, which are asymptotically optimal for given P^* and H, hold asymptotically.

Theorem 4.6. *Let H be a periodic space, $P^* = \{p, q\}$, $P^* \subset A_H$. Construct the asymptotically optimal sequence of approximations $\{N_s^{*(j)}\}_{s=p,q} \in \tilde{M}_1(P^*)$,*

$$N_s^{*(j)} = g(s)\left(\alpha L_p^{(j)} + \beta L_q^{(j)}\right),$$

in the space given (cf. (4.15)). Then for every $\tilde{D} > 0$ there exists a periodic space H' such that

$$\lim_{j\to\infty} \tilde{\Omega}_{H'}(1, P^*, N_s^{*(j)}) > \tilde{D}.$$

Proof. It is true in every periodic space H' that

$$\lim_{j\to\infty} \tilde{Q}_{H'}(1, P^*, N_s^{*(j)}) = \frac{\lim\limits_{j\to\infty} \omega_{H'}(P^*, N_s^{*(j)})}{\tilde{Q}_{H'}(1, P^*)} = \frac{\omega_{H'}(P^*, K_s^*)}{\Omega_{H'}(1, P^*)} = Q_{H'}(1, P^*, K_s^*),$$

as follows from Theorems 4.2, 4.3 and 3.5. The statement of Theorem 4.6 follows now from Theorem 4.5.

As a rule in practice, the information available on $f(x)$ is not so detailed as to enable us to determine the space in which the computation is to be carried out. Thus, we cannot use the optimal approximation without the risk illustrated by Theorems 4.5

and 4.6, the integrand being inserted only in an indefinite space from a class of spaces. It is of considerable importance, therefore, to search for such approximations that their Q is bounded on some class of periodic spaces.

This implies the purpose of the following

Definition 4.3. An approximation $\{G_p\}_{p\in P} \in M_n$ is termed a *universal approximation* for a given P and a class $\mathfrak{H}$ of periodic spaces if

$$Q_H(n, P, G_p) \leqq D(n, P)$$

holds in every space $H \in \mathfrak{H}$ and $D(n, P)$ is a constant independent of H.

This constant has also its quantitative meaning understandably. Similarly we have

Definition 4.4. A sequence of approximations $\{G_p^{(j)}\}_{p\in P}$, $j = 1, 2, \ldots$ from $\tilde{M}_n$ is termed a *universal sequence of approximations* for a given P and a class $\mathfrak{H}$ of periodic spaces if

$$\limsup_{j\to\infty} \tilde{Q}_H(n, P, G_p^{(j)}) \leqq \tilde{D}(n, P)$$

holds in every space $H \in \mathfrak{H}$ and $\tilde{D}(n, P)$ is a constant independent of H.

The following simple lemma will be of use in constructing the universal sequences of approximations.

Lemma 4.2. *Let $\{G_p\}$ be a universal approximation from M_n for a given set P and a given class $\mathfrak{H}$ of periodic spaces. Let $\{G_p^{(j)}\}$, $j = 1, 2, \ldots$ be a sequence of approximations from $\tilde{M}_n$. If*

$$\lim_{j\to\infty} \max_P \|I_p - G_p^{(j)}\| = \max_P \|I_p - G_p\| \tag{4.19}$$

is valid in all spaces from $\mathfrak{H}$, then $\{G_p^{(j)}\}$, $j = 1, 2, \ldots$ is a universal sequence of approximations and

$$\lim_{j\to\infty} \tilde{Q}_H(n, P, G_p^{(j)}) \leqq Q_H(n, P, G_p)$$

holds for every $H \in \mathfrak{H}$.

Proof. Given $H \in \mathfrak{H}$,

$$\tilde{Q}_H(n, P, G_p^{(j)}) \leqq Q_H(n, P, G_p^{(j)})$$

holds for every positive integer j. Using (4.19) we get

$$\limsup_{j\to\infty} \tilde{Q}_H(n, P, G_p^{(j)}) \leqq \lim_{j\to\infty} Q_H(n, P, G_p^{(j)}) = Q_H(n, P, G_p) \leqq D(n, P)$$

for every $H \in \mathfrak{H}$, as required.

In the following section we shall treat universal approximations in classes of periodic spaces, our interest being in the choice of the functions $g_k(p)$ primarily. The class of all periodic spaces will be denoted by $\mathfrak{H}_1$, the class of periodic spaces such that A_H is the set of all integers will be denoted by $\mathfrak{H}_2$ and the class of all strongly periodic spaces by $\mathfrak{H}_3$. Clearly, $\mathfrak{H}_1 \supset \mathfrak{H}_2 \supset \mathfrak{H}_3$.

5. THE UNIVERSAL APPROXIMATIONS IN PERIODIC SPACES

In the foregoing section we gave an example of an approximation not universal with respect to any of the classes $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3$. Naturally, the question arises whether for these classes a universal approximation exists at all.

Theorem 5.1. *For any n and P there exists no approximation in $M_n(P)$ universal with respect to $\mathfrak{H}_1$ or $\mathfrak{H}_2$.*

Proof. Since $\mathfrak{H}_2 \subset \mathfrak{H}_1$, it is sufficient to prove the theorem for the class $\mathfrak{H}_2$. We shall show that for an arbitrary approximation $\{G_p\} \in M_n$ and an arbitrary number D there exists a space $H \in \mathfrak{H}_2$ such that

$$Q_H(n, P, G_p) > D\,.$$

Choose $(n + 1)$ elements $p_1, p_2, \ldots, p_{n+1}$ from P arbitrarily. Now, the approximation $\{G_p\}$ determines some numbers λ_j, $j = 1, 2, \ldots, n + 1$ satisfying (4.11), (4.12) and independent of H. The λ_j's cannot vanish simultaneously; denote by λ_s a non-zero one. By (4.14), we get for the approximation given

$$\omega(P, G_p) \geqq \left(\sum_{j=1}^{n+1} \frac{|\lambda_j|^2}{\eta_{p_j}^2}\right)^{1/2} \geqq \frac{|\lambda_s|}{\eta_{p_s}}\,.$$

By (4.17),

$$\Omega(n, P) \leqq \max_{p\in P_2} \frac{1}{\eta_p}\,,$$

having split P arbitrarily into two disjoint subsets P_1 and P_2 such that P_1 has n elements.

Make this division in such a way that $p_s \in P_1$. Then

$$Q_H(n, P, G_p) \geqq |\lambda_s| \frac{\eta_q}{\eta_{p_s}}$$

where $1/\eta_q = \max\limits_{p\in P_2} 1/\eta_p$ and the subscript q depends on H of course. In virtue of Theorem 2.3, for every D it is possible to find a space $H \in \mathfrak{H}_2$ such that

$$\frac{\eta_q}{\eta_{p_s}} > \frac{D}{|\lambda_s|}\,,$$

which completes the proof.

Remark 5.1. The proofs of further theorems regarding the non-existence of universal approximation (or the conditions necessary for a given approximation to be universal) are analogous to the proof of Theorem 5.1 in outline. The basis of the latter proof is formed by the bounds (4.14) and (4.17). But, (4.14) holds also for the approximations from $\tilde{M}_n$, and using (3.20) we have

$$\Omega(n, P) \leqq \tilde{\Omega}(n, P) \leqq \max_{P_2} \frac{1}{\eta_p} \tag{5.1}$$

where P_2 was described in the proof of the foregoing theorem. Therefore, if we prove some theorem of the kind mentioned above for the universal approximations from M_n, we can simply obtain an analogous asymptotic statement regarding the sequences of approximations from $\tilde{M}_n$.

The proof of the following theorem concerning $\tilde{M}_n$ will be yet indicated, the passage from M_n to $\tilde{M}_n$ being left to the reader throughout the remainder of the section.

Theorem 5.2. *For any n and P there exists no sequence of approximations from $\tilde{M}_n(P)$ universal with respect to $\mathfrak{H}_1$ or $\mathfrak{H}_2$.*

Proof. Again, it is sufficient to prove the theorem for $\mathfrak{H}_2$. As we have stated, by (3.20) the bound (4.17) used in the proof of Theorem 5.1 is true also for $\tilde{\Omega}(n, P)$. Besides this bound we used only the properties of the functions $g_k(p)$ (i.e. those of the λ_j's and the bound (4.14)) in the above proof. Thus, for an arbitrary approximation $\{G_p^{(j)}\} \in \tilde{M}_n$ and number D we can find a space $H \in \mathfrak{H}_2$ such that

$$\tilde{Q}_H(n, P, G_p^{(j)}) > D$$

independently of j. Therefore, in this space

$$\limsup_{j \to \infty} \tilde{Q}_H(n, P, G_p^{(j)}) \geqq D ,$$

as required.

Before we shall consider the existence of a universal approximation for the class $\mathfrak{H}_3$, we rewrite Theorem 4.1 for the case of strongly periodic spaces. Throughout the remainder of the section we shall suppose the set P to be arranged in such a manner that $|p_i| \geqq |p_j|$ holds whenever $i \geqq j$. It follows then from Definition 2.4

$$\eta_{p_i} \geqq \eta_{p_j} \quad \text{for} \quad i \geqq j$$

in strongly periodic spaces.

Theorem 5.3. *Let there be given a strongly periodic space H and the set P. Then*

$$\Omega(n, P) \geqq \left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{-1/2} . \tag{5.2}$$

Owing to the arrangement of P introduced above, this bound is the best one from those obtainable by Theorem 4.1.

Proving Theorem 5.1 on the non-existence of a universal approximation with respect to $\mathfrak{H}_1$ and $\mathfrak{H}_2$ we made use of that in various spaces from these classes the norms η_k may behave arbitrarily to some extent. With the class $\mathfrak{H}_3$, however, the η_k's have the properties (c), (d). Thus we may expect some change in the situation now.

Theorem 5.4. *Given the set P and a positive integer n, there exists an approximation from M_n universal with respect to $\mathfrak{H}_3$. It is the approximation $\{B_p\}$ from Sec. 3 given by the division $P = P_1 \cup P_2$ such that $P_1 = \{p_1, p_2, \dots, p_n\}$, $P_2 = \{p_{n+1}, p_{n+2}, \dots, p_r\}$, and (3.17). Moreover,*

$$Q(n, P, B_p) \leqq \left(1 + \sum_{k=1}^{n} \left(\frac{\eta_{p_k}}{\eta_{p_{n+1}}}\right)^2\right)^{1/2} \leqq (1 + n)^{1/2} \tag{5.3}$$

holds for every $H \in \mathfrak{H}_3$.

Proof. By (3.18) and using the properties (c), (d) we get

$$\omega(P, B_p) = \max_{p \in P_2} \frac{1}{\eta_p} = \frac{1}{\eta_{p_{n+1}}}. \tag{5.4}$$

This and (5.2) yield

$$Q(n, P, B_p) \leqq \frac{1}{\eta_{p_{n+1}}} \left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{1/2}.$$

The statement of the theorem follows now readily by a trivial modification using (c) and (d).

A practical analogue is

Theorem 5.5. *Given the set P and a positive integer n, there exists a sequence of approximations from $\tilde{M}_n$ universal with respect to $\mathfrak{H}_3$. They are the approximations $\{B_p^{(j)}\}$ from Sec. 3 with the division $P = P_1 \cup P_2$ such as in Theorem 5.4. Moreover, for every $H \in \mathfrak{H}_3$ there exists an integer j_0 such that*

$$\tilde{Q}(n, P, B_p^{(j)}) \leqq \left(1 + \sum_{k=1}^{n} \left(\frac{\eta_{p_k}}{\eta^2_{p_{n+1}}}\right)^2\right)^{1/2} \leqq (1 + n)^{1/2} \tag{5.5}$$

holds for all $j \geqq j_0$.

Proof follows immediately from Theorems 5.4, 3.6 and Lemma 4.2.

Remark 5.2. We can see from (5.3) and (5.5) that if $\eta_{p_{n+1}}$ is sufficiently large in comparison with η_{p_n}, the quotients Q and $\tilde{Q}$ from Theorems 5.4, 5.5 may approach 1 arbitrarily closely.

Now, our interest is whether it is essential for the universality that there was

$$(5.6)\qquad g_k(p_j) = 1 \quad \text{for} \quad k = j\,, \qquad 0 \quad \text{for} \quad k \neq j\,, \quad k = 1, 2, \ldots, n\,; \quad j = 1, 2, \ldots, r$$

in Theorems 5.4 and 5.5 or whether there exist universal approximations (or sequences of approximations) employing other systems $\{g_k(p)\}$. If such approximations exist we shall be interested in bounding their Q (or $\tilde{Q}$). Before proceeding to considerations just suggested observe that, with the above arrangement of P, the upper bound (5.1) of the quantities Ω and $\tilde{\Omega}$ has the form

$$(5.7)\qquad \Omega(n, P) \leqq \tilde{\Omega}(n, P) \leqq \frac{1}{\eta_{p_{n+1}}}$$

in strongly periodic spaces.

Because of (4.14), it will prove reasonable to carry out further investigation not with the systems $\{g_k(p)\}_{k=1}^{n}$, $p \in P$, but right with the numbers λ_j occurring in (4.14).

For a given n, consider all the subsets of P of the form

$$P_s = \{p_1, p_2, \ldots, p_n, p_s\}\,, \quad n + 1 \leqq s \leqq r\,.$$

Each of these subsets together with the system $\{g_k(p)\}$ determines the numbers $\lambda_1^{(s)}, \lambda_2^{(s)}, \ldots, \lambda_{n+1}^{(s)}$ (cf. (4.11), (4.12)). There may be more than one such an $(n + 1)$-tuple for a given $\{g_k(p)\}$. Altogether, they represent all the solutions of the system

$$(5.8)\qquad \lambda_1^{(s)}\, g_k(p_1) + \lambda_2^{(s)}\, g_k(p_2) + \ldots + \lambda_n^{(s)}\, g_k(p_k) + \lambda_{n+1}^{(s)}\, g_k(p_s) = 0\,, \qquad k = 1, 2, \ldots, n$$

satisfying the condition

$$(5.9)\qquad \sum_{j=1}^{n+1} |\lambda_j^{(s)}| = 1\,.$$

We denote by W_s the linear space of all the solutions of (5.8), its dimension being designated by dim W_s. Further, we shall write

$$\mathbf{e}_j = [\delta_{1j}, \delta_{2j}, \ldots, \delta_{n+1,j}]\,, \quad j = 1, 2, \ldots, n + 1$$

where δ_{ij} is the Kronecker delta*). The vectors $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}$ form a basis of the $(n + 1)$-dimensional linear vector space W. Given a linearly independent system $\{\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_l\}$ of vectors from W, we denote by $V(\mathbf{d}_1, \mathbf{d}_2, \ldots, \mathbf{d}_l)$ the linear subspace of W with the above system as a basis. Finally, let $(\lambda_1^{(s)}, \lambda_2^{(s)}, \ldots, \lambda_{n+1}^{(s)})$ be a solution of (5.8) satisfying (5.9). We denote

$$\boldsymbol{\lambda}^{(s)} = [|\lambda_1^{(s)}|, |\lambda_2^{(s)}|, \ldots, |\lambda_{n+1}^{(s)}|]$$

*) $\delta_{ij} = 0$ unless $i = j$, in which case $\delta_{ij} = 1$.

and the system of vectors $\{\boldsymbol{\lambda}^{(s)}\}_{s=n+1}^{r}$ will be called a *system of vectors* $\boldsymbol{\lambda}^{(s)}$ *determined by the system of functions* $g_k(p)$.

Now, return to Theorems 5.4 and 5.5. From (5.6) we readily find that for the system $\{g_k(p)\}$ used there the following assertion is valid:

(5.10) The system $\{g_k(p)\}$ determines a unique system of vectors $\boldsymbol{\lambda}^{(s)}$, namely

$$\boldsymbol{\lambda}^{(n+1)} = \boldsymbol{\lambda}^{(n+2)} = \ldots = \boldsymbol{\lambda}^{(r)} = \mathbf{e}_{n+1} \,.$$

We shall investigate the systems $\{g_k(p)\}$ for which (5.10) does not hold. First, we give (5.10) a more lucid form and, moreover, we reformulate it in terms of the spaces W_s for later use.

Lemma 5.1. *The condition* (5.10) *is equivalent with*

$$\text{(5.11)} \qquad a)\ g_k(p_s) = 0\,, \quad k = 1, 2, \ldots, n\,; \quad s = n+1,\ n+2,\ \ldots,\ r\,;$$
$$b)\ \operatorname{rank}\left([g_k(p_j)]_{k,j=1}^{n}\right) = n$$

and with

$$\text{(5.12)} \qquad W_s = V(\mathbf{e}_{n+1}) \quad \textit{for each} \quad s = n+1,\ n+2,\ \ldots,\ r\,.$$

Proof. We shall show that (5.10) implies (5.11), (5.11) implies (5.12), and (5.12) implies (5.10).

I. Let (5.10) be true for a system $\{g_k(p)\}$. Substituting $\lambda_1^{(s)} = \lambda_2^{(s)} = \ldots = \lambda_n^{(s)} = 0$, $\lambda_{n+1}^{(s)} = 1$ into (5.8), $s = n+1, n+2, \ldots, r$, we get (5.11a) immediately. Therefore, (5.8) becomes

$$\text{(5.13)} \qquad \lambda_1^{(s)}\, g_k(p_1) + \lambda_2^{(s)}\, g_k(p_2) + \ldots + \lambda_n^{(s)}\, g_k(p_n) = 0\,,$$
$$k = 1, 2, \ldots, n\,,$$

for any $s = n+1,\ n+2,\ \ldots,\ r$. According to (5.10), this set of homogeneous linear equations has only the trivial solution. Thus, (5.11b) holds.

II. If (5.11) is valid for a system $\{g_k(p)\}$, then by (5.11b) $\dim W_s = 1$ for each $s = n+1,\ n+2,\ \ldots,\ r$. We have $\mathbf{e}_{n+1} \in W_s$ with respect to (5.11a) and thus $W_s = V(\mathbf{e}_{n+1})$.

III. Let (5.12) hold. Given an integer s such that $n+1 \leqq s \leqq r$, the space W_s contains exactly all the vectors of the form $\varrho\mathbf{e}_{n+1}$ where ϱ runs through the set of complex numbers. The vectors from W_s satisfying (5.9) are exactly those with $|\varrho| = 1$; thus (5.10) holds.

Theorem 5.6. *Let the set P be given and let n be a positive integer such that $|p_n| \neq |p_{n+1}|$. Then a necessary condition for an approximation $\{G_p\} \in M_n$,*

$$G_p(f) = \sum_{k=1}^{n} a_k(f)\, g_k(p)\,,$$

to be universal for the given P and with respect to $\mathfrak{H}_3$ *is that* $\{g_k(p)\}$ *should satisfy* (5.11).

Proof. Let us suppose that the system $\{g_k(p)\}$ does not satisfy (5.11). Then, by Lemma 5.1, it does not satisfy (5.12) either and, therefore, there exists a vector $\boldsymbol{\lambda}^{(s)}$, $n + 1 \leqq s \leqq r$, determined by $\{g_k(p)\}$ and such that

$$\lambda_q^{(s)} \neq 0 \tag{5.14}$$

for some q, $1 \leqq q \leqq n$.

From (4.14) we get

$$\omega(P, G_p) \geqq \left(\sum_{k=1}^{n} \frac{|\lambda_k^{(s)}|^2}{\eta_{p_k}^2} + \frac{|\lambda_{n+1}^{(s)}|^2}{\eta_{p_s}^2} \right)^{1/2} \geqq \frac{|\lambda_q^{(s)}|}{\eta_{p_q}}$$

for any approximation $\{G_p\} \in M_n$ with the given $\{g_k(p)\}$. Further, using (5.7) we have

$$Q(n, p, G_p) \geqq |\lambda_q^{(s)}| \frac{\eta_{p_{n+1}}}{\eta_{p_q}}.$$

Since $1 \leqq q \leqq n$ and $|p_n| \neq |p_{n+1}|$, and because of (5.14) and (d), for every $D > 0$ a space $H \in \mathfrak{H}_3$ such that

$$\frac{\eta_{p_{n+1}}}{\eta_{p_q}} > \frac{D}{|\lambda_q^{(s)}|}$$

holds in H may be found by means of Theorem 2.3. Therefore,

$$Q_H(n, P, G_p) > D$$

and $\{G_p\}$ is not universal with respect to $\mathfrak{H}_3$, as required.

An analogous theorem holds for the universal sequences of approximations.

Theorem 5.7. *Let the set P be given and let n be a positive integer such that* $|p_n| \neq |p_{n+1}|$. *Then a necessary condition for a sequence of approximations* $\{G_p^{(j)}\} \in \tilde{M}_n$, $j = 1, 2, \ldots$,

$$G_p^{(j)}(f) = \sum_{k=1}^{n} a_k^{(j)}(f)\, g_k(p),$$

to be universal for the given P and with respect to $\mathfrak{H}_3$ *is that* $\{g_k(p)\}$ *should satisfy* (5.11).

For the proof see Remark 5.1.

Theorem 5.6 enables us to judge the error of the universal approximation $\{B_p\}$ from Theorem 5.4 with regard to the class of universal approximations. Theorem 5.7 is of similar importance for the universal sequence of approximations from Theorem 5.5.

Denote by $U_n(P)$ the set of universal approximations from $M_n(P)$ for a given P and the class $\mathfrak{H}_3$. Given n such that $|p_n| \neq |p_{n+1}|$, then by Theorem 5.6, Lemma 5.1

and (4.14)

$$\Omega_H(U_n, P) \geqq \frac{1}{\eta_{p_{n+1}}} \tag{5.15}$$

holds in every $H \in \mathfrak{H}_3$. Because of (5.4) we have proved

Theorem 5.8. *Given the set P and a positive integer n such that $|p_n| \neq |p_{n+1}|$, the approximation $\{B_p\} \in U_n(P)$ is an optimal universal approximation from $M_n(P)$ for the class of strongly periodic spaces.*

Similarly, denote by $\tilde{U}_n(P)$ the set of all the approximations from $\tilde{M}_n(P)$ that are elements of a universal sequence of approximations for a given set P and the class $\mathfrak{H}_3$. Again, if n is such that $|p_n| \neq |p_{n+1}|$ then by Theorem 5.7, Lemma 5.1 and (4.14)

$$\Omega_H(\tilde{U}_n, P) \geqq \frac{1}{\eta_{p_{n+1}}} \tag{5.16}$$

holds in every $H \in \mathfrak{H}_3$. Using Theorem 5.5, (5.4) and Theorem 3.6 we obtain

Theorem 5.9. *Given the set P and a positive integer n such that $|p_n| \neq |p_{n+1}|$, the sequence of approximations $\{B_p^{(j)}\} \in \tilde{U}_n(P)$, $j = 1, 2, \ldots$, is an asymptotically optimal universal sequence of approximations from $\tilde{M}_n(P)$ for $\mathfrak{H}_3$.*

Now, we shall consider the approximations from M_n where n is such that $|p_n| = |p_{n+1}|$ in the set P given. We have then $\eta_{p_n} = \eta_{p_{n+1}}$ in all $H \in \mathfrak{H}_3$ by (c).

First we shall assume the functions $g_k(p)$, $k = 1, 2, \ldots, n$ to be linearly dependent on the set P, i.e.

$$\operatorname{rank}\left([g_k(p_j)],\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, r\right) < n .$$

Recalling Theorems 5.6 and 5.7 we can see that no approximation employing such a system $\{g_k(p)\}$ is universal with respect to $\mathfrak{H}_3$ in case that $|p_n| \neq |p_{n+1}|$ (and similarly for the sequences of approximations). If $|p_n| = |p_{n+1}|$, however, we may readily prove

Theorem 5.10. *Let the set P be given and let n be such that $|p_n| = |p_{n+1}|$. Then the approximation $\{C_p\} \in M_n$ given by*

$$\begin{aligned} a_k &= I_{p_k}, \quad k = 1, 2, \ldots, n-1, \\ a_n &= O \quad \textit{(the null functional)} \end{aligned} \tag{5.17}$$

and

$$\begin{aligned} g_k(p_j) &= \delta_{kj}, \quad k = 1, 2, \ldots, n-1;\ j = 1, 2, \ldots, r, \\ g_n(p_j) &= 0, \quad j = 1, 2, \ldots, r \end{aligned} \tag{5.18}$$

is universal with respect to $\mathfrak{H}_3$. Moreover,

$$Q(n, P, C_p) \leqq \left(2 + \sum_{k=1}^{n-1} \left(\frac{\eta_{p_k}}{\eta_{p_n}}\right)^2\right)^{1/2} \leqq (1 + n)^{1/2}$$

holds for every $H \in \mathfrak{H}_3$.

Proof might be carried out by an argument precisely analogous to that of Theorem 5.4, because

$$\omega(P, C_p) = \frac{1}{\eta_{p_n}} = \frac{1}{\eta_{p_{n+1}}} \tag{5.19}$$

in this case.

The formulation and proof of an analogous theorem regarding the universal sequence of approximations $\{C_p^{(j)}\}$ where we have replaced I_{p_k} by $L_{p_k}^{(j)}$ in (5.17) are left to the reader.

The system $\{g_k(p)\}$ from Theorem 5.10 is linearly dependent on P. From (5.18) we easily find that the following assertion concerning this $\{g_k(p)\}$ is valid:

(5.20) The system $\{g_k(p)\}$ determines exactly all the systems of vectors $\boldsymbol{\lambda}^{(s)}$ such that

$$\boldsymbol{\lambda}^{(s)} = \mu_s \mathbf{e}_n + (1 - \mu_s)\, \mathbf{e}_{n+1}\,, \quad n + 1 \leqq s \leqq r\,,$$

where $0 \leqq \mu_s \leqq 1$.

The systems $\{g_k(p)\}$ satisfying (5.20) are characterized by

Lemma 5.2. *The condition* (5.20) *is equivalent with*

(5.21) *a*) $g_k(p_s) = 0\,, \quad k = 1, 2, \ldots, n\,; \quad s = n, n + 1, \ldots, r\,;$

b) *If* $n > 1$ *then*

$$\operatorname{rank}\left([g_k(p_j)],\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n - 1\right) = n - 1$$

and with the condition

$$W_s = V(\mathbf{e}_n, \mathbf{e}_{n+1}) \quad \textit{for each} \quad s = n + 1, n + 2, \ldots, r\,. \tag{5.22}$$

Proof. We shall show that (5.20) implies (5.21), (5.21) implies (5.22), and (5.22) implies (5.20).

I. Let (5.20) hold for a system $\{g_k(p)\}$. Setting

$$\lambda_1^{(s)} = \lambda_2^{(s)} = \ldots = \lambda_n^{(s)} = 0,\ \lambda_{n+1}^{(s)} = 1$$

and

$$\lambda_1^{(s)} = \lambda_2^{(s)} = \ldots = \lambda_{n-1}^{(s)} = \lambda_{n+1}^{(s)} = 0,\ \lambda_n^{(s)} = 1$$

in (5.8), we get (5.21a) immediately. The sets of equations (5.8) thus become

$$\lambda_1^{(s)}\, g_k(p_1) + \lambda_2^{(s)}\, g_k(p_2) + \ldots + \lambda_{n-1}^{(s)}\, g_k(p_{n-1}) = 0\,, \quad k = 1, 2, \ldots, n - 1 \tag{5.23}$$

for all $s = n + 1, n + 2, \ldots, r$ (if $n > 1$). According to (5.20), this set of equations has only the trivial solution and, therefore, (5.21b) holds.

II. If (5.21) is true for a system $\{g_k(p)\}$ then the rank of the matrices of (5.8) is $n - 1$ and thus $\dim W_s = 2$ for any $s = n + 1, n + 2, \ldots, r$. By (5.21a), $\mathbf{e}_n \in W_s$, $\mathbf{e}_{n+1} \in W_s$ for all s and we have $W_s = V(\mathbf{e}_n, \mathbf{e}_{n+1})$, as required.

III. Let (5.22) be true. Let there be given an integer s, $n + 1 \leqq s \leqq r$. The space W_s contains exactly all the vectors of the form $\varrho\mathbf{e}_n + \sigma\mathbf{e}_{n+1}$ where ϱ and σ run through the set of complex numbers. The vectors from W_s which satisfy (5.9) are exactly those with $|\varrho| + |\sigma| = 1$. Thus, (5.20) is valid.

The lemma is proved.

Now we are able to formulate

Theorem 5.11. *Let the set P be given and let n be a positive integer such that $|p_n| = |p_{n+1}|$. Let there be given a linearly dependent system $\{g_k(p)\}_{k=1}^n$. Then a necessary condition for an approximation $\{G_p\} \in M_n$ or a sequence of approximations $\{G_p^{(j)}\} \in \tilde{M}_n$, $j = 1, 2, \ldots$ employing the system $\{g_k(p)\}$ to be universal for the given P and with respect to $\mathfrak{H}_3$ is that $\{g_k(p)\}$ should satisfy* (5.21).

Proof. If the system $\{g_k(p)\}$ does not satisfy (5.21) then, by Lemma 5.2, it does not satisfy (5.22) either. Since $\{g_k(p)\}$ is linearly dependent, we have $\dim W_s \geqq 2$ for each $s = n + 1, n + 2, \ldots, r$. Thus there exists a vector $\lambda^{(s)}$, $n + 1 \leqq s \leqq r$, determined by $\{g_k(p)\}$ and such that

$$\lambda_q^{(s)} \neq 0 \quad \text{for some } q\,, \quad 1 \leqq q \leqq n - 1\,. \tag{5.24}$$

By (4.14) and (5.7) we get similarly as in the proof of Theorem 5.6

$$Q(n, P, G_p) \geqq |\lambda_q^{(s)}| \frac{\eta_{p_{n+1}}}{\eta_{p_q}}$$

for any approximation $\{G_p\} \in M_n$ employing the system $\{g_k(p)\}$. We have $1 \leqq q \leqq \leqq n - 1$ and $|p_{n-1}| \neq |p_{n+1}|$. Owing to (5.24) and (d), for any $D > 0$ it is possible to find such a space $H \in \mathfrak{H}_3$ by means of Theorem 2.3 that

$$Q_H(n, P, G_p) > D$$

holds in H. Hence, $\{G_p\}$ is not universal.

The statement of the theorem regarding the sequences of approximations may be readily obtained by the argument indicated in Remark 5.1. The theorem is proved.

Let $V_n(P) \subset U_n(P)$ be the set of approximations employing linearly dependent systems $\{g_k(p)\}$. Let n be an integer such that $|p_n| = |p_{n+1}|$. Then from Theorem 5.11, Lemma 5.2 and (4.14) we get immediately

$$\Omega_H(V_n, P) \geqq \frac{1}{\eta_{p_n}} \tag{5.25}$$

in every space $H \in \mathfrak{H}_3$. Owing to (5.19), we have proved

Theorem 5.12. *Let the set P be given and let n be such that* $|p_n| = |p_{n+1}|$. *Then the approximation* $\{C_p\} \in V_n(P)$ (*cf.* (5.17), (5.18)) *is optimal in* $V_n(P)$ *for the whole class of strongly periodic spaces.*

Similarly, with the set of approximations $\tilde{V}_n(P) \subset \tilde{U}_n(P)$ employing linearly dependent systems $\{g_k(p)\}$ and with n such that $|p_n| = |p_{n+1}|$, it holds

$$\Omega_H(\tilde{V}_n, P) \geqq \frac{1}{\eta_{p_n}} . \tag{5.26}$$

Considering the sequence of approximations $\{C_p^{(j)}\} \in \tilde{V}_n(P), j = 1, 2, \ldots,$

$$\lim_{j \to \infty} \omega(P, C_p^{(j)}) = \frac{1}{\eta_{p_n}} \tag{5.27}$$

follows immediately from Theorem 3.4 by the same procedure as that of the proof of Theorem 3.6. Owing to (5.26), we have proved

Theorem 5.13. *Let the set P be given and let n be such that* $|p_n| = |p_{n+1}|$. *Then the sequence of approximations* $\{C_p^{(j)}\} \in \tilde{V}_n(P)$, $j = 1, 2, \ldots$ *is asymptotically optimal in* $\tilde{V}_n(P)$ *for the whole class* $\mathfrak{H}_3$.

Provided $n > 1$ one may easily find that the approximation $\{C_p\}$ and the sequence of approximations $\{C_p^{(j)}\}$ can be treated also as an approximation from $M_{n-1}(P)$ and a sequence of approximations from $\tilde{M}_{n-1}(P)$, respectively. Since the elements of P are distinct, $|p_n| = |p_{n+1}|$ implies $|p_{n-1}| \neq |p_n|$ and $\{C_p\}$ is an optimal universal approximation from $M_{n-1}(P)$ with respect to $\mathfrak{H}_3$ (similarly for $\{C_p^{(j)}\}, j = 1, 2, \ldots$). To put other way round, if $|p_{n+1}| = |p_{n+2}|$ then the optimal universal approximation $\{B_p\} \in M_n$ is a universal approximation from the set M_{n+1} as well. An analogous statement is true regarding the sequence of approximations $\{B_p^{(j)}\}$.

In the remainder of the section we shall treat the approximations with linearly independent systems $\{g_k(p)\}$. The situation here is more complicated than that in the case of $|p_n| \neq |p_{n+1}|$. (5.11) is no longer a necessary condition for the universality and there exist universal approximations with various systems $\{g_k(p)\}$ not satisfying the above condition. An example of such an approximation is described in the following theorem.

Theorem 5.14. *Let the set P be given and let n be a positive integer such that* $|p_n| = |p_{n+1}|$. *Then the approximation* $\{\hat{B}_p\} \in M_n$ *described below is a universal approximation from* M_n *for the set P and with respect to* $\mathfrak{H}_3$:

$$\begin{aligned} a_k &= I_{p_k} , \quad k = 1, 2, \ldots, n - 1 , \\ a_n &= I_{p_n} + I_{p_{n+1}} ; \end{aligned} \tag{5.28}$$

$$\begin{aligned} g_k(p_j) &= \delta_{kj} , \quad k = 1, 2, \ldots, n - 1 ; \quad j = 1, 2, \ldots, r , \\ g_n(p_j) &= 0 , \quad j = 1, 2, \ldots, r , \quad j \neq n , \quad j \neq n + 1 , \\ g_n(p_n) &= g_n(p_{n+1}) = \tfrac{1}{2} . \end{aligned} \tag{5.29}$$

Moreover, it holds for every $H \in \mathfrak{H}_3$:
If $r = n + 1$, *or* $r > n + 1$ *and* $\eta_{p_{n+1}}\sqrt{2} \leqq \eta_{p_{n+2}}$ *then*

$$Q(n, P, \hat{B}_p) \leqq \frac{1}{\sqrt{2}}\left(2 + \sum_{j=1}^{n-1}\left(\frac{\eta_{p_j}}{\eta_{p_{n+1}}}\right)^2\right)^{1/2} \leqq \frac{1}{\sqrt{2}}(1 + n)^{1/2}. \tag{5.30}$$

If $r > n + 1$ *and* $\eta_{p_{n+1}}\sqrt{2} > \eta_{p_{n+2}}$ *then*

$$Q(n, P, \hat{B}_p) \leqq \left(\sum_{j=1}^{n+1}\left(\frac{\eta_{p_j}}{\eta_{p_{n+2}}}\right)^2\right)^{1/2} \leqq (1 + n)^{1/2}. \tag{5.31}$$

Proof. Denote

$$P_0 = \{p_1, p_2, \ldots, p_{n-1}\}, \quad P_1 = \{p_n, p_{n+1}\}, \quad P_2 = (P - P_0) - P_1.$$

Obviously

$$\hat{B}_p = \begin{cases} I_p & \text{for } p \in P_0, \\ \frac{1}{2}(I_{p_n} + I_{p_{n+1}}) & \text{for } p \in P_1, \\ O & \text{for } p \in P_2. \end{cases} \tag{5.32}$$

We shall examine $r = n + 1$ and $r > n + 1$ separately.

I. Let $r = n + 1$. Then $P_2 = \emptyset$. In virtue of (5.32) we have

$$\omega(P, \hat{B}_p) = \omega(P_1, \hat{B}_p).$$

By Theorem 3.3, we get

$$\omega(P, \hat{B}_p) = \frac{1}{\eta_{p_n}\sqrt{2}} \tag{5.33}$$

after setting $p = p_n$, $q = p_{n+1}$, $\alpha = \beta = 1$ and $g(p_n) = g(p_{n+1}) = \frac{1}{2}$. Thus, with respect to (5.2),

$$Q(n, P, \hat{B}_p) \leqq \frac{1}{\eta_{p_n}\sqrt{2}}\left(\sum_{j=1}^{n+1}\eta_{p_j}^2\right)^{1/2},$$

which yields (5.30) easily.

II. Let $r > n + 1$. Then $P_2 \neq \emptyset$. By (5.32),

$$\omega(P, \hat{B}_p) = \max\left(\omega(P_1, \hat{B}_p), \omega(P_2, \hat{B}_p)\right).$$

In the same fashion as in case I we have

$$\omega(P_1, \hat{B}_p) = \frac{1}{\eta_{p_n}\sqrt{2}}.$$

From (5.32) and Theorem 3.2 it follows

$$\omega(P_2, \hat{B}_p) = \frac{1}{\eta_{p_{n+2}}}$$

and finally

$$\omega(P, \hat{B}_p) = \max\left(\frac{1}{\eta_{p_n}\sqrt{2}}, \frac{1}{\eta_{p_{n+2}}}\right). \tag{5.34}$$

In case that

$$\eta_{p_n}\sqrt{2} \leqq \eta_{p_{n+2}}$$

is valid in the space H, the remainder of the proof is the same as in case I. If this is not the case, using (5.2) we obtain

$$Q(n, P, \hat{B}_p) \leqq \frac{1}{\eta_{p_{n+2}}}\left(\sum_{j=1}^{n+1} \eta_{p_j}^2\right)^{1/2},$$

which proves (5.31) and completes the proof of the theorem.

Actually, the used system $\{g_k(p)\}$ does not satisfy (5.11), which may be easily verified. With the sequences of approximations, the situation is similar.

Theorem 5.15. *Let the set P be given and let n be a positive integer such that $|p_n| = |p_{n+1}|$. Then the sequence of approximations $\{\hat{B}_p^{(j)}\} \in \tilde{M}_n$, $j = 1, 2, \ldots$ obtained through replacing I_p by $L_p^{(j)}$ in (5.28) is universal for the set P and with respect to $\mathfrak{H}_3$.*

In addition, the inequalities (5.30) and (5.31) hold for every $H \in \mathfrak{H}_3$ with

$$\lim_{j\to\infty} \tilde{Q}(n, P, \hat{B}_p^{(j)})$$

instead of

$$Q(n, P, \hat{B}_p)$$

on the left-hand sides.

Proof. By Theorem 3.4,

$$\lim_{j\to\infty} \omega(P_0, \hat{B}_p^{(j)}) = 0$$

(unless $P_0 = \emptyset$). In virtue of Theorem 3.5

$$\lim_{j\to\infty} \omega(P_1, \hat{B}_p^{(j)}) = \omega(P_1, B_p).$$

Since for $P_2 \neq \emptyset$

$$\omega(P_2, \hat{B}_p^{(j)}) = \omega(P_2, \hat{B}_p)$$

for all $j = 1, 2, \ldots$, it holds also

$$\lim_{j \to \infty} \omega(P, \hat{B}_p^{(j)}) = \omega(P, \hat{B}_p) . \tag{5.35}$$

From (5.35) and Theorem 5.14, and using Lemma 4.2, we establish the universality of the sequence in question and the bounds on $\lim_{j \to \infty} \tilde{Q}(n, P, \hat{B}_p^{(j)})$, proving the theorem.

The necessary condition for the universality in the case of $|p_n| = |p_{n+1}|$ and linearly independent systems $\{g_k(p)\}$ is formulated not with the values of $g_k(p)$ but by means of the spaces W_s. The reason is the attempt to avoid at this stage a detailed examination of the types of universal approximations possible and to give a statement though weaker, but lucid. The detailed examination of the conditions for $g_k(p)$ will be carried out in the remark following Theorem 5.16.

Theorem 5.16. *Let the set P be given and let n be a positive integer such that $|p_n| = |p_{n+1}|$. Let there be given a linearly independent system $\{g_k(p)\}_{k=1}^n$. Then a necessary condition for an approximation $\{G_p\} \in M_n$ or a sequence of approximations $\{G_p^{(j)}\} \in \tilde{M}_n$, $j = 1, 2, \ldots$ employing this $\{g_k(p)\}$ to be universal for the given P and with respect to $\mathfrak{H}_3$ is that*

$$W_s \subset V(\mathbf{e}_n, \mathbf{e}_{n+1}) \tag{5.36}$$

should hold for each $s = n + 1, n + 2, \ldots, r$.

Proof. The invalidity of (5.36) implies (5.24). The remainder of the proof would be the same as in the proof of Theorem 5.11. The statement regarding the sequences of approximations may be obtained on the basis of Remark 5.1 immediately.

Remark 5.3. The foregoing theorem requires that $\{g_k(p)\}$ be linearly independent on P. This assumption results in further restriction of the spaces W_s, which enables us to discuss the condition (5.36) in a rather detailed way.

First of all, $W_s = V(\mathbf{e}_n, \mathbf{e}_{n+1})$ cannot hold for all $s = n + 1, n + 2, \ldots, r$, because this would imply $g_k(p_n) = g_k(p_{n+1}) = \ldots = g_k(p_r) = 0$ for all $k = 1, 2, \ldots, n$, which contradicts the linear independence of $\{g_k(p)\}$. Thus there exists an integer t, $n + 1 \leqq \leqq t \leqq r$ such that $W_t = V(\mathbf{e}_n)$ or $W_t = V(\mathbf{e}_{n+1})$ or there exist non-zero complex numbers ϱ_t, σ_t such that $W_t = V(\varrho_t \mathbf{e}_n + \sigma_s \mathbf{e}_{n+1})$.

I. Let $W_t = V(\mathbf{e}_n)$. Then $g_k(p_n) = 0$ for any $k = 1, 2, \ldots, n$ as we may see from the set of equations (5.8) for $s = t$. Since $W_s \subset V(\mathbf{e}_n, \mathbf{e}_{n+1})$ for all s in question, we get

$$\lambda_{n+1}^{(s)}\, g_k(p_s) = 0\,, \quad k = 1, 2, \ldots, n\,; \quad s = n + 1, n + 2, \ldots, r$$

where $(0, 0, \ldots, 0, \lambda_n^{(s)}, \lambda_{n+1}^{(s)})$ is an arbitrary solution of (5.8). Therefore, for each s we have either $g_k(p_s) = 0$, $k = 1, 2, \ldots, n$ or $\lambda_{n+1}^{(s)} = 0$ for any solution of (5.8). If for a given s the former takes place we have $W_s = V(\mathbf{e}_n, \mathbf{e}_{n+1})$; if the latter, then $W_s = V(\mathbf{e}_n)$.

II. Let $W_t = V(\mathbf{e}_{n+1})$. Then (5.8) yields $g_k(p_t) = 0$, $k = 1, 2, \ldots, n$ and rank $([g_k(p_j)]_{k,j=1}^n) = n$. Thus dim $W_s = 1$, $W_s \neq V(\mathbf{e}_n)$ holds for any s in question. Hence, two cases are possible, namely $W_s = V(\mathbf{e}_{n+1})$ and $W_s = V(\varrho_s \mathbf{e}_n + \sigma_s \mathbf{e}_{n+1})$ where $\varrho_s \sigma_s \neq 0$.

III. Let $W_t = V(\varrho_t \mathbf{e}_n + \sigma_t \mathbf{e}_{n+1})$, $\varrho_t \sigma_t \neq 0$. Then we get

$$g_k(p_t) = -\frac{\varrho_t}{\sigma_t} g_k(p_n), \quad k = 1, 2, \ldots, n$$

from (5.8) and further

$$\text{rank}\,([g_k(p_j)]_{k,j=1}^n) = n.$$

Thus dim $W_s = 1$, $W_s \neq V(\mathbf{e}_n)$ for all s and the conclusion is the same as that in II.

Now, we have described the combinations of the spaces W_s possible and formulated (5.36) more precisely. From the above results we can also derive the equivalent necessary conditions that the system $\{g_k(p)\}$ of a universal approximation must satisfy.

There are two possibilities:

A. 1) $g_k(p_n) = 0$, $k = 1, 2, \ldots, n$;

2) there exists an integer t, $n + 1 \leqq t \leqq r$,

such that

$$\text{rank}\,([g_k(p_j)],\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n - 1, t) = n;$$

3) for $s \neq t$, $n + 1 \leqq s \leqq r$, either

$$g_k(p_s) = 0, \quad k = 1, 2, \ldots, n$$

or

$$\text{rank}\,([g_k(p_j)],\ k = 1, 2, \ldots, n;\ j = 1, 2, \ldots, n - 1, s) = n$$

holds.

B. 1) rank $([g_k(p_j)]_{k,j=1}^n) = n$;

2) for each s, $n + 1 \leqq s \leqq r$ either

$$g_k(p_s) = 0$$

or there exists a non-zero number L_s such that

$$g_k(p_s) = L_s\, g_k(p_n)$$

for all $k = 1, 2, \ldots, n$.

The condition A is equivalent to the case I, the condition B to the cases II and III. Moreover, it may be shown that there exist universal approximations the systems $\{g_k(p)\}$ of which cover all the possibilities given by the above two conditions. We shall not, however, give examples of such approximations. Our aim here is to prove some optimal properties of $\{\hat{B}_p\}$ and $\{\hat{B}_p^{(j)}\}$, $j = 1, 2, \ldots$

Recall that $U_n(P)$ is the set of universal approximations from $M_n(P)$ for a given P and the class $\mathfrak{H}_3$ in our notation. Further, $V_n(P) \subset U_n(P)$ is the set of approximations employing linearly dependent systems $\{g_k(p)\}$. Obviously

$$\Omega_H(U_n, P) = \min\left(\Omega_H(V_n, P),\ \Omega_H(U_n - V_n, P)\right). \tag{5.37}$$

If $|p_n| = |p_{n+1}|$ and $\{G_p\}$ is an arbitrary approximation from $U_n - V_n$ then using Theorem 5.16 and (4.14) we get

$$\omega_H(P, G_p) \geqq \max_{s=n+1,\dots,r} \left(\frac{|\lambda_n^{(s)}|^2}{\eta_{p_n}^2} + \frac{|\lambda_{n+1}^{(s)}|^2}{\eta_{p_s}^2}\right)^{1/2}$$

where $\lambda_n^{(s)}, \lambda_{n+1}^{(s)}$ are the corresponding components of the solution of (5.8) and $|\lambda_n^{(s)}| + |\lambda_{n+1}^{(s)}| = 1$ for each $s = n+1, n+2, \dots, r$. By an argument precisely similar to that in the last part of the proof of Theorem 4.1 it may be shown that

$$\left(\frac{|\lambda_n^{(s)}|^2}{\eta_{p_n}^2} + \frac{|\lambda_{n+1}^{(s)}|^2}{\eta_{p_s}^2}\right)^{1/2} \geqq (\eta_{p_n}^2 + \eta_{p_s}^2)^{-1/2}$$

and therefore

$$\omega_H(P, G_p) \geqq \max_s (\eta_{p_n}^2 + \eta_{p_s}^2)^{-1/2},$$

which implies

$$\Omega_H(U_n - V_n, P) \geqq \frac{1}{\eta_{p_n}\sqrt{2}}$$

in every space $H \in \mathfrak{H}_3$. On account of (5.25) the relation (5.37) yields

$$\Omega_H(U_n, P) \geqq \frac{1}{\eta_{p_n}\sqrt{2}} \tag{5.38}$$

in any $H \in \mathfrak{H}_3$. The same bound is true also for $\Omega_H(\tilde{U}_n, P)$ evidently.

Hence, for the approximation $\{\hat{B}_p\}$ and the sequence of approximations $\{\hat{B}_p^{(j)}\}$, $j = 1, 2, \dots$ we have by (5.33), (5.34) and (5.35)

Theorem 5.17. *Let the set P be given and let n be a positive integer such that $|p_n| = |p_{n+1}|$. Let $r = n + 1$. Then the approximation $\{\hat{B}_p\} \in U_n(P)$ is an optimal universal approximation from $M_n(P)$ for the class $\mathfrak{H}_3$. Further, the sequence of approximations $\{\hat{B}_p^{(j)}\} \in \tilde{U}_n(P)$, $j = 1, 2, \dots$ is an asymptotically optimal universal sequence of approximations from $\tilde{M}_n(P)$ with respect to $\mathfrak{H}_3$.*

If $r > n + 1$ the optimal properties of the above approximations take place only in those spaces $H \in \mathfrak{H}_3$ where $\eta_{p_{n+1}}\sqrt{2} \leqq \eta_{p_{n+2}}$.

The assertions (5.33) to (5.35) enable us to compare the error of $\{\hat{B}_p\}$ with the error of the optimal universal approximation even in the cases not covered by Theorem 5.17.

Theorem 5.18. *Let the set P be given and let n be such that $|p_n| = |p_{n+1}|$. Let $r > n + 1$ and let there be given a space $H \in \mathfrak{H}_3$ such that $\eta_{p_{n+1}} \sqrt{2} > \eta_{p_{n+2}}$. Then*

$$\varrho_H(U_n, P, \hat{B}_p) \leqq \sqrt{2} \frac{\eta_{p_{n+1}}}{\eta_{p_{n+2}}} \leqq \sqrt{2}$$

and the same inequalities hold also for $\lim_{j \to \infty} \varrho_H(\tilde{U}_n, P, \hat{B}_p^{(j)})$.

If we return to the approximation $\{C_p\}$ from Theorem 5.10 we get with the help of (5.38)

Theorem 5.19. *Let the set P be given and let n be such that $|p_n| = |p_{n+1}|$. Then for every $H \in \mathfrak{H}_3$ it holds:*

I. *If $r = n + 1$, or $r > n + 1$ and $\eta_{p_{n+1}} \sqrt{2} \leqq \eta_{p_{n+2}}$ then*

$$\lim_{j \to \infty} \varrho_H(\tilde{U}_n, P, C_p^{(j)}) = \varrho_H(U_n, P, C_p) = \sqrt{2}.$$

II. *If $r > n + 1$ and $\eta_{p_{n+1}} \sqrt{2} > \eta_{p_{n+2}}$ then*

$$\varrho_H(U_n, P, C_p) \leqq \sqrt{2}$$

and also

$$\lim_{j \to \infty} \varrho_H(\tilde{U}_n, P, C_p^{(j)}) \leqq \sqrt{2}.$$

Proof. If I is the case then from (5.38) and Theorem 5.17 it follows

$$\Omega_H(U_n, P) = \Omega_H(\tilde{U}_n, P) = \frac{1}{\eta_{p_n} \sqrt{2}}.$$

The assertion of the theorem is obtained immediately from (5.19) and (5.27).

The statement of the theorem in the case II follows again from (5.19), (5.27) and from (5.38). The theorem is proved.

6. THE PRACTICAL ASPECTS

In this section we summarize briefly the practical conclusions resulting from the theorems of the paper.

Recall the formulation of our problem. Suppose that the function f is known to belong to some indefinite strongly periodic space. Our task is to compute the values of its Fourier coefficients $I_p(f)$ for $p \in P$. If there is no additional information about f available, then the theorems of Sec. 5 imply that the best strategy from our point of view (that is from the standpoint of universality and optimality as defined in the paper) is very simple. We shall concentrate our effort on approximating $I_p(f)$ for the p's with a small absolute value $(p \in P_1)$ and replace $I_p(f)$ simply by zero for the other $p \in P$ (the approximation $\{B_p\}$ from Theorem 5.4).

In addition, if we start replacing by zero from $I_{p_{n+1}}$, there is no reason to choose p_{n+1} in such a way that (with the arrangement of the set P introduced in Sec. 5) $p_{n+1} = -p_n$. In such a case it is better either to set $I_{p_n} \approx 0$ as well (the approximation $\{C_p\}$ from Theorem 5.10) or to use the approximation $\{\hat{B}_p\}$ from Theorem 5.14. The latter of the two possibilities means to evaluate one functional $a_k(f)$ more in comparison with the former, but it may decrease the error of the approximation by a factor of $1/\sqrt{2}$.

The question is to what extent this gain is substantial. Another question, which is of more importance and has not yet been settled, is what information about f should be at our disposal in order to approve the use of more complex systems $\{g_k(p)\}$.

To approximate I_p on P_1 we have used the trapezoidal rule functionals $L_p^{(j)}$ throughout the paper. Obviously, if we replace $L_p^{(j)}$ in $\{B_p^{(j)}\}$, $\{C_p^{(j)}\}$ or $\{\hat{B}_p^{(j)}\}$ by other functionals converging to I_p in the norm, we obtain an asymptotically optimal universal sequence of approximations again. The choice of $L_p^{(j)}$ was implied by the availability of the proof of convergence in this case [2].

Moreover, if we use not right I_p in $\{B_p\}$ etc., but some other approximating functionals such that the decisive part of the error resulted will be that of replacing by zero, we shall find that most results of Sec. 5 concerning $\{B_p\}$ etc. hold for the new approximations without any change.

Finally, we point out that it may be readily seen that, with a fixed set P, the conclusions of Sec. 5 are valid not only for the class $\mathfrak{H}_3$, but also for the class of periodic spaces $\mathfrak{H}_P$ such that every space $H \in \mathfrak{H}_P$ has the properties (c) and (d) for all $k \in P$, $j \in P$.

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ON AN INTEGRAL OPERATOR IN THE SPACE OF FUNCTIONS WITH BOUNDED VARIATION

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1. BASIC NOTATIONS

Let R^n be the n-dimensional real vector space. Let $\mathbf{A} = (a_{ij})$, $i = 1, \ldots, k$, $j = 1, \ldots, l$ be a $k \times l$-matrix (a_{ij} is the element of $\mathbf{A}$ from the i-th row and j-th column). By $\mathbf{A}'$ let us denote the transposed matrix of $\mathbf{A}$. For a $k \times l$-matrix $\mathbf{A}$ we define the number $\|\mathbf{A}\| = \max_{i=1,\ldots,k} \sum_{j=1}^{l} |a_{ij}|$. The elements $\mathbf{x}$ of R^n are column vectors (i.e. $n \times 1$-matrices). $\|.\|$ is a norm in R^n. The space of all $n \times n$-matrices let be denoted by $L(R^n \to R^n)$. $\|.\|$ is a norm in $L(R^n \to R^n)$ (the obvious operator norm corresponding to the given norm in R^n). We have evidently $\|\mathbf{A}'\| \leqq n\|\mathbf{A}\|$, $\|\mathbf{x}'\| \leqq n\|\mathbf{x}\|$ for $\mathbf{A} \in L(R^n \to R^n)$, $\mathbf{x} \in R^n$ respectively.

Let $\langle a, b\rangle \subset R = R^1$ be a bounded closed interval, $a < b$. For a given vector function $\mathbf{x}(t)$, $\mathbf{x} : \langle a, b\rangle \to R^n$ we define the (total) variation of $\mathbf{x}$ on $\langle a, b\rangle$ as usual:

$$\operatorname{var}_a^b \mathbf{x} = \sup_D \sum_{i=1}^{m} \|\mathbf{x}(t_i) - \mathbf{x}(t_{i-1})\|$$

where the supremum is taken over all finite decompositions $D : a = t_0 < t_1 < \ldots \ldots < t_m = b$ of $\langle a, b\rangle$.

We denote

$$V_n(a, b) = \{\mathbf{x} : \langle a, b\rangle \to R^n;\ \operatorname{var}_a^b \mathbf{x} < +\infty\}\,.$$

If no misunderstanding may occur, we write simply V_n instead of $V_n(a, b)$. If $n = 1$ we write $V(a, b)$ or V instead of $V_1(a, b)$ or V_1.

The following statement is obvious: $\mathbf{x} \in V_n(a, b)$ if and only if $x_i \in V(a, b)$ for all $i = 1, \ldots, n$, $\mathbf{x}' = (x_1, \ldots, x_n)$. The inequality

$$(1,1) \qquad \operatorname{var}_a^b x_i \leqq \operatorname{var}_a^b \mathbf{x}$$

is satisfied for all $i = 1, 2, \ldots, n$.

For $\mathbf{x} \in V_n(a, b)$ the limits $\lim_{\tau \to t+} \mathbf{x}(\tau) = \mathbf{x}(t+)$, $\lim_{\tau \to t-} \mathbf{x}(\tau) = \mathbf{x}(t-)$ exist for all $t \in \langle a, b \rangle$. We use the notations

$$\Delta^+ \mathbf{x}(t) = \mathbf{x}(t+) - \mathbf{x}(t), \quad \Delta^- \mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}(t-),$$
$$\Delta \mathbf{x}(t) = \mathbf{x}(t+) - \mathbf{x}(t-) = \Delta^+ \mathbf{x}(t) + \Delta^- \mathbf{x}(t).$$

2. THE INTEGRAL

For our purpose we use the concept of the generalized Perron-Stieltjes integral introduced by J. KURZWEIL in [1].

Let $f : \langle a, b \rangle \to R$, $g : \langle a, b \rangle \to R$ be given. If $g(t)$ is not defined for $t < a$ and $t > b$ then we suppose that $g(t) = g(a)$ for $t < a$ and $g(t) = g(b)$ for $t > b$. For $\langle c, d \rangle \subset \langle a, b \rangle$ we denote

$$\int_c^d f(t)\, \mathrm{d}g(t) = \int_c^d \mathrm{D}f(\tau)\, g(t)$$

where the right hand side is the Kurzweil integral ([1]) of the function $U(\tau, t) = f(\tau)\, g(t)$. In [1] the following is shown:

If $f : \langle a, b \rangle \to R$ is finite and $g \in V(c, d)$ then the integral $\int_c^d f(t)\, \mathrm{d}g(t)$ exists if and only if the Perron-Stieltjes integral (P.S.) $\int_c^d f(t)\, \mathrm{d}g(t)$ exists (in the usual sense) and both integrals are equal.

Further we have: If $f, g : \langle a, b \rangle \to R$, $\langle c, d \rangle \subset \langle a, b \rangle$, $f \in V(c, d)$, $g \in V(c, d)$ then the integral $\int_c^d f(t)\, \mathrm{d}g(t)$ exists.

This follows essentially from the same statement which holds for the Perron-Stieltjes integral and the above quoted equivalence of both concepts of integral.

If $|f(t)| \leqq M$ for $t \in \langle a, b \rangle$ and $g \in V(c, d)$, $\langle c, d \rangle \subset \langle a, b \rangle$ then

$$|f(\tau)\, (g(t_2) - g(t_1))| \leqq M \operatorname{var}_{t_1}^{t_2} g \leqq M |\operatorname{var}_c^{t_2} g - \operatorname{var}_c^{t_1} g|$$

for all $t_1, t_2, \tau \in \langle c, d \rangle$. From this inequality and from Lemma 2,1 in [3] the following proposition immediately follows:

If $f : \langle a, b \rangle \to R$, $|f(t)| \leqq M$ for all $t \in \langle a, b \rangle$, $g \in V(c, d)$ and if $\int_c^d f(t)\, \mathrm{d}g(t)$ exists, then

$$\left| \int_c^d f(t)\, \mathrm{d}g(t) \right| \leqq M \int_c^d \mathrm{d}(\operatorname{var}_c^t g) = M \operatorname{var}_c^d g. \tag{2,1}$$

If $f \in V(a, b)$ then obviously the inequality $|f(t)| \leqq |f(a)| + \operatorname{var}_a^b f$ holds for all $t \in \langle a, b \rangle$. Let $g \in V(c, d)$, $\langle c, d \rangle \subset \langle a, b \rangle$. By (2,1) we obtain

$$\left| \int_c^d f(t)\, \mathrm{d}g(t) \right| \leqq (|f(a)| + \operatorname{var}_a^b f) \operatorname{var}_c^d g. \tag{2,2}$$

From (2,2) we can easily obtain that for $f, g \in V(a, b)$ the function $\int_a^t f(\tau)\,dg(\tau)$: : $\langle a, b\rangle \to R$ belongs to $V(a, b)$, namely, the inequality

$$\operatorname{var}_a^b\left(\int_a^t f(\tau)\,dg(\tau)\right) \leqq (|f(a)| + \operatorname{var}_a^b f)\operatorname{var}_a^b g < +\infty$$

holds.

Proposition 2,1. *Let $f, g \in V(a, b)$. For $\alpha \in \langle a, b\rangle$, $t \in \langle a, b\rangle$ we define $\psi_\alpha^+(t) = 0$ for $t \leqq \alpha$, $\psi_\alpha^+(t) = 1$ for $\alpha < t$ if $\alpha < b$ and $\psi_\alpha^-(t) = 0$ for $t < \alpha$ if $a < \alpha$, $\psi_\alpha^-(t) = 1$ for $\alpha \leqq t$.*

Then we have

$$\int_a^b \psi_\alpha^+(t)\,dg(t) = \begin{cases} g(b) - g(\alpha+) & \text{if } \alpha < b \\ 0 & \text{if } \alpha = b \end{cases} \tag{2,3}$$

$$\int_a^b \psi_\alpha^-(t)\,dg(t) = \begin{cases} g(b) - g(\alpha-) & \text{if } a < \alpha \\ g(b) - g(a) & \text{if } a = \alpha \end{cases}$$

and

$$\int_a^b f(t)\,d\psi_\alpha^+(t) = \begin{cases} f(\alpha) & \text{if } \alpha < b \\ 0 & \text{if } \alpha = b \end{cases} \tag{2,4}$$

$$\int_a^b f(t)\,d\psi_\alpha^-(t) = \begin{cases} f(\alpha) & \text{if } \alpha > a \\ 0 & \text{if } \alpha = a\,. \end{cases}$$

Proof. If $\alpha = b$ then $\int_a^b \psi_\alpha^+(t)\,dg(t) = \int_a^b 0\,.\,dg(t) = 0$. Let $\alpha < b$. Then we have

$$\int_a^b \psi_\alpha^+(t)\,dg(t) = \int_a^{\alpha+\delta} \psi_\alpha^+(t)\,dg(t) + g(b) - g(\alpha + \delta)$$

for all $0 < \delta < b - \alpha$ because

$$\int_{\alpha+\delta}^b \psi_\alpha^+(t)\,dg(t) = \int_{\alpha+\delta}^b dg(t) = g(b) - g(\alpha + \delta)\,.$$

By Theorem 1, 3, 6 [1] there is

$$\lim_{\delta\to 0+}\int_a^{\alpha+\delta} \psi_\alpha^+(t)\,dg(t) = \int_a^\alpha \psi_\alpha^+(t)\,dg(t) + \lim_{\delta\to 0+}\psi_\alpha^+(\alpha)\,[g(\alpha + \delta) - g(\alpha)] = 0$$

since $\int_a^\alpha \psi_\alpha^+(t)\,dg(t) = \int_a^\alpha 0\,dg(t) = 0$ and $\psi_\alpha^+(\alpha) = 0$. For $\delta \to 0+$ so we obtain the first equation from (2,3). The second one can be proved similarly.

We verify for example the second formula from (2,4). The first one can be verified in a similar manner. If $\alpha = a$ then $\psi_\alpha^-(t) = 1$ in $\langle a, b\rangle$. For every $0 < \delta < b - a$ we have $f(\tilde{\tau}_0)(\psi_\alpha^-(\tau) - \psi_\alpha^-(\tau_0)) = 0$ for each $\tau_0 \in \langle a + \delta, b\rangle$ and $\tau_0 - \delta < \tau < +\infty$. By Lemma 1, 3, 1 [1] we obtain therefore $\int_{a+\delta}^b f(t)\, d\psi_\alpha^-(t) = 0$. This implies

$$\int_a^b f(t)\, \mathrm{d}\psi_\alpha^-(t) = \int_a^{a+\delta} f(t)\, \mathrm{d}\psi_\alpha^-(t)$$

for each $0 < \delta < b - a$ and by Theorem 1, 3, 6 [1] we have

$$\int_a^b f(t)\, \mathrm{d}\psi_\alpha^-(t) = f(a)\, \Delta^+\psi_\alpha^-(a) = 0\,.$$

Let $a < \alpha \leqq b$. By the same reason as above we have $\int_a^b f(t)\, \mathrm{d}\psi_\alpha^-(t) = \int_{\alpha-\delta}^{\alpha+\delta} f(t)\,.$ $.\, \mathrm{d}\psi_\alpha^-(t)$ if $\alpha < b$ and $\int_a^b f(t)\, \mathrm{d}\psi_\alpha^-(t) = \int_{\alpha-\delta}^b f(t)\, \mathrm{d}\psi_\alpha^-(t)$ if $\alpha = b$ for all sufficiently small $\delta > 0$. Using Theorem 1, 3, 6 [1] we can evaluate

$$\lim_{\delta\to 0+} \int_{\alpha-\delta}^{\alpha+\delta} f(t)\, \mathrm{d}\psi_\alpha^-(t) = \lim_{\delta\to 0+} \left(\int_{\alpha-\delta}^{\alpha} + \int_{\alpha}^{\alpha+\delta}\right) f(t)\, \mathrm{d}\psi_\alpha^-(t) =$$
$$= f(\alpha)\, \Delta^-\psi_\alpha^-(\alpha) + f(\alpha)\, \Delta^+\psi_\alpha^-(\alpha) = f(\alpha)\, \Delta\psi_\alpha^-(\alpha) = f(\alpha)$$

if $\alpha < b$ and similarly

$$\lim_{\delta\to 0+} \int_{\alpha-\delta}^{b} f(t)\, \mathrm{d}\psi_\alpha^-(t) = f(b)\, \Delta^-\psi_b^-(b) = f(b)$$

if $\alpha = b$. Therefore the second equation in (2,4) holds.

Corollary 2,1. *If $\alpha \in \langle a, b\rangle$, $\psi_\alpha(t) = 0$ for $t \in \langle a, b\rangle$, $t \neq \alpha$, $\psi_\alpha(\alpha) = 1$, $g \in V(a, b)$ then we have by* (2,3)

$$\text{(2,5)} \qquad \int_a^b \psi_\alpha(t)\, \mathrm{d}g(t) = g(\alpha+) - g(\alpha-) = \Delta g(\alpha)$$

since $\psi_\alpha(t) = \psi_\alpha^-(t) - \psi_\alpha^+(t)$ for $t \in \langle a, b\rangle$.

Let a countable set $(t_1, t_2, \ldots)$ of points in $\langle a, b\rangle$ be given, $t_i \neq t_j$ for $i \neq j$ and let us have two sequences c_i^+, c_i^-, $i = 1, 2, \ldots$ of real numbers such that the series $\sum_{a \leqq t_i < b} c_i^+$, $\sum_{a < t_i \leqq b} c_i^-$ converge absolutely. The function

$$g_B(t) = \sum_{a \leqq t_i < t} c_i^+ + \sum_{a < t_i \leqq t} c_i^-$$

will be called a break function in $\langle a, b\rangle$. For this break function $g_B(t)$ we have evidently

$$\operatorname{var}_a^b g_B = \sum_{a < t_i \leqq b} |c_i^-| + \sum_{a \leqq t_i < b} |c_i^+| < +\infty\,,$$

$\Delta^+ g_B(t) = \Delta^- g_B(t) = 0$ if $t \neq t_i$, $i = 1, 2, \dots$ and $\Delta^+ g_B(t_i) = c_i^+$, $\Delta^- g_B(t_i) = c_i^-$, $i = 1, 2, \dots$

Using the functions ψ_α^+ and ψ_α^- introduced in Proposition 2,1 we can evidently express the break function in the form

$$g_B(t) = \sum_{i=1}^{\infty} [c_i^+ \psi_{t_i}^+(t) + c_i^- \psi_{t_i}^-(t)] =$$

$$= \sum_{i=1}^{\infty} [\Delta^+ g_B(t_i) \psi_{t_i}^+(t) + \Delta^- g_B(t_i) \psi_{t_i}^-(t)] .$$

Remark 2,1. The notion of a break function can be similarly introduced for n-vector functions too; for this case it is sufficient to take $c_i^+ \in R^n$, $c_i^- \in R^n$ and repeat the above procedure where instead of $|.|$ should be written $\|.\|$.

Proposition 2,2. *Let $g_B \in V(a, b)$ be a break function. If $f \in V(a, b)$ then*

$$\int_a^b f(t)\, \mathrm{d}g_B(t) = f(a)\, \Delta^+ g_B(a) + \sum_{a<\tau<b} f(\tau)\, \Delta g_B(\tau) + f(b)\, \Delta^- g_B(b) . \tag{2,6}$$

Proof. Since g_B is a break function there exists a sequence $\{t_i\}$, $t_i \in \langle a, b \rangle$, $t_i \neq t_j$ for $i \neq j$ such that

$$g_B(t) = \sum_{i=1}^{\infty} [\Delta^+ g_B(t_i) \psi_{t_i}^+(t) + \Delta^- g_B(t_i) \psi_{t_i}^-(t)] .$$

We put

$$g_B^N(t) = \sum_{i=1}^{N} [\Delta^+ g_B(t_i) \psi_{t_i}^+(t) + \Delta^- g_B(t_i) \psi_{t_i}^-(t)] .$$

We have

$$\operatorname{var}_a^b (g_B - g_B^N) = \operatorname{var}_a^b \Big(\sum_{i=N+1}^{\infty} [\Delta^+ g(t_i) \psi_{t_i}^+(t) + \Delta^- g(t_i) \psi_{t_i}^-(t)] =$$

$$= \sum_{i=N+1}^{\infty} [|\Delta^+ g(t_i)| + |\Delta^- g(t_i)|] .$$

The relation $g \in V(a, b)$ implies the convergence of the series $\sum_{i=1}^{\infty} [|\Delta^+ g(t_i)| + |\Delta^- g(t_i)|]$ and therefore we have

$$\lim_{N\to\infty} \operatorname{var}_a^b (g_B - g_B^N) = 0 .$$

Hence by (2,2) we obtain

$$\lim_{N\to\infty} \left| \int_a^b f(t)\, \mathrm{d}g_B(t) - \int_a^b f(t)\, \mathrm{d}g_B^N(t) \right| \leqq \lim_{N\to\infty} [|f(a)| + \operatorname{var}_a^b f] \operatorname{var}_a^b (g_B - g_B^N) = 0$$

i.e.

$$\int_a^b f(t)\,\mathrm{d}g_B(t) = \lim_{N\to\infty} \int_a^b f(t)\,\mathrm{d}g_B^N(t)\,. \tag{2,7}$$

Using (2,4) we have

$$\int_a^b f(t)\,\mathrm{d}g_B^N(t) = \sum_{i=1}^N \int_a^b f(t)\,\mathrm{d}[\Delta^+ g_B(t_i)\,\psi_{t_i}^+(t) + \Delta^- g_B(t_i)\,\psi_{t_i}^-(t)] =$$

$$= \sum_{i=1}^N \left[\Delta^+ g_B(t_i) \int_a^b f(t)\,\mathrm{d}\psi_{t_i}^+(t) + \Delta^- g_B(t_i) \int_a^b f(t)\,\mathrm{d}\psi_{t_i}^-(t)\right] =$$

$$= \sum_{i=1}^N [\Delta^+ g_B(t_i) f(t_i) + \Delta^- g_B(t_i) f(t_i)] =$$

$$= \sum_{i=1}^N f(t_i)\,[\Delta^+ g_B(t_i) + \Delta^- g_B(t_i)]\,.$$

This and (2,7) give (2,6) and Proposition 2,2 is proved.

Corollary 2,2. *If $g \in V(a, b)$ is a break function such that $\Delta g(t) = 0$ for all $t \in \in (a, b)$, $\Delta^+ g(a) = \Delta^- g(b) = 0$ then $\int_a^b f(t)\,\mathrm{d}g(t) = 0$ for all $f \in V(a, b)$.*

The proof follows immediately from (2,6).

In [2] the following theorem on integration by parts is proved:

Let $f, g \in V(a, b)$ then for any interval $\langle c, d\rangle \subset \langle a, b\rangle$ we have

$$\int_c^d f(t)\,\mathrm{d}g(t) + \int_c^d g(t)\,\mathrm{d}f(t) = \tag{2,8}$$

$$= f(d)\,g(d) - f(c)\,g(c) - \sum_{c \leqq \tau < d} \Delta^+ f(\tau)\,\Delta^+ g(\tau) + \sum_{c < \tau \leqq d} \Delta^- f(\tau)\,\Delta^- g(\tau)\,.$$

Let $\mathbf{z}, \mathbf{w} \in V_n(a, b)$; we denote for $\langle c, d\rangle \subset \langle a, b\rangle$

$$\int_c^d \mathbf{z}'(t)\,\mathrm{d}\mathbf{w}(t) = \int_c^d \mathrm{d}[\mathbf{w}'(t)]\,\mathbf{z}(t) = \sum_{i=1}^n \int_c^d z_i(t)\,\mathrm{d}w_i(t)\,.$$

Using this notation and the integration by parts formula (2,8) we can easily derive the integration by parts formula for n-vector functions $\mathbf{z}, \mathbf{w} \in V_n(a, b)$, $\langle c, d\rangle \subset \subset \langle a, b\rangle$ in the form

$$\int_c^d \mathbf{z}'(t)\,\mathrm{d}\mathbf{w}(t) + \int_c^d \mathbf{w}'(t)\,\mathrm{d}\mathbf{z}(t) = \int_c^d \mathbf{z}'(t)\,\mathrm{d}\mathbf{w}(t) + \int_c^d \mathrm{d}[\mathbf{z}'(t)]\,\mathbf{w}(t) = \tag{2,9}$$

$$= \mathbf{z}'(d)\,\mathbf{w}(d) - \mathbf{z}'(c)\,\mathbf{w}(c) - \sum_{c \leqq \tau < d} \Delta^+ \mathbf{z}'(\tau)\,\Delta^+ \mathbf{w}(\tau) + \sum_{c < \tau \leqq d} \Delta^- \mathbf{z}'(\tau)\,\Delta^- \mathbf{w}(\tau)\,.$$

Remark 2,2. In a similar manner can be obtained the result of Corollary 2,2 for n-vector functions: If $\boldsymbol{w} \in V_n(a, b)$ is a break function (cf. Remark 2,1) such that $\Delta \boldsymbol{w}(t) = 0$ for all $t \in (a, b)$, $\Delta^+ \boldsymbol{w}(a) = \Delta^- \boldsymbol{w}(b) = 0$ then $\int_a^b \boldsymbol{z}'(t) \, d\boldsymbol{w}(t) = 0$ for all $\boldsymbol{z} \in V_n(a, b)$.

Now let a nondegenerate interval $I = \langle a, b \rangle \times \langle c, d \rangle$ in R^2 be given; $\boldsymbol{K}(s, t) : I \to L(R^n \to R^n)$ let be a matrix function defined on the interval I. The elements of the matrix $\boldsymbol{K}(s, t)$ are denoted by $k_{ij}(s, t)$, i.e. $\boldsymbol{K}(s, t) = (k_{ij}(s, t))$, $i, j = 1, 2, \ldots, n$.

For a given subinterval $J = \langle \bar{a}, \bar{b} \rangle \times \langle \bar{c}, \bar{d} \rangle \subset I$ we set $m_{\boldsymbol{K}}(J) = \boldsymbol{K}(\bar{b}, \bar{d}) - \boldsymbol{K}(\bar{b}, \bar{c}) - \boldsymbol{K}(\bar{a}, \bar{d}) + \boldsymbol{K}(\bar{a}, \bar{c}) \in L(R^n \to R^n)$ and define

$$v_I(\boldsymbol{K}) = \sup \sum_i \|m_{\boldsymbol{K}}(J_i)\| \tag{2,10}$$

where the supremum is taken over all finite systems of subintervals $J_i \subset I$ such that for the interiors J_i^0 of J_i (in the topology of R^2) we have $J_i^0 \cap J_j^0 = \emptyset$ when $i \neq j$. The norm $\|.\|$ used in (2,10) is the operator norm in $L(R^n \to R^n)$ (see Sec. 1.).

The number $v_I(\boldsymbol{K})$ established in (2,10) is a kind of twodimensional variation of the matrix function $\boldsymbol{K}(s, t)$ in the interval I. This notion of a twodimensional variation is considered in the book of T. H. Hildebrandt [5] (for the case $n = 1$).

For a real function $k(s, t) : I \to R$ we can define the number $v_I(k)$ as above if we take $n = 1$. The properties of our operator norm imply

$$v_I(k_{ij}) \leqq v_I(\boldsymbol{K}) \tag{2,11}$$

for all $i, j = 1, 2, \ldots, n$.

If $I_j \subset I$ is a rectangle for each $j = 1, 2, \ldots, m$ and $I_i^0 \cap I_k^0 = \emptyset$ for $j \neq k$, then we can define the number $v_{I_j}(\boldsymbol{K})$ for each $j = 1, 2, \ldots, m$ as above and by definition we easily obtain

$$\sum_{j=1}^{m} v_{I_j}(\boldsymbol{K}) \leqq v_I(\boldsymbol{K}) \,. \tag{2,12}$$

We define as usual

$$\operatorname{var}_c^d \boldsymbol{K}(s, .) = \sup \sum_i \|\boldsymbol{K}(s, t_i) - \boldsymbol{K}(s, t_{i-1})\|$$

for fixed $s \in \langle a, b \rangle$ and

$$\operatorname{var}_a^b \boldsymbol{K}(., t) = \sup \sum_j \|\boldsymbol{K}(s_j, t) - \boldsymbol{K}(s_{j-1}, t)\|$$

for fixed $t \in \langle c, d \rangle$ where the supremums are taken over all finite decompositions of the interval $\langle c, d \rangle$, $\langle a, b \rangle$ respectively.

The properties of the used operator norm imply

$$\operatorname{var}_c^d k_{ij}(s, .) \leqq \operatorname{var}_c^d \boldsymbol{K}(s, .) \,, \tag{2,13a}$$

$$\operatorname{var}_a^b k_{ij}(., t) \leqq \operatorname{var}_a^b \boldsymbol{K}(., t) \tag{2,13b}$$

for any $i, j = 1, 2, \ldots, n$, $s \in \langle a, b \rangle$, $t \in \langle c, d \rangle$.

For any $s, s_0 \in \langle a, b\rangle$, $t_{j-1}, t_j \in \langle c, d\rangle$ we have

$$\|\mathbf{K}(s, t_j) - \mathbf{K}(s, t_{j-1})\| \leqq \|m_{\mathbf{K}}(J_j)\| + \|\mathbf{K}(s_0, t_j) - \mathbf{K}(s_0, t_{j-1})\|$$

where $J_i = \langle s_0, s\rangle \times \langle t_{j-1}, t_j\rangle$. Hence for each decomposition $D : c = t_0 < < t_1 < \ldots < t_m = d$ of the interval $\langle c, d\rangle$ the inequality

$$\sum_{j=1}^{m} \|\mathbf{K}(s, t_j) - \mathbf{K}(s, t_{j-1})\| \leqq \sum_{j=1}^{m} \|m_{\mathbf{K}}(J_j)\| + $$
$$+ \sum_{j=1}^{m} \|\mathbf{K}(s_0, t_j) - \mathbf{K}(s_0, t_{j-1})\| \leqq v_I(\mathbf{K}) + \operatorname{var}_c^d \mathbf{K}(s_0, .)$$

holds; therefore

(2,14a)
$$\operatorname{var}_c^d \mathbf{K}(s, .) \leqq v_I(\mathbf{K}) + \operatorname{var}_c^d \mathbf{K}(s_0, .)$$

for each $s \in \langle a, b\rangle$. Similarly can be proved for $t_0 \in \langle c, d\rangle$ the inequality

(2,14b)
$$\operatorname{var}_a^b \mathbf{K}(., t) \leqq v_I(\mathbf{K}) + \operatorname{var}_a^b \mathbf{K}(., t_0)$$

which holds for each $t \in \langle c, d\rangle$.

Therefore, if we suppose that $v_I(\mathbf{K}) < +\infty$, $\operatorname{var}_c^d \mathbf{K}(s_0, .) < +\infty$ for some $s_0 \in \in \langle a, b\rangle$ then we have $\operatorname{var}_c^d \mathbf{K}(s, .) < +\infty$ for all $s \in \langle a, b\rangle$ and symmetrically if $v_I(\mathbf{K}) < +\infty$, $\operatorname{var}_a^b \mathbf{K}(., t_0) < +\infty$ for some $t_0 \in \langle c, d\rangle$, then $\operatorname{var}_a^b \mathbf{K}(., t) < +\infty$ for all $t \in \langle c, d\rangle$.

Let us put

(2,15a)
$$\varphi(\sigma) = v_{\langle a,\sigma\rangle \times \langle c,d\rangle}(\mathbf{K})$$

for $\sigma \in \langle a, b\rangle$; $\varphi(\sigma) : \langle a, b\rangle \to R$ is evidently a nondecreasing function in $\langle a, b\rangle$, $\varphi(a) = 0$, $\varphi(b) = v_I(\mathbf{K})$. In the same way we can define

(2,15b)
$$\psi(\tau) = v_{\langle a,b\rangle \times \langle c,\tau\rangle}(\mathbf{K})$$

for $\tau \in \langle c, d\rangle$; $\psi(\tau) : \langle c, d\rangle \to R$ is nondecreasing, $\psi(c) = 0$, $\psi(d) = v_I(\mathbf{K})$.

Note that for an arbitrary decomposition of the interval $\langle a, b\rangle : a = s_0 < s_1 < \ldots \ldots < s_l = b$ and any two points $t_1, t_2 \in \langle c, d\rangle$ we have

$$|\operatorname{var}_c^{t_2} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .)) - \operatorname{var}_c^{t_1} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .))| =$$
$$= |\operatorname{var}_{t_1}^{t_2} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .))| \leqq v_{\langle s_{i-1},s_i\rangle \times \langle t_1,t_2\rangle}(\mathbf{K})$$

for $i = 1, 2, \ldots, l$, i.e.

(2,16a)
$$\left|\sum_{i=1}^{l} [\operatorname{var}_c^{t_2} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .)) - \operatorname{var}_c^{t_1} (\mathbf{K}(s_i, .) - \mathbf{K}(s_{i-1}, .))]\right| \leqq$$
$$\leqq \sum_{i=1}^{l} v_{\langle s_{i-1},s_i\rangle \times \langle t_1,t_2\rangle}(\mathbf{K}) \leqq v_{\langle a,b\rangle \times \langle t_1,t_2\rangle}(\mathbf{K}) \leqq |\psi(t_2) - \psi(t_1)| .$$

Symmetrically for an arbitrary decomposition $c = t_0 < t_1 < \ldots < t_m = d$ and any two points $s_1, s_2 \in \langle a, b\rangle$ it is

$$\text{(2,16b)} \qquad \left|\sum_{j=1}^{m} [\operatorname{var}_a^{s_2} (\mathbf{K}(.,t_j) - \mathbf{K}(.,t_{j-1})) - \operatorname{var}_a^{s_1} (\mathbf{K}(.,t_j) - \mathbf{K}(.,t_{j-1}))]\right| \leqq$$
$$\leqq |\varphi(s_2) - \varphi(s_1)|\,.$$

Lemma 2,1. *Let $\mathbf{K}(s, t) : I = \langle a, b\rangle \times \langle c, d\rangle \to L(R^n \to R^n)$ satisfy $v_I(\mathbf{K}) < +\infty$ and $\operatorname{var}_c^d \mathbf{K}(s_*, .) < +\infty$ for some $s_* \in \langle a, b\rangle$. Let further for some $s_0 \in \langle a, b\rangle$*

$$\text{(2,17)} \qquad \lim_{s \to s_0+} \|\mathbf{K}(s, t) - \mathbf{K}(s_0, t)\| = 0$$

or

$$\lim_{s \to s_0-} \|\mathbf{K}(s, t) - \mathbf{K}(s_0, t)\| = 0$$

for all $t \in \langle c, d\rangle$ then

$$\text{(2,18)} \qquad \lim_{s \to s_0+} \varphi(s) = \varphi(s_0)\,, \quad \lim_{s \to s_0-} \varphi(s) = \varphi(s_0)$$

respectively where $\varphi : \langle a, b\rangle \to R$ is the function defined in (2,15a). *(If $s_0 = a$ ($s_0 = b$) then we consider the first (second) case only.)*

Proof. We prove the first case only, the other one is symmetric. The function φ is nondecreasing, i.e. $\varphi(s) - \varphi(s_0) \geqq 0$ for $s \geqq s_0$. Let us suppose that our Lemma is not valid and that there is a number $\eta > 0$ such that for all $s > s_0$ we have

$$\text{(2,19)} \qquad \varphi(s) - \varphi(s_0) \geqq \eta > 0\,.$$

By definition of $\varphi(s)$ there exists a finite system of intervals $J_l \subset \langle a, s\rangle \times \langle c, d\rangle$, $J_l^0 \cap J_j^0 = \emptyset$, $l \neq j$ such that

$$\sum_l \|m_{\mathbf{K}}(J_l)\| > \varphi(s) - \frac{\eta}{8}$$

We put $I_l^* = J_l \cap \langle a, s_0\rangle \times \langle c, d\rangle$ and $I_l = J_l \cap \langle s_0, s\rangle \times \langle c, d\rangle$. Obviously

$$\|m_{\mathbf{K}}(J_l)\| \leqq \|m_{\mathbf{K}}(I_l^*)\| + \|m_{\mathbf{K}}(I_l)\|$$

and

$$\sum_l \|m_{\mathbf{K}}(I_l)\| \leqq \varphi(s_0)\,.$$

Hence

$$\sum_l \|m_{\mathbf{K}}(I_l)\| + \sum_l \|m_{\mathbf{K}}(I_l^*)\| - \varphi(s_0) > \varphi(s) - \varphi(s_0) - \frac{\eta}{8}$$

and we obtain

$$\text{(2,20)} \qquad \sum_l \|m_{\mathbf{K}}(I_l)\| > \varphi(s) - \varphi(s_0) - \frac{\eta}{8}\,.$$

At the same time I_l is a finite system of intervals in $\langle s_0, s\rangle \times \langle c, d\rangle$ and $I_l^0 \cap I_j^0 = \emptyset$ for $j \neq l$.

Let $0 < \delta < s - s_0$ and put $\tilde{I}_l = I_l \cap \langle s_0, s_0 + \delta \rangle \times \langle c, d \rangle$, $\hat{I}_l = I_l \cap \langle s_0 + \delta, s \rangle \times \langle c, d \rangle$. If for example $I_l = \langle s_0, s_0 + \delta \rangle \times \langle t', t'' \rangle$ then we have

$$\|m_{\mathbf{K}}(\tilde{I}_l)\| = \|\mathbf{K}(s_0 + \delta, t'') - \mathbf{K}(s_0 + \delta, t') - \mathbf{K}(s_0, t'') + \mathbf{K}(s_0, t')\| \leqq$$
$$\leqq \|\mathbf{K}(s_0 + \delta, t'') - \mathbf{K}(s_0, t'')\| + \|\mathbf{K}(s_0 + \delta, t') - \mathbf{K}(s_0, t')\|$$

and we see that a choice of sufficiently small $\delta > 0$ implies by (2,17) that $\|m_{\mathbf{K}}(\tilde{I}_l)\|$ is arbitrarily small. This procedure can be repeated for all possible forms of $\tilde{I}_l$ in a similar manner. Hence for sufficiently small $\delta > 0$ we can obtain

$$\sum_l \|m_{\mathbf{K}}(\tilde{I}_l)\| < \frac{\eta}{8}.$$

This inequality together with $\sum_l \|m_{\mathbf{K}}(I_l)\| \leqq \sum_l \|m_{\mathbf{K}}(\hat{I}_l)\| + \sum_l \|m_{\mathbf{K}}(\tilde{I}_l)\|$ and (2,20) implies

$$\sum_l \|m_{\mathbf{K}}(\hat{I}_l)\| > \varphi(s) - \varphi(s_0) - \frac{\eta}{4}. \tag{2,21}$$

As (2,19) is assumed there is a finite system of intervals $I_k^{**} \subset \langle a, s_0 + \delta \rangle \times \langle c, d \rangle$, $(I_k^{**})^0 \cap (I_j^{**})^0 = \emptyset$ for $j \neq k$ such that

$$\sum_k \|m_{\mathbf{K}}(I_k)\| - \varphi(s) > \frac{\eta}{2}.$$

From this and from (2,21) we have

$$\sum_l \|m_{\mathbf{K}}(\hat{I}_l)\| + \sum_k \|m_{\mathbf{K}}(I_k^{**})\| - \varphi(s_0) > \varphi(s) - \varphi(s_0) + \frac{\eta}{4}. \tag{2,22}$$

Since the union of intervals $\hat{I}_l$ and I_k^{**} forms a finite system of intervals in $\langle a, s \rangle \times \langle c, d \rangle$ with mutually disjoint interiors, we obtain from (2,22) by definition of $\varphi(s)$ the contradictory inequality $\varphi(s) - \varphi(s_0) > \varphi(s) - \varphi(s_0) + \eta/4$. Thus our Lemma is proved.

Remark 2,3. By definition we have for $s_0 \in \langle a, b \rangle$, $0 < \delta < b - s_0$

$$\operatorname{var}_c^d (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0, .)) \leqq v_{\langle s_0, s_0 + \delta \rangle \times \langle c, d \rangle}(\mathbf{K}) \leqq \varphi(s_0 + \delta) - \varphi(s_0) \tag{i}$$

where φ is given in (2,15a). Hence if the first equation in (2,17) holds, we have by (2,18)

$$\lim_{\delta \to 0+} \operatorname{var}_c^d (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0, .)) = 0.$$

If an arbitrary $\mathbf{K}(s, t) : \langle a, b \rangle \times \langle c, d \rangle \to L(R^n \to R^n)$ is given with $v_I(\mathbf{K}) < +\infty$, $\operatorname{var}_c^d \mathbf{K}(s_*, .) < +\infty$ for some $s_* \in \langle a, b \rangle$ and $\lim_{s \to s_0+} \mathbf{K}(s, t) = \mathbf{K}(s_0+, t)$ exists for every $t \in \langle c, d \rangle$ then we can define $\mathbf{K}^\circ(s, t) = \mathbf{K}(s, t)$ if $(s, t) \in I = \langle a, b \rangle \times$

$\times \langle c, d\rangle$, $s \neq s_0$, $\mathbf{K}^\circ(s_0, t) = \mathbf{K}(s_0+, t)$. Easily can be obtained $v_I(\mathbf{K}^\circ) < +\infty$ and $\operatorname{var}_c^d \mathbf{K}^\circ(s_{**}, .) < +\infty$ for some $s_{**} \in \langle a, b\rangle$. We have further evidently $\lim_{s \to s_0+} \|\mathbf{K}^\circ(s, t) - \mathbf{K}^\circ(s_0, t)\| = 0$ for every $t \in \langle c, d\rangle$ and in the same way as above we obtain

$$\text{(ii)} \qquad \lim_{\delta \to 0+} \operatorname{var}_c^d (\mathbf{K}^\circ(s_0 + \delta, .) - \mathbf{K}^\circ(s_0, .)) =$$
$$= \lim_{\delta \to 0+} \operatorname{var}_c^d (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0+, .)) = 0 .$$

A similar appointment gives the same result for left hand side limits. This implies the following

Corollary 2,3. *If for* $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$, $v_I(\mathbf{K}) < +\infty$, $\operatorname{var}_c^d \mathbf{K}(s_*, .) < +\infty$ *for some* $s_* \in \langle a, b\rangle$ *and if* $\lim_{s \to s_0+} \mathbf{K}(s, t) = \mathbf{K}(s_0+, t)$ *exists for any* $t \in \langle c, d\rangle$ *then we have*

$$\operatorname{var}_c^d (\mathbf{K}(s_0+, .) - \mathbf{K}(s_0, .)) = \operatorname{var}_c^d \Delta^+ \mathbf{K}(s_0, .) \leqq \varphi(s_0+) - \varphi(s_0) = \Delta^+ \varphi(s_0) .$$

(*A similar statement for left hand side limits holds.*)

Proof. For any $\delta > 0$ we have

$$\operatorname{var}_c^d (\mathbf{K}(s_0+, .) - \mathbf{K}(s_0, .)) =$$
$$= \operatorname{var}_c^d (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0, .) + \mathbf{K}(s_0+, .) - \mathbf{K}(s_0 + \delta, .) \leqq$$
$$\leqq \operatorname{var}_c^d (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0, .)) + \operatorname{var}_c^d (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0+, .)) \leqq$$
$$\leqq \varphi(s_0 + \delta) - \varphi(s_0) + \operatorname{var}_c^d (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0+, .)) .$$

Hence by the limiting process $\delta \to 0+$ we obtain our inequality by means of (i) and (ii).

Now we define integrals of vector functions. If an n-vector $\mathbf{U}(\tau, t) = (U_1(\tau, t), \ldots, \ldots, U_n(\tau, t))$ is given, $\mathbf{U}(\tau, t) : S \to R^n$, $S = \{(\tau, t) \in R^2;\ c \leqq \tau \leqq d,\ \tau - \delta(\tau) \leqq t \leqq \leqq \tau + \delta(\tau),\ \delta(\tau) > 0 \text{ for every } \tau \in \langle c, d\rangle\}$ then by definition (cf. [1])

$$\int_c^d \mathrm{D}\mathbf{U}(\tau, t) = \left(\int_c^d \mathrm{D}U_1(\tau, t), \ldots, \int_c^d \mathrm{D}U_n(\tau, t)\right).$$

Given $\mathbf{x} : \langle c, d\rangle \to R^n$, we put for any $s \in \langle a, b\rangle$

$$\mathbf{U}(\tau, t) = \mathbf{K}(s, t)\,\mathbf{x}(\tau) = \Big(\sum_{j=1}^n k_{1j}(s, t)\, x_j(\tau), \ldots, \sum_{j=1}^n k_{nj}(s, t)\, x_j(\tau)\Big)'$$

and denote

$$\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) = \int_c^d \mathrm{D}\mathbf{U}(\tau, t) =$$

$$= \left(\int_c^d \mathrm{D}\big(\sum_{j=1}^n k_{1j}(s, t)\, x_j(\tau)\big), \ldots, \int_c^d \mathrm{D}\big(\sum_{j=1}^n k_{nj}(s, t)\, x_j(\tau)\big)\right)' =$$

$$= \left(\sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t k_{1j}(s, t), \ldots, \sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t k_{nj}(s, t)\right)'.$$

Remark 2,4. For this definition we need to know the values $\mathbf{K}(s, t)$ for $t < c$, $t > d$. We suppose therefore $\mathbf{K}(s, t) = \mathbf{K}(s, c)$ for $t < c$ and $\mathbf{K}(s, t) = \mathbf{K}(s, d)$ for $t > d$.

For $\mathbf{y} : \langle a, b\rangle \to R^n$ we define in a similar manner the integral $\int_a^b \mathrm{d}_s[\mathbf{K}(s, t)]\, \mathbf{y}(s)$ and as above we suppose $\mathbf{K}(s, t) = \mathbf{K}(a, t)$ for $s < a$ and $\mathbf{K}(s, t) = \mathbf{K}(b, t)$ for $s > b$.

Proposition 2,3. *Let* $\mathbf{K}(s, t) : I = \langle a, b\rangle \times \langle c, d\rangle \to L(R^n \to R^n)$ *be given and let* $\mathbf{K}(s, t) = \mathbf{K}(s, c)$ *for* $t < c$, $\mathbf{K}(s, t) = \mathbf{K}(s, d)$ *for* $t > d$. *Let us suppose that* $v_I(\mathbf{K}) < +\infty$ *and* $\mathrm{var}_c^d \mathbf{K}(s_0, .) < +\infty$ *for some* $s_0 \in \langle a, b\rangle$.

If $\mathbf{x}(t) \in V_n(c, d)$ *then the integral* $\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t)$ *exists for any* $s \in \langle a, b\rangle$. *The inequality*

$$\left\|\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t)\right\| \leqq \sup_{t\in\langle c,d\rangle} \|\mathbf{x}(t)\| \,.\, \mathrm{var}_c^d \mathbf{K}(s, .) \tag{2,23}$$

holds for any $s \in \langle a, b\rangle$. *Further we have*

$$\mathrm{var}_a^b\left(\int_c^d \mathrm{d}_t \mathbf{K}(s, t)\, \mathbf{x}(t)\right) \leqq \int_c^d \|\mathbf{x}(t)\|\, \mathrm{d}\psi(t) \leqq \sup_{t\in\langle c,d\rangle} \|\mathbf{x}(t)\| \,.\, v_I(\mathbf{K}) \tag{2,24}$$

where the function ψ *is defined in* (2,15b). *Thus the integral* $\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t)$ *as a function of the variable s belongs to* $V_n(a, b)$.

Proof. By the assumption and by (2,14a) we obtain $\mathrm{var}_c^d \mathbf{K}(s, .) < +\infty$ for all $s \in \langle a, b\rangle$; this implies the existence of the integral $\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t)$ for any $s \in \langle a, b\rangle$. Further we have for each $s \in \langle a, b\rangle$

$$\|\mathbf{K}(s, t_2)\, \mathbf{x}(\tau) - \mathbf{K}(s, t_1)\, \mathbf{x}(\tau)\| \leqq \|\mathbf{x}(\tau)\| \,.\, \mathrm{var}_{t_1}^{t_2} \mathbf{K}(s, .) \leqq$$
$$\leqq \|\mathbf{x}(\tau)\| \,.\, |\mathrm{var}_c^{t_2} \mathbf{K}(s, .) - \mathrm{var}_c^{t_1} \mathbf{K}(s, .)|$$

for any t_1, t_2 and for $\tau \in \langle c, d\rangle$. Hence Lemma 2,1 [3] implies

$$\left\|\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t)\right\| \leqq \int_c^d \|\mathbf{x}(t)\|\, \mathrm{d}(\mathrm{var}_c^t \mathbf{K}(s, .))$$

and by (2,1) we obtain (2,23).

Let an arbitrary finite decomposition $a = s_0 < s_1 < \ldots < s_l = b$ of the interval $\langle a, b \rangle$ be given. We have

$$\sum_{i=1}^{l} \left\| \int_c^d \mathrm{d}_t[\boldsymbol{K}(s_i, t) - \boldsymbol{K}(s_{i-1}, t)]\, \boldsymbol{x}(t) \right\| \leqq$$
$$\leqq \sum_{i=1}^{l} \int_c^d \|\boldsymbol{x}(t)\|\, \mathrm{d}(\mathrm{var}_c^t\, [\boldsymbol{K}(s_i, .) - \boldsymbol{K}(s_{i-1}, .)]) \leqq$$
$$\leqq \int_c^d \|\boldsymbol{x}(t)\|\, \mathrm{d}(\sum_{i=1}^{l} \mathrm{var}_c^t\, [\boldsymbol{K}(s_i, .) - \boldsymbol{K}(s_{i-1}, .)])\,.$$

From this inequality we obtain using (2,16a) and Lemma 2,1 [3] the inequality

$$\sum_{i=1}^{l} \left\| \int_c^d \mathrm{d}_t[\boldsymbol{K}(s_i, t) - \boldsymbol{K}(s_{i-1}, t)]\, \boldsymbol{x}(t) \right\| \leqq \int_c^d \|\boldsymbol{x}(t)\|\, \mathrm{d}\psi(t)\,.$$

Passing to the supremum on the left hand side in this inequality we obtain immediately the first inequality in (2,24). The other one follows from the relation $\int_c^d d\psi(t) = \psi(d) - \psi(c) = v_I(\boldsymbol{K})$ and from (2,1). Therefore we have $\int_c^d \mathrm{d}_t[\boldsymbol{K}(s, t)]\, \boldsymbol{x}(t) \in V_n(a, b)$.

Remark 2,5. A similar proposition can also be proved for $\boldsymbol{y} \in V_n(a, b)$ and the integral $\int_a^b \mathrm{d}_s[\boldsymbol{K}(s, t)]\, \boldsymbol{y}(s)$.

Lemma 2,2. *Let $k(s, t) : I = \langle a, b \rangle \times \langle c, d \rangle \to R$ be given. Suppose that $k(s, t) = k(a, t)$ for $s < a$, $k(s, t) = k(b, t)$ for $s > b$, $k(s, t) = k(s, c)$ for $t < c$ and $k(s, t) = k(s, d)$ for $t > d$. Further let $v_I(k) < +\infty$, $\mathrm{var}_c^d\, k(s_0, .) < +\infty$ for some $s_0 \in \langle a, b \rangle$ and $\mathrm{var}_a^b\, k(., t_0) < +\infty$ for some $t_0 \in \langle c, d \rangle$. If $f(s) \in V(a, b)$, $g \in V(c, d)$ then*

$$\int_c^d g(t)\, \mathrm{d}_t \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, t)] \right) = \int_a^b f(s)\, \mathrm{d}_s \left(\int_c^d g(t)\, \mathrm{d}_t[k(s, t)] \right) \tag{2,25}$$

and

$$\int_c^d g(t)\, \mathrm{d}_t \left(\int_a^b k(s, t)\, \mathrm{d}f(s) \right) = \int_a^b \left(\int_c^d g(t)\, \mathrm{d}_t[k(s, t)] \right) \mathrm{d}f(s) \tag{2,26}$$

hold and the integrals on both sides of (2,25) and (2,26) exist.

Proof. By Proposition 2,3 we have $\int_a^b f(s)\, \mathrm{d}_s[k(s, t)] \in V(c, d)$, $\int_c^d g(t)\, \mathrm{d}_t[k(s, t)] \in V(a, b)$ and the existence of the integrals on both sides of (2,25) follows from this fact immediately.

Let us put $f(s) = \psi_\alpha^+(s)$, $g(t) = \psi_\beta^-(t)$ for $\alpha \in \langle a, b \rangle$, $\beta \in \langle c, d \rangle$ (cf. Proposition 2,1). Using (2,3) from Proposition 2,1 we obtain by easy computation

$$\left(\int_c^d g(t)\, \mathrm{d}_t \int_a^b f(s)\, \mathrm{d}_s[k(s, t)] \right) = k(b, d) - k(b, \beta-) - k(\alpha +, d) + k(\alpha+, \beta-) =$$
$$= \int_a^b f(s)\, \mathrm{d}_s \left(\int_c^d g(t)\, \mathrm{d}_t[k(s, t)] \right)$$

i.e. (2,25) holds for this choice of f and g. We note that the term $k(\alpha+, \beta-)$ in the above computation is in one case obtained as $\lim_{t\to\beta-} \lim_{s\to\alpha+} k(s, t)$ and as $\lim_{s\to\alpha+} \lim_{t\to\beta-} k(s, t)$ in the other one. By Theorem III. 5.3 in [5] both iterated limits are equal since by assumption the existence of all quadrantal limits in any point of I is quaranted. Similarly it can be proved by direct computation that (2,25) is true if $f(s)$ equals $\psi_\alpha^-(s)$ or $\psi_\alpha^+(s)$ and $g(t)$ equals $\psi_\beta^+(t)$ or $\psi_\beta^-(t)$ for some $\alpha \in \langle a, b\rangle$, $\beta \in \langle c, d\rangle$ (c.f. Proposition 2,1). Hence from the linearity of the integral we obtain that (2,25) holds if $f(s)$, $g(t)$ are step functions because it is clear that every step function can be expressed as a finite linear combination of functions of the type ψ_α^+ and ψ_α^-. It is known that if $f \in V(a, b)$ and $g \in V(c, d)$ then there exist sequences $\{f_l(s)\}$ $f_l : \langle a, b\rangle \to R$ and $\{g_l(t)\}$, $g_l : \langle c, d\rangle \to R$, $l = 1, 2, \ldots, f_l, g_l$ are step functions such that $\lim_{l\to\infty} f_l(s) = f(s)$, $\lim_{l\to\infty} g_l(t) = g(t)$ uniformly in $\langle a, b\rangle$, $\langle c, d\rangle$ respectively. We denote

$$I_{1,l} = \int_c^d g_l(t)\, \mathrm{d}_t \left(\int_a^b f_l(s)\, \mathrm{d}_s[k(s, t)]\right),$$

$$I_{2,l} = \int_a^b f_l(s)\, d_s\left(\int_c^d g_l(t)\, \mathrm{d}_t[k(s, t)]\right);$$

since (2,25) holds for step functions we have

$$(2,27) \qquad I_{1,l} = I_{2,l} \quad \text{for every} \quad l = 1, 2, \ldots$$

Further by (2,2) and (2,24) we have

$$\left|\int_c^d g(t)\, \mathrm{d}_t \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, t)]\right) - I_{1,l}\right| \leqq \left|\int_c^d (g(t) - g_l(t))\, \mathrm{d}_t \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, t)]\right)\right| +$$

$$+ \left|\int_c^d g_l(t)\, \mathrm{d}_t \left(\int_a^b (f(s) - f_l(s))\, \mathrm{d}_s[k(s, t)]\right)\right| \leqq$$

$$\leqq \sup_{t\in\langle c,d\rangle} |g(t) - g_l(t)| \operatorname{var}_c^d \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, .)]\right) +$$

$$+ \sup_{t\in\langle c,d\rangle} |g_l(t)| \operatorname{var}_c^d \left(\int_a^b (f(s) - f_l(s))\, \mathrm{d}_s[k(s, .)]\right) \leqq$$

$$\leqq [\sup_{t\in\langle c,d\rangle} |g(t) - g_l(t)| \left(|f(a)| + \operatorname{var}_a^b f\right) + \sup_{t\in\langle c,d\rangle} |g_l(t)| \sup_{s\in\langle a,b\rangle} |f(s) - f_l(s)|]\, v_I(k)$$

and therefore

$$\lim_{l\to\infty} I_{1,l} = \int_c^d g(t)\, \mathrm{d}_t \left(\int_a^b f(s)\, \mathrm{d}_s[k(s, t)]\right).$$

Similarly it also holds

$$\lim_{l\to\infty} I_{2,l} = \int_a^b f(s)\, \mathrm{d}_s \left(\int_c^d g(t)\, \mathrm{d}_t[k(s, t)]\right).$$

In this way (2,27) implies (2,25) for arbitrary $f \in V(a, b)$, $g \in V(c, d)$.

The integral on the right hand side of (2,26) exists evidently ($\int_c^d g(t)\,\mathrm{d}_t[k(s,t)] \in V(a,b)$). It can be easily proved that the inequality

$$\operatorname{var}_c^d\left(\int_a^b k(s,.)\,\mathrm{d}f(s)\right) \leqq [v_I(k) + \operatorname{var}_c^d k(s_0,.)]\operatorname{var}_a^b f < +\infty$$

holds. Thus $\int_a^b k(s,t)\,\mathrm{d}f(s) \in V(c,d)$ and the integral on the left hand side of (2,26) also exists.

The equality (2,26) holds if we set $g(t) = \psi_\alpha^+(t)$ (cf. Proposition 2,1). In fact we have by (2,3)

$$\int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t[k(s,t)] = k(s,d) - k(s,\alpha+)\,.$$

Hence

$$\int_a^b\left(\int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t[k(s,t)]\right)\mathrm{d}f(s) = \int_a^b (k(s,d) - k(s,\alpha+))\,\mathrm{d}f(s)\,.$$

By (2,3) we have also

$$\int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t\left(\int_a^b k(s,t)\,\mathrm{d}f(s)\right) = \int_a^b (k(s,d) - k(s,\alpha+))\,\mathrm{d}f(s)\,.$$

Therefore

$$\int_a^b\left(\int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t[k(s,t)]\right)\mathrm{d}f(s) = \int_c^d \psi_\alpha^+(t)\,\mathrm{d}_t\left(\int_a^b k(s,t)\,\mathrm{d}f(s)\right).$$

Similarly can be proved that (2,26) holds if $g(t) = \psi_\alpha^-(t)$ and we obtain that (2,26) holds for every step function $g(t)$. Let for $g \in V(c,d)$ a sequence of step functions $g_l : \langle c,d\rangle \to R$ be given, $\lim_{l\to\infty} g_l(t) = g(t)$ uniformly in $\langle c,d\rangle$. Then we have

$$\left|\int_c^d (g(t) - g_l(t))\,\mathrm{d}_t\left(\int_a^b k(s,t)\,\mathrm{d}f(s)\right)\right| \leqq \sup_{t\in\langle c,d\rangle}|g(t) - g_l(t)|\,(v_I(k) + \operatorname{var}_c^d k(s_0,.))\operatorname{var}_a^b f$$

and

$$\left|\int_a^b\left(\int_c^d (g(t) - g_l(t))\,\mathrm{d}_t[k(s,t)]\right)\mathrm{d}f(s)\right| \leqq$$
$$\leqq \sup_{t\in\langle c,d\rangle}|g(t) - g_l(t)|\,(v_I(k) + \operatorname{var}_c^d k(s_0,.))\operatorname{var}_a^b f\,.$$

Thus in the same way as in the case of (2,25) we obtain that (2,26) holds for any $g \in V(c,d)$.

Let us now denote

$$\langle \mathbf{z}, \mathbf{w}\rangle_{(c,d)} = \int_c^d \mathbf{z}'(t)\,\mathrm{d}\mathbf{w}(t) = \int_c^d \mathrm{d}[\mathbf{w}'(t)]\,\mathbf{z}(t) = \sum_{j=1}^n \int_c^d z_j(t)\,\mathrm{d}w_j(t)$$

if $\mathbf{z}, \mathbf{w} \in V_n(c, d)$ and

$$\langle \mathbf{z}, \mathbf{w} \rangle_{(a,b)} = \int_a^b \mathbf{z}'(s)\, \mathrm{d}\mathbf{w}(s) = \int_a^b \mathrm{d}[\mathbf{w}'(s)]\, \mathbf{z}(s) = \sum_{j=1}^n \int_a^b z_j(s)\, \mathrm{d}w_j(s)$$

if $\mathbf{z}, \mathbf{w} \in V_n(a, b)$.

Proposition 2,4. *Let $\mathbf{K}(s, t) : I = \langle a, b \rangle \times \langle c, d \rangle \to L(R^n \to R^n)$ be given and let $\mathbf{K}(s, t)$ be extended for $s < a$, $s > b$, $t < c$, $t > d$ as in Remark 2,4. Let us suppose that $v_I(\mathbf{K}) < +\infty$, $\operatorname{var}_c^d \mathbf{K}(s_0, .) < +\infty$ for some $s_0 \in \langle a, b \rangle$, $\operatorname{var}_a^b \mathbf{K}(., t_0) < < +\infty$ for some $t_0 \in \langle c, d \rangle$. Let $\mathbf{x} \in V_n(c, d)$, $\mathbf{y} \in V_n(a, b)$. Then*

$$\left\langle \mathbf{y}, \int_c^d \mathrm{d}_t[\mathbf{K}(., t)]\, \mathbf{x}(t) \right\rangle_{(a,b)} = \left\langle \mathbf{x}, \int_a^b \mathrm{d}_s[\mathbf{K}'(s, .)]\, \mathbf{y}(s) \right\rangle_{(c,d)} \tag{2,28}$$

and

$$\left\langle \int_c^d \mathrm{d}_t[\mathbf{K}(., t)]\, \mathbf{x}(t), \mathbf{y} \right\rangle_{(a,b)} = \left\langle \mathbf{x}, \int_a^b \mathbf{K}'(s, .)\, \mathrm{d}\mathbf{y}(s) \right\rangle_{(c,d)} . \tag{2,29}$$

Proof. By (1.1), (2,11), (2,13a), (2,13b) all assumptions of Lemma 2,2 are satisfied for $k_{ij}(s, t)$, $x_l(t)$, $y_m(s)$, $i, j, l, m = 1, \ldots, n$. Therefore from (2,25) we obtain

$$\int_a^b y_m(s)\, \mathrm{d}_s \left(\int_c^d x_l(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) = \int_c^d x_l(t)\, \mathrm{d}_t \left(\int_a^b y_m(s)\, \mathrm{d}_s[k_{ij}(s, t)] \right) \tag{2,30}$$

for every $i, j, l, m = 1, 2, \ldots, n$. Hence

$$\begin{aligned}
\left\langle \mathbf{y}, \int_c^d \mathrm{d}_t[\mathbf{K}(., t)]\, \mathbf{x}(t) \right\rangle_{(a,b)} &= \int_a^b \mathbf{y}'(s)\, \mathrm{d}_s \left(\int_c^d \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) \right) = \\
&= \sum_{i=1}^n \int_a^b y_i(s)\, \mathrm{d}_s \left(\sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) = \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_a^b y_i(s)\, \mathrm{d}_s \left(\int_c^d x_j(t)\, \mathrm{d}_t[k_{ij}(s, t)] \right) = \\
&= \sum_{i=1}^n \sum_{j=1}^n \int_c^d x_j(t)\, \mathrm{d}_t \left(\int_a^b y_i(s)\, \mathrm{d}_s[k_{ij}(s, t)] \right) = \\
&= \sum_{j=1}^n \int_c^d x_j(t)\, d_t \left(\sum_{i=1}^n \int_a^b y_i(s)\, \mathrm{d}_s[k_{ij}(s, t)] \right) = \\
&= \int_c^d \mathbf{x}'(t)\, \mathrm{d}_t \left(\int_a^b \mathrm{d}_s[\mathbf{K}'(s, t)]\, \mathbf{y}(s) \right) = \\
&= \left\langle \mathbf{x}, \int_a^b \mathrm{d}_s[\mathbf{K}'(s, .)]\, \mathbf{y}(s) \right\rangle_{(c,d)} .
\end{aligned}$$

Thus the equality (2,28) is proved.

From (2,26) we have

$$(2,31) \qquad \int_c^d x_l(t)\,\mathrm{d}_t\left(\int_a^b k_{ij}(s,t)\,\mathrm{d}y_m(s)\right) = \int_a^b \left(\int_c^d x_l(t)\,\mathrm{d}_t[k_{ij}(s,t)]\right)\mathrm{d}y_m(s)$$

for every $i, j, l, m = 1, 2, \ldots, n$. Hence

$$\left\langle \int_c^d \mathrm{d}_t[\mathbf{K}(.,t)]\,\mathbf{x}(t), \mathbf{y} \right\rangle_{(a,b)} =$$

$$= \sum_{i=1}^n \int_a^b \left(\sum_{j=1}^n \int_c^d x_j(t)\,\mathrm{d}_t[k_{ij}(s,t)]\right)\mathrm{d}y_i(s) =$$

$$= \sum_{i=1}^n \sum_{j=1}^n \int_a^b \left(\int_c^d x_j(t)\,\mathrm{d}_t[k_{ij}(s,t)]\right)\mathrm{d}y_i(s) =$$

$$= \sum_{i=1}^n \sum_{j=1}^n \int_c^d x_j(t)\,\mathrm{d}_t\left(\int_a^b k_{ij}(s,t)\,\mathrm{d}y_i(s)\right) =$$

$$= \sum_{j=1}^n \int_c^d x_j(t)\,\mathrm{d}_t\left(\sum_{i=1}^n \int_a^b k_{ij}(s,t)\,\mathrm{d}y_i(s)\right) =$$

$$= \int_c^d \mathbf{x}'(t)\,\mathrm{d}_t\left(\int_a^b \mathbf{K}'(s,t)\,\mathrm{d}\mathbf{y}(s)\right) = \left\langle \mathbf{x}, \int_a^b \mathbf{K}'(s,.)\,\mathrm{d}\mathbf{y}(s)\right\rangle_{(c,d)}$$

and (2,29) is also proved.

Remark 2,6. The equality (2,30) (resp. (2,31)) enables us also to derive the following relations which are symmetric to the relations (2,28) (resp. (2,29))

$$(2,32) \qquad \left\langle \mathbf{x}, \int_a^b \mathrm{d}_s[\mathbf{K}(s,.)]\,\mathbf{y}(s)\right\rangle_{(c,d)} = \left\langle \mathbf{y}, \int_c^d \mathrm{d}_t\mathbf{K}'(.,t)\,\mathbf{x}(t)\right\rangle_{(a,b)}$$

and

$$(2,33) \qquad \left\langle \int_a^b \mathrm{d}_s[\mathbf{K}(s,.)]\,\mathbf{y}(s), \mathbf{x}\right\rangle_{(c,d)} = \left\langle \mathbf{y}, \int_c^d \mathbf{K}'(.,t)\,\mathrm{d}\mathbf{x}(t)\right\rangle_{(a,b)}.$$

We note that (2,29) can be written in the form

$$\int_a^b \mathrm{d}\mathbf{y}'(s)\left(\int_c^d \mathrm{d}_t\mathbf{K}(s,t)\,\mathbf{x}(t)\right) = \int_c^d \mathrm{d}_t\left[\int_a^b \mathrm{d}\mathbf{y}'(s)\,\mathbf{K}(s,t)\right]\mathbf{x}(t).$$

In the sequel we denote $V_n = V_n(0, 1)$. For $x \in V_n$ we denote

$$\|x\|_{V_n} = \|x(0)\| + \operatorname{var}_0^1 x\,. \tag{3,1}$$

$\|\cdot\|_{V_n}$ is the usual norm in V_n, V_n with this norm forms a Banach space.

Let $K(s, t) : I = \langle 0, 1\rangle \times \langle 0, 1\rangle \to L(R^n \to R^n)$ be given. Let us suppose that $K(s, t) = K(s, 0)$ for $t < 0$ and $K(s, t) = K(s, 1)$ for $t > 1$.

Further we assume in this section that

$$v(K) = v_I(K) < +\infty \tag{3,2}$$

and

$$\operatorname{var}_0^1 K(0, .) < +\infty\,. \tag{3,3}$$

Proposition 2,3 quarantees for every $x \in V_n$, $x'(t) = (x_1(t), \ldots, x_n(t))$ the existence of the integral

$$\int_0^1 d_t[K(s, t)]\, x(t) = y(s)\,, \quad s \in \langle 0, 1\rangle\,. \tag{3,4}$$

By (2,24) from the same Proposition we obtain the inequality

$$\operatorname{var}_0^1 y = \operatorname{var}_0^1 \left(\int_0^1 d_t[K(s, t)]\, x(t) \right) \leqq \|x\|_{V_n} v(K)\,. \tag{3,5}$$

The map

$$Kx = y \tag{3,6}$$

(y is determined by (3,4)) is evidently a linear operator on the space V_n because (3,5) implies $y \in V_n$.

Further it is

$$\|Kx\|_{V_n} = \|y(0)\| + \operatorname{var}_0^1 y \leqq \left\| \int_0^1 d_t[K(0, t)]\, x(t) \right\| + \|x\|_{V_n} v(K) \leqq$$

$$\leqq \sup_{t \in \langle 0,1\rangle} \|x(t)\| \operatorname{var}_0^1 K(0, .) + \|x\|_{V_n} v(K)$$

and therefore

$$\|Kx\|_{V_n} \leqq (\operatorname{var}_0^1 K(0, .) + v(K))\, \|x\|_{V_n} \tag{3,7}$$

i.e. $K : V_n \to V_n$ is a continuous linear operator on V_n.

Theorem 3,1. *The linear operator* $K : V_n \to V_n$ *from* (3,6) *is completely continuous if* (3,2) *and* (3,3) *is satisfied.*

Proof. We denote by $B = \{x \in V_n; \|x\|_{V_n} < 1\}$ the unit ball in V_n. Let a sequence x^l, $l = 1, 2, \dots$ be given such that $x^l \in B$ for $l = 1, 2, \dots$. By Helly's Choice Theorem it is possible to select from the sequence x^l a subsequence x^{l_k} such that

$$\lim_{k \to \infty} x^{l_k}(t) = x^*(t)$$

for any $t \in \langle 0, 1 \rangle$ so that at the same time $\operatorname{var}_0^1 x^* \leqq 1$ (i.e. $x^* \in V_n$). We put $z^k(t) = x^{l_k}(t) - x^*(t)$ for $t \in \langle 0, 1 \rangle$. Evidently $z^k \in V_n$ for all $k = 1, 2, \dots$, $\|z^k\|_{V_n} \leqq \|x^{l_k}\|_{V_n} + \|x^*(0)\| + \operatorname{var}_0^1 x < 3$ and

$$\lim_{k \to \infty} z^k(t) = 0 \quad \text{for all} \quad t \in \langle 0, 1 \rangle\,. \tag{3,8}$$

(2,24) implies

$$\operatorname{var}_0^1 \left(\int_0^1 \mathrm{d}_t[K(s, t)]\, z^k(t) \right) \leqq \int_0^1 \|z^k(t)\| \,\mathrm{d}\psi(t)$$

where $\psi : \langle 0, 1 \rangle \to R$ is a nondecreasing function $\psi(0) = 0$, $\psi(1) = v(K)$ (cf. (2,15b) where $\langle a, b \rangle = \langle c, d \rangle = \langle 0, 1 \rangle$). Clearly $0 \leqq \|z^k(t)\| \leqq 3$ for $t \in \langle 0, 1 \rangle$ and the function $\|z^k(t)\|$ belongs to $V(0, 1)$ for all $k = 1, 2, \dots$, hence the integral $\int_0^1 \|z^k(t)\| \,\mathrm{d}\psi(t)$ exists (as Kurzweil integral or equivalently as Perron-Stieltjes integral). The Dominated Convergence Theorem for the Perron-Stieltjes integral implies

$$\lim_{k \to \infty} \int_0^1 \|z^k(t)\| \,\mathrm{d}\psi(t) = 0\,.$$

Hence we have

$$\lim_{k \to \infty} \operatorname{var}_0^1 \left(\int_0^1 \mathrm{d}_t[K(s, t)]\, x^{l_k}(t) - \int_0^1 \mathrm{d}_t[K(s, t)]\, x^*(t) \right) = 0\,. \tag{3,9}$$

Similarly we can show that

$$\lim_{k \to \infty} \left\| \int_0^1 \mathrm{d}_t[K(0, t)]\, x^{l_k}(t) - \int_0^1 \mathrm{d}_t[K(0, t)]\, x^*(t) \right\| = 0\,. \tag{3,10}$$

From (3,9) and (3,10) we have

$$\lim_{k \to \infty} \left\| \int_0^1 \mathrm{d}_t[K(s, t)]\, x^{l_k}(t) - y^*(s) \right\|_{V_n} = 0$$

where $y^*(s) = \int_0^1 \mathrm{d}_t[K(s, t)]\, x^*(t) \in V_n$ since $x^* \in V_n$. Hence we conclude that the image of the unit ball B is precompact in V_n and consequently the operator K is completely continuous.

We shall derive now some special analytic properties of the operator $\boldsymbol{K}$ if $\mathbf{K}(s, t)$: $: I \to L(R^n \to R^n)$ satisfies some additional assumptions.

Proposition 3,1. *Let* $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ *satisfy* (3,2) *and* (3,3), $\mathbf{x} \in V_n$. *Let further for some* $s_0 \in \langle 0, 1 \rangle$

$$\lim_{s \to s_0+} \|\mathbf{K}(s, t) - \mathbf{K}(s_0, t)\| = 0 \quad \text{or} \quad \lim_{s \to s_0-} \|\mathbf{K}(s, t) - \mathbf{K}(s_0, t)\| = 0$$

for all $t \in \langle 0, 1 \rangle$. *Then*

$$\lim_{s \to s_0+} \mathbf{y}(s) = \mathbf{y}(s_0), \quad \lim_{s \to s_0-} \mathbf{y}(s) = \mathbf{y}(s_0)$$

respectively where $\mathbf{y} \in V_n$ *is given in* (3,4).

Proof. The statement follows in an easy way from the inequality (see (2,23) from Proposition 2,3 and (2,15a))

$$\|\mathbf{y}(s) - \mathbf{y}(s_0)\| = \left\| \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t) - \mathbf{K}(s_0, t)]\, \mathbf{x}(t) \right\| \leqq$$

$$\leqq \|\mathbf{x}\|_{V_n} \operatorname{var}_0^1 (\mathbf{K}(s, .) - \mathbf{K}(s_0, .)) \leqq \|\mathbf{x}\|_{V_n} |\varphi(s) - \varphi(s_0)|$$

and from Lemma 2,1.

Corollary 3,1. *If* $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ *satisfies* (3,2), (3,3) *and* $\mathbf{K}(s, t)$ *is continuous in the variable* s *for any* $t \in \langle 0, 1 \rangle$ *then the vector function* $\mathbf{y}(s) : \langle 0, 1 \rangle \to R^n$ *from* (3,4) *is continuous for any* $\mathbf{x} \in V_n$, *i.e.* $\boldsymbol{K}$ *maps* V_n *into* CV_n (*the space of continuous n-vector functions with bounded variation*).

Lemma 3,1. *Let* $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ *satisfy* (3,2) *and* (3,3), $\mathbf{x} \in V_n$. *Let* $s_0 \in \langle 0, 1 \rangle$. *If the limit*

$$\lim_{s \to s_0+} \mathbf{K}(s, t) = \mathbf{K}(s_0+, t) \quad \text{or} \quad \lim_{s \to s_0-} \mathbf{K}(s, t) = \mathbf{K}(s_0-, t) \tag{3,11}$$

exists for all $t \in \langle 0, 1 \rangle$ *then we have*

$$\mathbf{y}(s_0+) = \lim_{s \to s_0+} \mathbf{y}(s) = \int_0^1 \mathrm{d}_t[\mathbf{K}(s_0+, t)]\, \mathbf{x}(t), \tag{3,12}$$

$$\mathbf{y}(s_0-) = \lim_{s \to s_0-} \mathbf{y}(s) = \int_0^1 \mathrm{d}_t[\mathbf{K}(s_0-, t)]\, \mathbf{x}(t)$$

respectively.

Proof. We have

$$\mathbf{y}(s_0+) - \int_0^1 \mathrm{d}_t[\mathbf{K}(s_0+, t)]\, \mathbf{x}(t) = \lim_{s \to s_0+} \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t) - \mathbf{K}(s_0+, t)]\, \mathbf{x}(t) = 0$$

because for $\delta > 0$

$$\left\| \int_0^1 \mathrm{d}_t[\mathbf{K}(s_0 + \delta, t) - \mathbf{K}(s_0+, t)]\, \mathbf{x}(t) \right\| \leqq \|\mathbf{x}\|_{V_n} \operatorname{var}_0^1 (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0+, .))$$

and by Corollary 2, 3 it is

$$\lim_{\delta \to 0+} \operatorname{var}_0^1 (\mathbf{K}(s_0 + \delta, .) - \mathbf{K}(s_0+, .)) = 0\,.$$

The second statement can be proved similarly.

Remark 3,1. If we suppose that $\operatorname{var}_0^1 \mathbf{K}(., t_*) < +\infty$ for some $t_* \in \langle 0, 1 \rangle$ for $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ in addition to the conditions (3,2), (3,3) then $\operatorname{var}_0^1 \mathbf{K}(., t) < < +\infty$ for all $t \in \langle 0, 1 \rangle$ and the limits (3,11) exist for all $s_0 \in \langle 0, 1 \rangle$ and all $t \in \in \langle 0, 1 \rangle$, and (3,12) holds for all $s_0 \in \langle 0, 1 \rangle$.

Proposition 3,2. *Let $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfy (3,2) and (3,3). If $\mathbf{K}(s, t)$ is for every $t \in \langle 0, 1 \rangle$ regular at $s_0 \in \langle 0, 1 \rangle$, i.e. the limits (3,11) exist for every $t \in \langle 0, 1 \rangle$ and*

$$\mathbf{K}(s_0+, t) + \mathbf{K}(s_0-, t) - 2\mathbf{K}(s_0, t) = 0 \tag{3,13}$$

for all $t \in \langle 0, 1 \rangle$ then

$$\mathbf{y}(s_0+) + \mathbf{y}(s_0-) - 2\mathbf{y}(s_0) = 0\,, \tag{3,14}$$

i.e. $\mathbf{y}(s)$ is a regular function at $s_0 \in \langle 0, 1 \rangle$.

Proof follows immediately from Lemma 3,1.

Corollary 3,2. *If $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfies (3,2), (3,3), the limits (3,11) exist for every $s_0 \in \langle 0, 1 \rangle$ and $t \in \langle 0, 1 \rangle$ and (3,13) holds for every $t \in \langle 0, 1 \rangle$ then the n-vector function $\mathbf{y}(s) : \langle 0, 1 \rangle \to R^n$ from (3,4) is regular for any $\mathbf{x} \in V_n$ (this means that (3,22) holds for all $s_0 \in \langle 0, 1 \rangle$), i.e. the operator $\boldsymbol{K}$ maps V_n into the space RV_n of all regular n-vector functions of bounded variation.*

Proposition 3,3. *Let $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ satisfy (3,2), (3,3) and*

$$\operatorname{var}_0^1 \mathbf{K}(., 0) < +\infty\,. \tag{3,15}$$

If $\alpha, \beta \in \langle 0, 1\rangle$ then for $\mathbf{x} \in V_n$ the expressions

$$(3,16) \qquad \begin{aligned} \boldsymbol{K}_1\mathbf{x} &= \mathbf{K}(s, \alpha)\, \mathbf{x}(\beta) = \mathbf{y}_1(s)\,, \\ \boldsymbol{K}_2\mathbf{x} &= \Delta_t^+ \mathbf{K}(s, \alpha)\, \Delta^+\mathbf{x}(\beta) = \mathbf{y}_2(s)\,, \\ \boldsymbol{K}_3\mathbf{x} &= \Delta_t^- \mathbf{K}(s, \alpha)\, \Delta^-\mathbf{x}(\beta) = \mathbf{y}_3(s) \end{aligned}$$

define completely continuous operators on V_n.

Proof. We have (by (2,14b))

$$\begin{aligned} \|\boldsymbol{K}_1\mathbf{x}\|_{V_n} &= \|\mathbf{K}(0, \alpha)\, \mathbf{x}(\beta)\| + \operatorname{var}_0^1 (\mathbf{K}(., \alpha)\, \mathbf{x}(\beta)) \leqq \\ &\leqq \{\|\mathbf{K}(0, \alpha)\| + \operatorname{var}_0^1 \mathbf{K}(., \alpha)\}\, \|\mathbf{x}(\beta)\| \leqq \\ &\leqq \{\|\mathbf{K}(0, \alpha)\| + \operatorname{var}_0^1 \mathbf{K}(., 0) + v(\mathbf{K})\}\, \|\mathbf{x}\|_{V_n}\,. \end{aligned}$$

Further it is (cf. Corollary 2,3)

$$\begin{aligned} \|\boldsymbol{K}_2\mathbf{x}\|_{V_n} &\leqq \|\Delta^+\mathbf{K}(0, \alpha)\|\, \|\Delta^+\mathbf{x}(\beta)\| + \operatorname{var}_0^1 \Delta_t^+\mathbf{K}(., \alpha)\, \|\Delta^+\mathbf{x}(\beta)\| = \\ &= \{\operatorname{var}_0^1 \mathbf{K}(0, .) + v(\mathbf{K})\}\, \|\Delta^+\mathbf{x}(\beta)\| \leqq \\ &\leqq \{\operatorname{var}_0^1 \mathbf{K}(0, .) + v(\mathbf{K})\}\, \|\mathbf{x}\|_{V_n} \end{aligned}$$

and similarly

$$\|\boldsymbol{K}_3\mathbf{x}\|_{V_n} \leqq \{\operatorname{var}_0^1 \mathbf{K}(0, .) + v(\mathbf{K})\}\, \|\Delta^-\mathbf{x}(\beta)\| \leqq \{\operatorname{var}_0^1 \mathbf{K}(0, .) + v(\mathbf{K})\}\, \|\mathbf{x}\|_{V_n}\,.$$

Let B be the unit ball in the space V_n and let $\mathbf{x}_l \in B$, $l = 1, 2, \ldots$ be given. The sequence $\mathbf{x}_l(\beta)$ $(\Delta^+\mathbf{x}_l(\beta), \Delta^-\mathbf{x}_l(\beta))$ is bounded in R^n. Hence there is a point $\mathbf{z}(\beta)$ $(\mathbf{z}^+(\beta), \mathbf{z}^-(\beta))$ in R^n and a subsequence $\mathbf{x}_{l_j}(\beta)$ $(\Delta^+\mathbf{x}_{l_j}(\beta), \Delta^-\mathbf{x}_{l_j}(\beta))$ such that $\lim_{j\to\infty} \mathbf{x}_{l_j}(\beta) = \mathbf{z}(\beta)$ $(\lim_{j\to\infty} \Delta^+\mathbf{x}_{l_j}(\mathrm{b}) = \mathbf{z}^+(\beta),\ \lim_{j\to\infty} \Delta^-\mathbf{x}_{l_j}(\beta) = \mathbf{z}^-(\beta))$.

Let us set $\mathbf{y}_1^*(s) = \mathbf{K}(s, \alpha)\, \mathbf{z}(\beta) \in V_n$, then we have $\|\boldsymbol{K}_1\mathbf{x}_{l_j} - \mathbf{y}_1^*\|_{v_n} \leqq \{\mathbf{K}(0, 0) + \operatorname{var}_0^1 \mathbf{K}(0, .) + \operatorname{var}_0^1 \mathbf{K}(., 0) + v(\mathbf{K})\}\| \mathbf{x}_{l_j}(\beta) - \mathbf{z}(\beta)\|$, hence $\lim_{j\to\infty} \|\boldsymbol{K}_1\mathbf{x}_{l_j} - \mathbf{y}_1^*\|_{V_n} = 0$, i.e. the operator $\boldsymbol{K}_1$ maps B into precompact set and therefore $\boldsymbol{K}_1$ is completely continuous.

By setting $\mathbf{y}_2^*(s) = \Delta_t^+\mathbf{K}(s, \alpha)\, \mathbf{z}^+(\beta)$ we obtain

$$\|\boldsymbol{K}_2\mathbf{x}_{l_j} - \mathbf{y}_2^*\|_{V_n} \leqq \{\Delta_t^+\mathbf{K}(0, \alpha) + \operatorname{var}_0^1 \Delta_t^+\mathbf{K}(., \alpha)\}\, \Delta^+\mathbf{x}_{l_j}(\beta) - \mathbf{z}^+(\mathrm{b})\|\,,$$

hence $\lim_{j\to\infty} \|\boldsymbol{K}_2\mathbf{x}_{l_j} - \mathbf{y}_2^*\|_{V_n} = 0$ and $\boldsymbol{K}_2$ is a completely continuous operator. Similarly the complete continuity of $\boldsymbol{K}_3$ can be obtained.

Proposition 3,4. *If* $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ *satisfies* (3,2), (3,3) *and* (3,15) *then the series*

$$\sum_{0 \leqq \tau < 1} \Delta_t^+ \mathbf{K}(s, \tau)\, \Delta^+ \mathbf{x}(\tau)\,, \quad \sum_{0 < \tau \leqq 1} \Delta_t^- \mathbf{K}(s, \tau)\, \Delta^- \mathbf{x}(\tau) \tag{3,17}$$

define completely continuous operators on V_n.

Proof. For any $t', t'' \in \langle 0, 1\rangle$ and all $s \in \langle 0, 1\rangle$ we have

$$\begin{aligned} \|\mathbf{K}(s, t') - \mathbf{K}(s, t'')\| &\leqq \|\mathrm{K}(0, t') - \mathbf{K}(0, t'')\| + \\ &+ \|\mathbf{K}(s, t') - \mathbf{K}(0, t') - \mathbf{K}(s, t'') + \mathbf{K}(0, t'')\| \leqq \\ &\leqq |\mathrm{var}_0^{t'} \mathbf{K}(0, .) - \mathrm{var}_0^{t''} \mathbf{K}(0, .)| + v_{\langle 0,1\rangle \times \langle t', t''\rangle}(\mathbf{K}) \leqq \\ &\leqq |\mathrm{var}_0^{t'} \mathbf{K}(0, .) - \mathrm{var}_0^{t''} \mathbf{K}(0, .)| + |\psi(t') - \psi(t'')| \end{aligned}$$

(cf. (2,15b) for ψ). This implies that the set of discontinuities of $\mathbf{K}(s, t)$ in the variable t lies on a denumerable family of lines parallel to the s-axis: $t = t_l$, $l = 1, 2, \ldots$ since $\mathrm{var}_0^1 \mathbf{K}(0, .)$ and $\psi(t)$ are functions $(\langle 0, 1\rangle \to R)$ of bounded variation. In this way it is possible to rewrite the first expression from (3,17) in the form

$$\sum_{l=1}^{\infty} \Delta_t^+ \mathbf{K}(s, t_l)\, \Delta^+ \mathbf{x}(t_l) \tag{3,18}$$

and similarly the second one.

The operator (3,18) is defined as the limit of the sequence of operators $U_N : V_n \to V_n$ where

$$U_N \mathbf{x} = \sum_{l=1}^{N} \Delta_t^+ \mathbf{K}(s, t_l)\, \Delta_t^+ \mathbf{x}(t_l)\,, \tag{3,19}$$

i.e.

$$U\mathbf{x} = \sum_{l=2}^{\infty} \Delta_t^+ \mathbf{K}(s, t_l)\, \Delta^+ \mathbf{x}(t_l) = \lim_{N \to \infty} U_N \mathbf{x}\,.$$

By Proposition 3,3 for any integer N the operator U_N from (3,19) is completely continuous because U_N is a finite sum of completely continuous operators.

Let us denote $[V_n \to V_n]$ the space of all linear operators acting on V_n, $[V_n \to V_n]$ is a normed linear space with the norm

$$\|U\|_{[V_n \to V_n]} = \sup_{\|\mathbf{x}\|_{V_n} = 1} \|U\mathbf{x}\|_{V_n}\,.$$

The completeness of V_n implies that the space $[V_n \to V_n]$ is complete. Further we have

$$\begin{aligned} \|U_M \mathbf{x} - U_N \mathbf{x}\|_{V_n} &= \Big\|\sum_{l=N+1}^{M} \Delta_t^+ \mathbf{K}(s, t_1)\, \Delta^+ \mathbf{x}(t_1)\Big\|_{V_n} = \\ = \Big\|\sum_{l=N+1}^{M} \Delta_t^+ \mathbf{K}(0, t_l)\, \Delta^+ \mathbf{x}(t_l)\Big\| &+ \mathrm{var}_0^1 \sum_{l=N+1}^{M} \Delta_t^+ \mathbf{K}(., t_l)\, \Delta^+ \mathbf{x}(t_l) \leqq \\ &\leqq \mathrm{var}_0^1 \mathbf{x}\Big(\sum_{l=N+1}^{M} \|\Delta_t^+ \mathbf{K}(0, t_l)\| + \sum_{l=N+1}^{M} \mathrm{var}_0^1 \Delta_t^+ \mathbf{K}(., t_l)\Big)\,. \end{aligned}$$

Hence

$$(3,20)\quad \|U_M - U_N\|_{[V_n \to V_n]} \leqq \sum_{l=N+1}^{M} \|\Delta_t^+ \mathbf{K}(0, t_l)\| + \sum_{l=N+1}^{M} \operatorname{var}_0^1 \Delta_t^+ \mathbf{K}(., t_l) .$$

The assumption (3,3) yields the convergence of the series $\sum_{l=1}^{\infty} \|\Delta_t^+ \mathbf{K}(0, t_l)\|$. Further we have (cf. Corollary 2,3)

$$\operatorname{var}_0^1 \Delta_t^+ \mathbf{K}(., t_l) \leqq \psi(t_l+) - \psi(t_l) = \Delta^+ \psi(t_l)$$

where $\psi : \langle 0, 1\rangle \to R$ is a non-decreasing function, $\psi(0) = 0$, $\psi(1) = v(\mathbf{K})$. Since the series $\sum_{l=1}^{\infty} \Delta^+ \psi(t_l)$ evidently converges we obtain that the series $\sum_{l=1}^{\infty} \operatorname{var}_0^1 \Delta_t^+ \mathbf{K}(., t_l)$ converges as well. This implies by (3,20) that U_N, $N = 1, 2, \ldots$ forms a fundamental sequence in the (complete) Banach space $[V_n \to V_n]$ and that $\lim_{N \to \infty} U_N = U$ exists, hence the operator

$$U\mathbf{x} = \sum_{l=1}^{\infty} \Delta_t^+ \mathbf{K}(s, t_l)\, \Delta^+ \mathbf{x}(t_l) = \sum_{0 \leqq \tau < 1} \Delta_t^+ \mathbf{K}(s, \tau)\, \Delta^+ \mathbf{x}(\tau)$$

is completely continuous. The proof of complete continuity of the second operator in (3,17) can be carried out in the same manner.

Theorem 3,2. *If* $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ *satisfies* (3,2), (3,3) *and* (3,15) *then the expression*

$$(3,20)\qquad \int_0^1 \mathbf{K}(s, t)\, \mathrm{d}\mathbf{x}(t) = \hat{\mathbf{K}}\mathbf{x}, \quad \mathbf{x} \in V_n$$

defines a completely continuous operator on V_n.

Proof. Using the integration by parts formula (2,9) for row vectors of $\mathbf{K}(s, t)$ for any $s \in \langle 0, 1\rangle$ we can write

$$\int_0^1 \mathbf{K}(s, t)\, \mathrm{d}\mathbf{x}(t) = -\int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) + \mathbf{K}(s, 1)\, \mathbf{x}(1) - \mathbf{K}(s, 0)\, \mathbf{x}(0) -$$
$$- \sum_{0 \leqq \tau < 1} \Delta_t^+ \mathbf{K}(s, \tau)\, \Delta^+ \mathbf{x}(\tau) + \sum_{0 < \tau \leqq 1} \Delta_t^- \mathbf{K}(s, \tau)\, \Delta^- \mathbf{x}(\tau) .$$

By Theorem 3,1, Propositions 3,3 and 3,4 we see from this expression that the operator $\hat{\mathbf{K}}$ from (3,20) is a linear combination of completely continuous operators and we obtain in this way our Theorem.

Remark 3,2. We note that for the operator $\hat{\mathbf{K}} : V_n \to V_n$ given in (3,20) it is possible to derive further analytic properties (continuity, regularity of the result of the operation) if some additional conditions for $\mathbf{K}(s, t) : I \to L(R^n \to R^n)$ are assumed. This can be made if we employ the properties of our integral.

4. AUXILIARY STATEMENT FROM FUNCTIONAL ANALYSIS

In this Section we give a simple general statement based on the well known Riesz theory from functional analysis which will be useful in our considerations about Fredholm-Stieltjes integral equations in Section 5.

Let X, Y be normed spaces with the norms $\|\,.\,\|_X$, $\|\,.\,\|_Y$ respectively. By X', Y' we denote the dual spaces to X, Y respectively. Let a bilinear form $\langle x, y\rangle$ on $X \times Y$ be given which separates points of X, i.e.

(i) for $x \in X$, $x \neq 0$ there exists $y \in Y$ such that $\langle x, y\rangle \neq 0$

and which separates points of Y, i.e.

(ii) for $y \in Y$, $y \neq 0$ there exists $x \in X$ such that $\langle x, y\rangle \neq 0$.

Further we assume that

$$|\langle x, y\rangle| \leqq C\|x\|_X \|y\|_Y \tag{4,1}$$

for any $x \in X$, $y \in Y$ where $C \geqq 0$ is a constant.

For any fixed $y \in Y$ we denote by $[y]$ the linear functional on X which corresponds to $y \in Y$ in terms of the bilinear form $\langle .\,, .\rangle$; $[y]$ is defined by the relation

$$[y](x) = \langle x, y\rangle\,. \tag{4,2}$$

The inequality (4,1) quarantees the continuity of $[y]$, i.e. we have $[y] \in X'$.

We denote by $[Y]$ the linear set in X' of all continuous linear functionals of the form (4,2).

Since the bilinear form $\langle x, y\rangle$ separates points of Y we have $[y] = 0 \in X'$ ($[y] \in \in [Y]$) if and only if $y = 0$ ($y \in Y$), i.e. $[y] \neq [\tilde{y}]$ if and only if $y \neq \tilde{y}$, $y, \tilde{y} \in Y$.

In this way a one-to-one correspondence between elements of Y and $[Y]$ is given, in other words we have a one-to-one correspondence between the space Y and its immersion $[Y]$ into X' which is given by the bilinear form $\langle x, y\rangle$.

For a set $M \subset Y$ we denote by $[M]$ the set of all elements in X' which are determined by an element in M, i.e.

$$[M] = \{f \in X';\ f = [y],\ y \in M\}\,.$$

In the same way for any fixed $x \in X$ a continuous linear functional $[x] \in Y'$ is given and we have a one-to-one correspondence between X and $[X] \subset Y'$. This follows from the fact that the bilinear form $\langle x, y\rangle$ separates points of X.

For a given operator $\boldsymbol{K} : X \to X$ we denote by $\boldsymbol{K}^* : X' \to X'$ the adjoint operator which is defined by the obvious relation $f(\boldsymbol{K}x) = \boldsymbol{K}^*f(x)$ for $f \in X'$, $x \in X$ (similarly for operators $\boldsymbol{L} : Y \to Y$). If a linear operation $\boldsymbol{T} : X \to X$ is given then $\boldsymbol{T}^{-1}(0)$ means the null – space of this operator, i.e.

$$\boldsymbol{T}^{-1}(0) = \{x \in X;\ \boldsymbol{T}x = 0\}\,.$$

Proposition 4,1. *Let X, Y be normed spaces, $\boldsymbol{K}: X \to X$, $\boldsymbol{L}: Y \to Y$ completely continuous operators in X, Y respectively. Further let $\langle x, y\rangle$ be a bilinear form on $X \times Y$ which separates points of X and Y such that for any $x \in X$, $y \in Y$ the inequality (4,1) holds, and let*

$$\langle \boldsymbol{K}x, y\rangle = \langle x, \boldsymbol{L}y\rangle \tag{4,3}$$

for any $x \in X$, $y \in Y$.

We denote $\boldsymbol{T} = \boldsymbol{I}_X - \boldsymbol{K}$, $\boldsymbol{S} = \boldsymbol{I}_Y - \boldsymbol{L}$, $\boldsymbol{T}^* = \boldsymbol{I}_{X'} - \boldsymbol{K}^*$, $\boldsymbol{S}^* = \boldsymbol{I}_{Y'} - \boldsymbol{L}^*$ ($\boldsymbol{I}_X$, $\boldsymbol{I}_Y$, $\boldsymbol{I}_{X'}$, $\boldsymbol{I}_{Y'}$ are the identity operators in X, Y, X', Y' respectively). Then we have

$$\dim \boldsymbol{T}^{-1}(0) = \dim \boldsymbol{T}^{*-1}(0) = \dim \boldsymbol{S}^{-1}(0) = \dim \boldsymbol{S}^{*-1}(0) = r \tag{4,4}$$

where r is a nonnegative integer (by dim *the dimension of a linear set is denoted) and*

$$\boldsymbol{T}^{*-1}(0) \subset [Y], \quad \boldsymbol{S}^{*-1}(0) \subset [X]. \tag{4,5}$$

Proof. We have (see VIII.2 in [6]): The equation

$$\boldsymbol{T}x = x - \boldsymbol{K}x = \tilde{x}, \quad \tilde{x} \in X \tag{4,6}$$

has a solution $x \in X$ if and only if for any solution $f \in X'$ of the equation

$$\boldsymbol{T}^* f = f - \boldsymbol{K}^* f = 0 \tag{4,7}$$

the relation

$$f(\tilde{x}) = 0 \tag{4,8}$$

holds and the dimension of the linear set

$$\boldsymbol{T}^{-1}(0) = \{x \in X;\ \boldsymbol{T}x = x - \boldsymbol{K}x = 0\}$$

is finite and equal to the dimension of the linear set

$$\boldsymbol{T}^{*-1}(0) = \{f \in X';\ \boldsymbol{T}^* f = f - \boldsymbol{K}^* f = 0\},$$

i.e. we have

$$\dim \boldsymbol{T}^{-1}(0) = \dim \boldsymbol{T}^{*-1}(0) = r. \tag{4,9}$$

Observe that (4,3) can be written in the form $[y](\boldsymbol{K}x) = [\boldsymbol{L}y](x)$ hence we have

$$\boldsymbol{K}^*[y] = [\boldsymbol{L}y] \tag{4,10}$$

for any functional $[y] \in [Y] \subset X'$. Further by (4,10) it is

$$\begin{aligned} \boldsymbol{T}^{*-1}(0) \cap [Y] &= \{[y] \in [Y];\ \boldsymbol{T}^*[y] = [y] - \boldsymbol{K}^*[y] = [y] - [\boldsymbol{L}y] = \\ &= [y - \boldsymbol{L}y] = 0\} = [\{y \in Y;\ \boldsymbol{S}y = y - \boldsymbol{L}y = 0\}] = [\boldsymbol{S}^{-1}(0)] \end{aligned}$$

and evidently

$$(4,11) \qquad \dim [S^{-1}(0)] = \dim (T^{*-1}(0) \cap [Y]) = p \leqq r\,.$$

With respect to the one-to-one correspondence between Y and $[Y]$ we have evidently

$$(4,12) \qquad \dim [S^{-1}(0)] = \dim S^{-1}(0) = p\,.$$

Since $L : Y \to Y$ is a completely continuous operator, the Riesz theory yields

$$(4,13) \qquad \dim S^{-1}(0) = \dim S^{*-1}(0) = \mathrm{p}\,.$$

The equation (4,3) can be also rewritten in the form

$$[x]\,(Ly) = [Kx]\,(y)$$

for all $x \in X$, $y \in Y$, i.e. we have

$$(4,14) \qquad L^*[x] = [Kx]$$

for any functional $[x] \in [X] \subset Y'$.

Analoguously as above we obtain by (4,14)

$$S^{*-1}(0) \cap [X] =$$
$$= \{[x] \in [X];\ [x] - L^*[x] = [x] - [Kx] = [x - Kx] = 0\} = [T^{-1}(0)]\,.$$

Hence (by (4,13)) we have

$$(4,15) \qquad \dim [T^{-1}(0)] = \dim (S^{*-1}(0) \cap [X]) = q \leqq p\,.$$

Further we have evidently by (4,9)

$$q = \dim [T^{-1}(0)] = \dim T^{-1}(0) = r\,.$$

From this equality together with (4,11) and (4,15) we obtain $r = p$; thus (4,9) and (4,11) imply

$$\dim T^{*-1}(0) = \dim (T^{*-1}(0) \cap [Y]) = r$$

and hence we have

$$T^{*-1}(0) \subset [Y]\,.$$

The second relation from (4,5) can be derived similarly.

Proposition 4,1 enables us to derive the following

Theorem 4,1. *Let the assumptions of Proposition* 4,1 *be satisfied. Then either the equation*

$$(4,16) \qquad Tx = x - Kx = \tilde{x}\,, \quad \tilde{x} \in X$$

admits a unique solution $x \in X$ for any $\tilde{x} \in X$, in particular $x = 0$ for $\tilde{x} = 0$; or the homogeneous equation

$$x - \boldsymbol{K}x = 0 \tag{4,17}$$

admits r linearly independent solutions $x_1, \ldots, x_r$ in X.

In the first case the equation

$$\boldsymbol{S}y = y - \boldsymbol{L}y = \tilde{y}\,, \quad \tilde{y} \in Y \tag{4,18}$$

has also a unique solution $y \in Y$ for any $\tilde{y} \in Y$. In the second case the equation

$$y - \boldsymbol{L}y = 0 \tag{4,19}$$

admits r linearly independent solutions $y_1, \ldots, y_r$ in Y. Moreover in the second case the equation (4,16) has a solution in X if and only if

$$\langle \tilde{x}, y \rangle = 0 \tag{4,20}$$

for any solution $y \in Y$ of (4,19) and symmetrically (4,18) has a solution in Y if and only if

$$\langle x, \tilde{y} \rangle = 0 \tag{4,21}$$

for any solution $x \in X$ of (4,17).

Proof. The first part of this theorem corresponds to the case when $r = 0$ in (4,4) from Proposition 4,1. The result of this part is a consequence of the well known Riesz theory of completely continuous operators (cf. 11.3 in [4] or Chapter VIII. in [6]).

For the second part of the theorem we have $r > 0$ in Proposition 4,1. Using the duality theory for completely continuous operators in normed spaces (see [6], VIII. 2) we know that (4,16) has a solution if and only if $f(\tilde{x}) = 0$ for any functional $f \in \boldsymbol{T}^{*-1}(0) = \{f \in X'; f - \boldsymbol{K}^*f = 0\}$ ($\boldsymbol{K}^*$ is the adjoint operator to $\boldsymbol{K}$). From (4,5) we have $\boldsymbol{T}^{*-1}(0) \subset [Y]$ and (4,3) implies $\boldsymbol{K}^*[y] = [\boldsymbol{L}y]$ for any $[y] \in [Y]$. Hence (4,16) has a solution if and only if $[y](\tilde{x}) = \langle \tilde{x}, y \rangle = 0$ for any $[y]$ from the set

$$\{[y] \in [Y];\ [y] - \boldsymbol{K}^*[y] = [y - \boldsymbol{L}y] = 0\} =$$
$$= [\{y \in Y;\ \boldsymbol{S}y = y - \boldsymbol{L}y = 0\}] = [\boldsymbol{S}^{-1}(0)]\,.$$

Further evidently $[y] \in [\boldsymbol{S}^{-1}(0)]$ if and only if y is a solution of (4,19) and we obtain in this way the result of the second part of the Theorem for Eq. (4,16). The result for Eq. (4,18) can be derived similarly.

Remark 4,1. The following statement is an easy consequence of Theorem 4,1: If (4,20) for all solutions of (4,19) is satisfied then the general solution $x \in X$ of (4,16) is written as

$$x = \hat{x} + \sum_{l=1}^{r} c_l x_l$$

where $\hat{x}$ is a particular solution of (4,18), $x_1, \ldots, x_r$ are the linearly independent solutions of (4,17) (the base of $\boldsymbol{T}^{-1}(0)$) and $c_1, \ldots, c_r$ are arbitrary constants. A similar statement for the general solution of (4,18) also holds.

5. ALTERNATIVE FOR FREDHOLM-STIELTJES INTEGRAL EQUATIONS IN V_n

We denote by S_n the set of all break functions $\boldsymbol{w}$ in the Banach space space $V_n = V_n(0, 1)$ such that $\Delta \boldsymbol{w}(t) = \boldsymbol{w}(t+) - \boldsymbol{w}(t-) = 0$ for all $t \in (0, 1)$, $\Delta^+ \boldsymbol{w}(0) = \Delta^- \boldsymbol{w}(1) = 0$. Obviously S_n is a linear set in V_n. The set S_n is closed in V_n. In fact, if $\boldsymbol{w} \in V_n$ is an adherent point of S_n then there exists a sequence $\boldsymbol{w}_l \in S_n$, $l = 1, 2, \ldots$ such that $\lim_{l \to \infty} \|\boldsymbol{w}_l - \boldsymbol{w}\|_{V_n} = 0$. For any $t \in (0, 1)$ and $l = 1, 2, \ldots$ we have

$$\|\Delta \boldsymbol{w}(t)\| = \|\Delta \boldsymbol{w}(t) - \Delta \boldsymbol{w}_l(t)\| \leqq \|\boldsymbol{w}_l - \boldsymbol{w}\|_{V_n}$$

thus $\Delta \boldsymbol{w}(t) = 0$. Similarly $\Delta^+ \boldsymbol{w}(0) = \Delta^- \boldsymbol{w}(1) = 0$.

The convergence in V_n implies that $\boldsymbol{w}_l$ converges uniformly on $\langle 0, 1 \rangle$ to $\boldsymbol{w}$. Let $A \subset \langle 0, 1 \rangle$ be the union of all discontinuity points of $\boldsymbol{w}_l$, $l = 1, 2, \ldots$ and $\boldsymbol{w}$; A is a countable set. Any $\boldsymbol{w}_l$ is a constant function in $\langle 0, 1 \rangle - A$. The uniform convergence implies that $\boldsymbol{w}$ is a constant function in $\langle 0, 1 \rangle - A$ and hence we have $\boldsymbol{w} \in S_n$.

Let us consider the quotient space V_n/S_n. An element of V_n/S_n is a class of functions in V_n such that their difference belongs to S_n. Elements of V_n/S_n let be denoted by capitals. The canonical mapping of V_n onto V_n/S_n let be denoted by $\varkappa$; for $\varphi \in V_n$ we have $\varkappa(\varphi) = \varphi + S_n = \Phi \in V_n/S_n$. Any element $\varphi \in V_n$ for which $\varkappa(\varphi) = \Phi$ will be called a representant of the class $\Phi \in V_n/S_n$.

The space V_n/S_n forms a Banach space with the norm

$$\|\Phi\|_{V_n/S_n} = \inf_{\varphi \in \Phi} \|\varphi\|_{V_n} = \inf_{\varkappa(\varphi) = \Phi} \|\varphi\|_{V_n} = \inf_{\varphi \in \Phi} \operatorname{var}_0^1 \varphi \,. \tag{5,1}$$

We have evidently

$$\|\Phi\|_{V_n/S_n} \leqq \operatorname{var}_0^1 \varphi \tag{5,2}$$

for all $\varphi \in V_n$, $\varkappa(\varphi) = \Phi$.

Theorem 5,1. *If $\boldsymbol{K}(s, t) : I = \langle 0, 1 \rangle \times \langle 0, 1 \rangle \to L(R^n \to R^n)$ satisfies $v(\boldsymbol{K}) < +\infty$, $\operatorname{var}_0^1 \boldsymbol{K}(0, .) < +\infty$, $\operatorname{var}_0^1 \boldsymbol{K}(., 0) < +\infty$ then the expression*

$$\boldsymbol{L}\Phi = \varkappa \left(\int_0^1 \boldsymbol{K}'(s, t) \, \mathrm{d}\varphi(s) \right), \quad \varphi \in V_n \,, \quad \varkappa(\varphi) = \Phi \tag{5,3}$$

defines a completely continuous linear operator on the Banach space V_n/S_n.

Proof. Let $B(V_n/S_n) = \{\Phi \in V_n/S_n;\ \|\Phi\|_{V_n/S_n} \leqq 1\}$ be the unit ball in V_n/S_n. Let

$$A = \{\Phi \in V_n/S_n;\ \Phi = \varkappa(\varphi),\ \varphi \in V_n,\ \varphi(0) = 0,\ \mathrm{var}_0^1\, \varphi \leqq 1\}\ ;$$

according to (5,2) it is $B(V_n/S_n) \subset A$. Let $\Phi_l \in B(V_n/S_n)$, $l = 1, 2, \ldots$ then there exist $\varphi_l \in V_n$ with $\mathrm{var}_0^1\, \varphi_l \leqq 1$, $\varphi_l(0) = 0$, $l = 1, 2, \ldots$ such that $\varkappa(\varphi_l) = \Phi_l$. The matrix $\boldsymbol{K}'(s, t) : I \to L(R^n \to R^n)$ satisfies evidently all conditions of Theorem 3,2 hence the operator $\int_0^1 \boldsymbol{K}'(s, t)\, \mathrm{d}\varphi(s)$ is completely continuous in V_n. This implies that there exist $\boldsymbol{z} \in V_n$ and a subsequence φ_{l_j}, $j = 1, 2, \ldots$ such that

$$\lim_{j \to \infty} \left\| \int_0^1 \boldsymbol{K}'(s, t)\, \mathrm{d}\varphi_{l_j}(s) - \boldsymbol{z}(t) \right\|_{V_n} = 0$$

i.e.

$$\lim_{j \to \infty} \mathrm{var}_0^1 \left(\int_0^1 \boldsymbol{K}'(s, t)\, \mathrm{d}\varphi_{l_j}(s) - \boldsymbol{z}(t) \right) = 0\,.$$

From (5,2) we have

$$\lim_{j \to \infty} \|\boldsymbol{L}\Phi_{l_j} - \varkappa(\boldsymbol{z})\|_{V_n/S_n} \leqq \lim_{j \to \infty} \mathrm{var}_0^1 \left(\int_0^1 \boldsymbol{K}'(s, t)\, \mathrm{d}\varphi_{l_j}(s) - \boldsymbol{z}(t) \right) = 0$$

and in this way we obtain the complete continuity of $\boldsymbol{L} : V_n/S_n \to V_n/S_n$.

Let further $\boldsymbol{x} \in V_n$, $\Phi \in V_n/S_n$. We denote

$$\langle \boldsymbol{x}, \Phi \rangle = \langle \boldsymbol{x}, \varphi \rangle_{(0,1)} = \int_0^1 \boldsymbol{x}'(t)\, \mathrm{d}\varphi(t) \tag{5,4}$$

where $\varphi \in V_n$, $\varkappa(\varphi) = \Phi$. The expression $\langle \boldsymbol{x}, \Phi \rangle$ is independent of the choice of the representant φ of the class Φ. Indeed, if we have $\varphi^\circ \in V_n$, $\varkappa(\varphi^\circ) = \Phi$ then (cf. Remark 2,2)

$$\langle \boldsymbol{x}, \varphi - \varphi^\circ \rangle_{(0,1)} = \int_0^1 \boldsymbol{x}'(t)\, \mathrm{d}(\varphi(t) - \varphi^\circ(t)) = 0$$

because $\varphi - \varphi^\circ \in S_n$. The expression $\langle ., . \rangle$ from (5,4) is a bilinear form on $V_n \times V_n/S_n$ and the following lemma holds:

Lemma 5,1. *The bilinear form* $\langle ., . \rangle$ *from* (5,4) *separates points of* V_n *and* V_n/S_n (cf. (i) *and* (ii) *in Section* 4.).

Proof. (i) Let $\boldsymbol{x} \in V_n$, $\boldsymbol{x} \neq 0$. Then there exists $\alpha \in \langle 0, 1 \rangle$ such that $\boldsymbol{x}(\alpha) \neq 0$, i.e. there is an index $i = 1, \ldots, n$ such that $x_i(\alpha) \neq 0$. We define $\varphi(t) \in V_n$ as follows: $\varphi_k(t) = 0$ for $t \in \langle 0, 1 \rangle$, $k \neq i$, $\varphi_i(t) = 0$ for $0 \leqq t < \alpha$, $\varphi_i(t) = 1$ for $\alpha \leqq t \leqq 1$ provided $\alpha > 0$; if $\alpha = 0$ then we set $\varphi_i(0) = 1$, $\varphi_i(t) = 0$, $0 < t \leqq 1$. Let us put $\Phi = \varkappa(\varphi)$. Then Proposition 2,1 yields $\langle \boldsymbol{x}, \Phi \rangle = \int_0^1 x_i(t)\, \mathrm{d}\varphi_i(t) = x_i(\alpha) \neq 0$ in the case $\alpha > 0$ and similarly $\langle \boldsymbol{x}, \Phi \rangle = -x_i(0) \neq 0$ if $\alpha = 0$. Hence $\langle ., . \rangle$ separates points of V_n.

(ii) Let $\Phi \neq 0$ (i.e. $\Phi \neq S_n$). For any $\varphi \in V_n$, $\varkappa(\varphi) = \Phi$ it holds either

1) there exists an $\alpha \in (0, 1)$ such that $\varphi_i(\alpha+) \neq \varphi_i(\alpha-)$ for some $i = 1, 2, \ldots, n$ or

2) for each $\alpha \in (0, 1)$ it is $\varphi(\alpha+) = \varphi(\alpha-)$ and there exist two points $\beta, \gamma \in \langle 0, 1\rangle$, $\beta < \gamma$ such that $\varphi_i(\beta) \neq \varphi_i(\gamma)$ for some $i = 1, \ldots, n$ where β, γ are points of continuity of $\varphi_i(t)$, i.e. $\varphi_i(\beta) = \varphi_i(\beta-)$, $\varphi_i(\gamma) = \varphi_i(\gamma-)$.

In the case 1) we set $x_i(t) = 0$ for $t \in \langle 0, 1\rangle$, $t \neq \alpha$, $x_i(\alpha) = 1$, $x_j(t) = 0$ for $t \in \langle 0, 1\rangle$ if $j \neq i$. Then Corollary 2,1 yields

$$\langle \mathbf{x}, \Phi\rangle = \int_0^1 x_i(t)\, \mathrm{d}\varphi_i(t) = \varphi_i(\alpha+) - \varphi_i(\alpha-) \neq 0\,.$$

In the case 2) it suffices to set $x_i(t) = 1$ for $t \in \langle \beta, \gamma\rangle$, $x_i(t) = 0$ for $t \in \langle 0, 1\rangle - \langle \beta, \gamma\rangle$, $x_j(t) = 0$ for $t \in \langle 0, 1\rangle$, $j \neq i$. Then we obtain from Proposition 2,1

$$\langle \mathbf{x}, \Phi\rangle = \int_0^1 x_i(t)\, \mathrm{d}\varphi_i(t) = \varphi_i(\gamma) - \varphi_i(\beta) \neq 0\,.$$

Hence $\langle .,. \rangle$ separates points of V_n/S_n.

Remark 5,1. Since the bilinear form $\langle .,. \rangle$ separates points of V_n/S_n we can subjoin the following addition to Corollary 2,2: If $g \in V(a, b)$ and $\int_a^b f(t)\, \mathrm{d}g(t) = 0$ for all $f \in V(a, b)$ then necessarily $g \in S(a, b)$, i.e. $\Delta g(t) = 0$ for all $t \in (a, b)$, $\Delta^+ g(a) = \Delta^- g(b) = 0$. This means that if $g \in V(a, b)$ then $\int_a^b f(t)\, \mathrm{d}g(t) = 0$ for every $f \in V(a, b)$ if and only if $g \in S(a, b)$.

Since $\langle \mathbf{x}, \Phi\rangle$ from (5,4) is independent of the choice of $\varphi \in V_n$, $\varkappa(\varphi) = \Phi$ and

$$\left|\int_0^1 \mathbf{x}'(t)\, \mathrm{d}\varphi(t)\right| \leqq \sup_{t \in \langle 0,1\rangle} \|\mathbf{x}'(t)\| \operatorname{var}_0^1 \varphi \leqq n\|\mathbf{x}\|_{V_n} \operatorname{var}_0^1 \varphi$$

holds we have

$$|\langle \mathbf{x}, \Phi\rangle| \leqq n\|\mathbf{x}\|_{V_n} \inf_{\varkappa(\varphi) = \Phi} \operatorname{var}_0^1 \varphi = n\|\mathbf{x}\|_{V_n} \|\Phi\|_{V_n/S_n} \tag{5,5}$$

Theorem 5,2. *Let* $\mathbf{K}(s, t) : I = \langle 0, 1\rangle \times \langle 0, 1\rangle \to L(R^n \to R^n)$, $v(\mathbf{K}) < +\infty$, $\operatorname{var}_0^1 \mathbf{K}(0, .) < +\infty$, $\operatorname{var}_0^1 \mathbf{K}(., 0) < +\infty$.

Then either the Fredholm-Stieltjes integral equation

$$\mathbf{x}(s) - \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) = \mathbf{x}^\circ(s)\,, \quad \mathbf{x}^\circ \in V_n \tag{5,6}$$

admits a unique solution for any $\mathbf{x}^\circ \in V_n$ *or the homogeneous equation*

$$\mathbf{x}(s) - \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) = 0 \tag{5,7}$$

admits r linearly independent solutions $\mathbf{x}_1, \ldots, \mathbf{x}_r \in V_n$.

In the first case the equation

$$(5,8) \qquad \varphi(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s) = \varphi^\circ(t)\,, \quad \varphi^\circ \in V_n$$

has a solution for any $\varphi^\circ \in V_n$ *(this solution is not necessarily unique). In the second case the equation* (5,6) *has a solution in* V_n *if and only if*

$$(5,9) \qquad \langle \mathbf{x}^\circ, \varphi \rangle_{(0,1)} = \int_0^1 \mathbf{x}'(t)\, \mathrm{d}\varphi(t) = 0$$

for any solution $\varphi \in V_n$ *of the equation*

$$(5,10) \qquad \varphi(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s) = 0$$

and symmetrically (5,8) *has a solution if and only if*

$$(5,11) \qquad \langle \mathbf{x}, \varphi^\circ \rangle_{(0,1)} = \int_0^1 \mathbf{x}'(t)\, \mathrm{d}\varphi^\circ(t) = 0$$

for any solution $\mathbf{x} \in V_n$ *of the equation* (5,7).

Proof. By Theorems 3,1 and 5,1 the operators

$$\boldsymbol{K}\mathbf{x} = \int_0^1 \mathrm{d}_t[\mathbf{K}(s, t)]\, \mathbf{x}(t) : V_n \to V_n\,, \quad \varkappa(\varphi) = \boldsymbol{\Phi}\,,$$

$$\boldsymbol{L\Phi} = \varkappa\left(\int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s)\right) : V_n/S_n \to V_n/S_n$$

are completely continuous. $\langle ., . \rangle$ from (5,4) represents a bilinear form on $V_n \times V_n/S_n$ which separates points of V_n and V_n/S_n (cf. Lemma 5,1) and (5,5) holds.

Further by (2,28) we have

$$\langle \boldsymbol{K}\mathbf{x}, \boldsymbol{\Phi} \rangle = \langle \boldsymbol{K}\mathbf{x}, \varphi \rangle_{(0,1)} = \left\langle \int_0^1 \mathrm{d}_t[\mathbf{K}(., t)]\, \mathrm{d}\mathbf{x}(t), \varphi \right\rangle_{(0,1)} =$$

$$= \left\langle \mathbf{x}, \int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\varphi(s) \right\rangle_{(0,1)} = \left\langle \mathbf{x}, \varkappa\left(\int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\varphi(s)\right) \right\rangle = \langle \mathbf{x}, \boldsymbol{L\Phi} \rangle$$

for any $\mathbf{x} \in V_n$, $\boldsymbol{\Phi} \in V_n/S_n$. All assumptions of Theorem 4,1 are satisfied and using this Theorem we obtain the first part of Theorem 5,2 viz. (the alternative for Eq. (5,6) resp. (5,7)). Further by Theorem 4,1 the equation

$$(5,12) \qquad \boldsymbol{\Phi} - \boldsymbol{L\Phi} = \boldsymbol{\Phi}^\circ\,, \quad \boldsymbol{\Phi}^\circ \in V_n/S_n$$

has a unique solution for any $\boldsymbol{\Phi}^\circ \in V_n/S_n$. For an arbitrary $\varphi^\circ \in V_n$ we denote $\boldsymbol{\Phi}^\circ = \varkappa(\varphi^\circ) \in V_n/S_n$. Let $\varphi \in V_n$ be a representant of the (unique) solution of (5,12) with this $\boldsymbol{\Phi}^\circ$. Then we have

$$\varkappa\left(\varphi - \int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\varphi(s)\right) = \varkappa(\varphi^\circ),$$

i.e.

$$\varkappa\left(\varphi - \int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\varphi(s) - \varphi^\circ\right) = 0 \in V_n/S_n .$$

Hence

$$\varphi(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s) - \varphi^\circ(t) = \mathbf{w}(t) \in S_n$$

for all $t \in \langle 0, 1\rangle$. Since $\int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\varphi(s) = \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}(\varphi(s) - \mathbf{w}(s))$ we have

$$\varphi(t) - \mathbf{w}(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}(\varphi(s) - \mathbf{w}(s)) = \varphi^\circ(t)$$

for all $t \in \langle 0, 1\rangle$, i.e. the function $\varphi - \mathbf{w} \in V_n$ is a solution of Eq. (5,8). (The unicity of $\mathbf{w} \in S_n$ is not quaranteed.)

For the second case we know by Theorem 4,1 that (5,6 has a solution if and only if for any solution $\boldsymbol{\Phi} \in V_n/S_n$ of the equation

$$\boldsymbol{\Phi} - \boldsymbol{L}\boldsymbol{\Phi} = 0 \tag{5,13}$$

we have $\langle \mathbf{x}^\circ, \boldsymbol{\Phi}\rangle = 0$.

Obviously the following assertion holds: for any solution $\boldsymbol{\Phi} \in V_n/S_n$ of Eq. (5,13) there is a $\varphi \in V_n$, $\varkappa(\varphi) = \boldsymbol{\Phi}$ such that φ is a solution of (5,10). In fact, for any representant $\psi \in V_n$ of $\boldsymbol{\Phi}$ $(\varkappa(\psi) = \boldsymbol{\Phi})$ we have

$$\varkappa\left(\psi - \int_0^1 \mathbf{K}'(s, .)\, \mathrm{d}\psi(s)\right) = 0 \in V_n/S_n , \quad \text{i.e.}$$

$$\psi(t) - \int_0^1 \mathbf{K}'(s, t)\, \mathrm{d}\psi(s) = \mathbf{w}(t) \in S_n .$$

If we set $\varphi = \psi - \mathbf{w}$, then $\varkappa(\varphi) = \varkappa(\psi) = \boldsymbol{\Phi}$ and φ is a solution of (5,10). It is easy to prove also the converse statement: If $\varphi \in V_n$ is a solution of (5,10), then $\boldsymbol{\Phi} = \varkappa(\varphi)$ is a solution of (5,13).

Since $\langle \mathbf{x}, \boldsymbol{\Phi}\rangle = \langle \mathbf{x}, \varphi\rangle_{(0,1)}$ is independent of the choice of the representant $\varphi \in V_n$, $\varkappa(\varphi) = \boldsymbol{\Phi}$, we conclude that in the second case (5,6) has a solution if and only if $\langle \mathbf{x}^\circ, \varphi\rangle_{(0,1)} = \int_0^1 \mathbf{x}^{\circ\prime}(t)\, \mathrm{d}\varphi(t) = 0$ for all solutions of Eq. (5,10). The symmetrical statement about Eq. (5,8) can be proved similarly.

Remark 5,2. Theorem 5,2 is a Fredholm type theorem for Fredholm-Stieltjes integral equations (5,6). Let us mention that Eq. (5,8) as well as the equation $\boldsymbol{\Phi} - \boldsymbol{L\Phi} = \boldsymbol{\Phi}^\circ$ are not the adjoint equations to (5,6) in the usual sense. We have not a satisfactory description of the dual space V_n' to V_n which would make it possible to derive the analytic form of the adjoint operator $\boldsymbol{K}^*$. Nevertheless conditions for solvability of Eq. (5,6) are obtained in a form which is closely related to the well known Fredholm alternative for Fredholm integral equations of the second kind in L_2 – spaces.

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STRUČNÉ CHARAKTERISTIKY ČLÁNKŮ UVEŘEJNĚNÝCH V TOMTO ČÍSLE V CIZÍM JAZYKU

MICHAL BUČKO, Košice: *Eine Benutzung der Zahlenzerlegung zur Bestimmung der Anzahl unisomorpher Zyklen in ξ-Turnieren.* (Použitie rozkladu čísel na určenie počtu neizomorfných cyklov v ξ-turnajoch.)

Na základe výsledkov odvodených o rozklade čísla $2n + 1$ na k čísel ($k = 3, 4$), z ktorých žiadne neprevyšuje číslo n, je odvodená horná a dolná hranica pre počet neizomorfných cyklov dĺžok 3 a 4 špeciálneho turnaja.

MICHAL DONT, Praha: *Non-tangential limits of the double layer potentials.* (Netangenciální limity potenciálu dvojvrstvy.)

V článku se odvozují nutné a postačující podmínky existence netangenciálních limit potenciálu dvojvrstvy se spojitou hustotou nebo s obecnou distribucí na hranici množiny konečného průměru. Východiskem úvah jsou výsledky J. Krále. Dále se studují vztahy mezi potenciálem dvojvrstvy na množině s konečným průměrem a logaritmickým potenciálem, definovaným J. Králem.

PETR PŘIKRYL, Praha: *Optimal universal approximations of Fourier coefficients in spaces of continuous periodic functions.* (Optimální universální aproximace Fourierových koeficientů v prostorech spojitých periodických funkcí.)

Článek pojednává o aproximaci a výpočtu r Fourierových koeficientů ($r > 1$) spojité 2π-periodické funkce. Chyba aproximace se měří jako maximální chyba počítaných koeficientů. Studují se optimální aproximace v periodických prostorech zavedených Babuškou. Jsou uvedeny příklady ukazující nestabilitu těchto aproximací vzhledem k prostoru. Hlavní úsilí je soustředěno na zkoumání universálních aproximací, jejichž chyba se v nějaké široké třídě prostorů „příliš" (v přesně definovaném smyslu) neliší od chyby optimální aproximace. Závěrem jsou ve třídě universálních aproximací hledány aproximace optimální.

ŠTEFAN SCHWABIK, Praha: *On an integral operator in the space of functions with bounded variation.* (O jistém integrálním operátoru v prostoru funkcí s konečnou variací.)

V práci se vyšetřuje Fredholm-Stieltjesův integrální operátor tvaru $\boldsymbol{K}\boldsymbol{x} = \int_0^1 \mathrm{d}_t(\boldsymbol{K}(s, t))\, \boldsymbol{x}(t)$, kde $\boldsymbol{x}(t)$ je n-vektorová funkce s konečnou variací v $\langle 0, 1\rangle$ a jádro $\boldsymbol{K}(s, t)$ je matice typu $n \times n$, jejíž dvoudimenzionální variace v $\langle 0, 1\rangle \times \langle 0, 1\rangle$ (v jistém smyslu) je konečná. Je dokázána věta Fredholmova typu pro příslušnou Fredholm-Stieltjesovu integrální rovnici $\boldsymbol{x} - \boldsymbol{K}\boldsymbol{x} = \boldsymbol{y}$.

Časopis pro pěstování matematiky, roč. 97 (1972), Praha

RŮZNÉ

O POSTUPNOSTIACH PRIRODZENÝCH ČÍSEL S OHRANIČENÝM POČTOM PRVOČÍSELNÝCH DELITEĽOV

PAVEL KOSTYRKO, Bratislava

(Došlo dňa 8. marca 1971)

V monografii [1], str. 332, príkl. 9 sa dokazuje nasledujúce tvrdenie: *Najväčší počet po sebe idúcich prirodzených čísel, ktoré majú nanajvýš dva rôzne prvočíselné delitele je 29. Jedinou 29-člennou postupnosťou s touto vlastnosťou je postupnosť* 1, 2, ..., 29. V tomto článku dokážeme nasledujúce obecnejšie tvrdenie.

Veta. *Nech q je prirodzené číslo a nech $m(q)$ je najväčšie číslo s touto vlastnosťou: existuje $m(q)$ po sebe idúcich prirodzených čísel, ktoré majú nanajvýš q rôznych prvočíselných deliteľov. Potom platí:*

1) $m(q) = \prod_{i=1}^{q+1} p_i - 1$, *kde* $\{p_i\}_{i=1}^{\infty}$ *je rastúca postupnosť všetkých prvočísel,*

2) *pre každé prirodzené q existuje práve jedna $m(q)$-členná postupnosť s uvedenou vlastnosťou a je ňou postupnosť* $1, 2, \ldots, m(q)$.

Dôkaz. Ľahko sa možno presvedčiť, že každý člen postupnosti $1, 2, \ldots, \prod_{i=1}^{q+1} p_i - 1$ má nanajvýš q rôznych prvočíselných deliteľov. V opačnom prípade by totiž existovala rastúca postupnosť $j_1, \ldots, j_r$ $(r > q)$ prirodzených čísel a prirodzené čísla $\alpha_1, \ldots, \alpha_r$ tak, že

$$p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_r}^{\alpha_r} \leqq \prod_{i=1}^{q+1} p_i - 1$$

čo vedie, vzhľadom na nerovnosti

$$\prod_{i=1}^{q+1} p_i - 1 < \prod_{i=1}^{q+1} p_i \leqq p_{j_1}^{\alpha_1} p_{j_2}^{\alpha_2} \cdots p_{j_r}^{\alpha_r}$$

ku sporu. Teda $m(q) \geqq \prod_{i=1}^{q+1} p_i - 1$. V poslednej nerovnosti však nemôže nastať ostrá nerovnosť, pretože v každej postupnosti po sebe idúcich prirodzených čísel majúcej aspoň $\prod_{i=1}^{q+1} p_i$ členov sa nachádza člen deliteľný $\prod_{i=1}^{q+1} p_i$, ktorý má aspoň $q + 1$ rôznych prvočíselných deliteľov. Teda $m(q) = \prod_{i=1}^{q+1} p_i - 1$.

Ak

$$s + 1,\ s + 2, \ldots, s + m(q) \tag{1}$$

je postupnosť, ktorej každý člen má nanajvýš q rôznych prvočíselných deliteľov, potom nutne $s = t \prod_{i=1}^{q+1} p_i$ (t – celé nezáporné číslo). V opačnom prípade by totiž postupnosť (1) obsahovala člen, ktorý by mal prvočíselné delitele $p_1, \ldots, p_{q+1}$. Ukážeme, že za predpokladu $t > 0$ má niektorý z členov postupnosti (1) viac než q rôznych prvočíselných deliteľov. Uvažujme o týchto $p_{q+1} - 1$ členoch postupnosti (1)

$$s + j \prod_{i=1}^{q} p_i = \prod_{i=1}^{q} p_i (t p_{q+1} + j) \quad j = 1, 2, \ldots, p_{q+1} - 1 \tag{2}$$

V ďalšom budeme používať nasledujúce známe tvrdenie: Ak $n > k$, tak v postupnosti prirodzených čísel $n, n + 1, \ldots, n + k - 1$ existuje aspoň jedno číslo, ktoré má prvočíselného deliteľa p, $p > k$ (pozri [1], str. 401). Ak položíme $n = t p_{q+1} + 1$ a $k = p_{q+1} - 1$ do hore uvedeného tvrdenia, tak podmienka $n > k$ bude splnená pretože $t > 0$. Z citovaného tvrdenia plynie, že existuje j_0, $1 \leqq j_0 \leqq p_{q+1} - 1$ tak, že $t p_{q+1} + j_0$ obsahuje prvočíselného deliteľa $p > p_q$, teda v dôsledku (2) $s + j_0 \prod_{i=1}^{q} p_i$ má viac než q rôznych prvočíselných deliteľov.

Literatúra

[1] *Sierpiński W.:* Teoria liczb II, Warszawa 1959.

Author's address: Bratislava, Mlynská dolina (Prírodovedecká fakulta UK).

Časopis pro pěstování matematiky, roč. 97 (1972), Praha

ÚLOHY A PROBLÉMY

Úloha č. 1. Nechť U je resolutivní množina s hranicí $U^* \neq \emptyset$ v harmonickém prostoru X (viz [1]) a označme pro každý kompakt $K \subset X$ symbolem $C(K)$ prostor všech spojitých (konečných) reálných funkcí na K. Každé funkci $f \in C(U^*)$ je tedy přiřazena harmonická funkce H_f^U na U, která je zobecněným řešením (v Perronově smyslu) Dirichletovy úlohy příslušné k množině U a okrajové podmínce f. Nechť U_r značí množinu všech $x \in U^*$, pro něž $\lim\limits_{\substack{y \to x \\ y \in U}} H_f^U(y) = f(x)$ pro každou funkci $f \in C(U^*)$. Množina U se nazývá semiregulární, jestliže pro každou funkci $f \in C(U^*)$ lze příslušnou funkci H_f^U rozšířit na $F \in C(U \cup U^*)$. Je-li U semiregulární, pak U_r je kompaktní. Obrácení tohoto tvrzení neplatí v Bauerových harmonických prostorech. Rozhodněte, zda obrácené tvrzení platí v Brelotových prostorech (nebo alespoň v harmonickém prostoru indukovaném klasickými harmonickými funkcemi na n-rozměrném euklidovském prostoru $X = R^n$), tj. rozhodněte o správnosti následujícího

Tvrzení. *Nechť X je Brelotův prostor a buď $U \subset X$ relativně kompaktní otevřená (a tedy resolutivní) množina, $U^* \neq \emptyset$. Pak U je semiregulární, právě když U_r je kompaktní.*

Literatura

[1] *C. Constantinescu:* Harmonic spaces and their connections with the semi-elliptic differential equations and with the Markov processes, Elliptische Differentialgleichungen (Symposium), Akademie-Verlag, Berlin 1969.

Josef Král, Praha

Časopis pro pěstování matematiky, roč. 97 (1972), Praha

RECENSE

Putnam, C. R.: COMMUTATION PROPERTIES OF HILBERT SPACE OPERATORS AND RELATED TOPICS. (Komutační vlastnosti operátorů v Hilbertově prostoru a příbuzná témata) Springer-Verlag, New York 1967, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 36.

Kniha je prvním systematickým výkladem teorie komutátorů $AB - BA$ operátorů A, B v Hilbertově prostoru. Obsahuje bohatý výběr aplikací na kvantovou mechaniku, perturbační teorii, Laurentovy a Toeplitzovy operátory a singulární integrální transformace. Aplikace na kvantovou fyziku jsou formulovány matematicky, takže k porozumění není třeba žádná znalost fyziky.

Kniha je rozdělena do šesti kapitol. První tři pojednávají obecně o vlastnostech komutátorů (aditivních i multiplikativních). Zvláštní pozornost je věnována spektrální teorii v případě poloomezeného samoadjungovaného A a nezáporného samoadjungovaného $C = AB - BA$. Celá jedna kapitola se zabývá tzv. seminormálními operátory T, tj. operátory pro něž $TT^* - T^*T$ je semidefinitní.

Kapitola 4 obsahuje věty o existenci a jednoznačnosti řešení komutačních relací, vyskytujících se v kvantové mechanice, zvláště vztahů $AB - BA = -iI$ a $AB^* - B^*A = I$. Následující kapitola vychází z teorie rozptylu kvantové mechaniky. Je v ní uvedena řada vlastností vlnových operátorů a operátorů rozptylu. Vedle toho se studuje i unitární ekvivalence samoadjungovaných operátorů s jejich perturbovanými formami. Nakonec se důkladně probírají Laurentovy, Toeplitzovy a singulární integrální operátory.

Autor nedokazuje všechny věty; některé důkazy jsou jen načrtnuty a jiné úplně vynechány. Je však vždy uveden odkaz, kde lze důkaz najít. Bibliografie je vypracována s příkladnou péčí. Hluboká znalost látky umožnila autorovi uvést u každého výsledku odkaz, takže čtenář má dojem, že čte historii oboru.

Jan Kučera, Pullman

Horst Sachs: EINFÜHRUNG IN DIE THEORIE DER ENDLICHEN GRAPHEN, Teil I., Mathematisch-Naturwissenschaftliche Bibliothek 43, B. G. Teubner, Leipzig 1970, stran 182, 108 obrázků, cena neuvedena.

Ještě před dvaceti lety bylo možné, aby jeden matematik obsáhl všechny práce, které tehdy byly napsány o grafech a existovala vlastně jen jediná monografie z tohoto oboru. Napsal ji maďarský matematik D. König a vyšla německy v Lipsku r. 1936. Dnes jde počet prací už do tisíců a také knižních publikací přibývá každým rokem. V těchto řádcích se podíváme na knihu, kterou nedávno sepsal H. Sachs s několika spolupracovníky. Je to vlastně první díl chystané dvousvazkové monografie a autorsky na něm spolupracovali H.-J. Finck, H. Hutschenreuther, E. Kaiser, R. Lang, M. Schäuble, H.-J. Voss a H. Walther. Zdálo by se, že se do knihy tak malého rozsahu nevejde příliš mnoho látky, ale Sachsovi se podařil dobrý výběr ze starších i zcela nových problémů této teorie. Aby čtenář nezabředl do triviálních větiček, soustředí se výklad jen na ty pojmy, jež jsou nutné k pochopení těchto problémů. Mám tu na mysli např. větu Turánovu, Mengerovu a Ford-Fulkersonovu, z nichž každé je věnována jedna kapitola. Někde se ovšem kniha vzdává důkazů, jak je tomu např. u Rédeiovy věty o počtu úplných drah v turnajích.

Krásný kombinatorický důkaz této věty, který roku 1943 podal T. Szele, se do knihy nevešel a je tu jen letmo zmíněn. Sachsův oblíbený problém o tom, jakou šířku v pase (Taillenweite) má pravidelný graf, se dostal ke slovu též v jedné kapitole. Šířkou v pase se zde rozumí délka nejkratší kružnice a Sachs i jeho žáci o ní publikovali v posledních letech několik prací. Vliv spolupracovníků je patrný někde více, někde méně. Tak např. známá determinantová metoda pro určení počtu koster daného grafu je tu zpracována podle H. Hutschenreuthera. V předmluvě se praví, že se druhý chystaný díl monografie bude věnovat studiu rovinných grafů. S celkovým zhodnocením díla si tedy počkáme, až si budeme moci pročíst i tento druhý svazek.

Jiří Sedláček, Praha

David Hilbert: GESAMMELTE ABHANDLUNGEN, I—III. (Zweite Auflage, Springer-Verlag Berlin—Heidelberg—New York 1970) Band I: Zahlentheorie, XVI + 539; Band II: Algebra, Invariantentheorie, Geometrie, VIII + 453; Band III: Analysis, Grundlagen der Matematik, Physik, Verschiedenes, Lebensgeschichte, VII + 435.

Prvý svazek obsahuje 11 Hilbertových prací z teorie čísel (převážně algebraické), včetně jeho rozsáhlého pojednání o teorii algebraických číselných těles z roku 1897. V doslovu H. Hasse popisuje další rozvoj teorie algebraických čísel až do roku 1932 (tj. do roku prvého vydání Hilbertových sebraných spisů). Druhý svazek zahrnuje 26 prací z algebry a teorie invariantů a 3 práce z geometrie, z nichž jedna je přetištěním poznámky o plochách s konstantní Gaussovou křivostí z Hilbertovy knihy o základech geometrie. V tomto svazku jsou umístěny stati B. L. van der Waerdena a A. Schmidta hodnotící Hilbertovy výsledky z algebry a z geometrie. Konečně třetí díl obsahuje 8 prací z analýzy, 4 práce o základech matematiky, 4 práce z fyziky, 23 daných problémů, formulovaných Hilbertem na Mezinárodním matematickém kongresu v Paříži roku 1900, články věnované památce K. Weierstrasse, H. Minkowského, G. Darbouxe a A. Hurwitze. A posléze populární článek o vztahu logiky a poznání přírody. V tomto svazku dále E. Hellinger vykládá teorii integrálních rovnic a nekonečných soustav rovnic na základě výsledků D. Hilberta a jeho pokračovatelů. Konečně je zde zařazena stať P. Bernayse o Hilbertových výzkumech v oblasti základů matematiky a Hilbertova biografie napsaná O. Blumenthalem.

Při rozsáhlosti a rozmanitosti Hilbertova díla je nemožné hodnotit jednotlivě byť i jen významnější práce. Je to snad zbytečné i proto, že význam Hilbertovy činnosti v jednotlivých matematických oborech je všeobecně znám. O tom, že odkaz D. Hilberta v moderní matematice je stále živý, svědčí jistě i ten fakt, že jeho sebraná pojednání vycházejí již po druhé během necelých čtyřiceti let.

Otto Vejvoda, Praha

A. F. Monna: ANALYSE NON-ARCHIMÉDIENNE. Ergebnisse der Mathematik und ihrer Grenzgebiete, Bd. 56; Sgringer Verlag, Berlin—Heidelberg—New York 1970. Stran VI + + 119, cena 38 DM.

Absolutní hodnota na komutativním tělese K je, jak známo, zobrazení $|.| : K \to R$ (R = těleso reálných čísel), které má pro všechna $a, b \in K$ následující vlastnosti: (i) $|a| \geqq 0$, (ii) $|a| = 0$ právě když $a = 0$, (iii) $|ab| = |a| \cdot |b|$, (iv) $|a + b| \leqq |a| + |b|$. Absolutní hodnota se nazývá archimedovskou, jestliže existuje přirozené n tak, že pro $n \cdot 1 = 1 + \ldots + 1 \in K$ máme $|n| > 1$; absolutní hodnota $|.|$ je nearchimedovská právě když místo požadavku (iv) splňuje silnější požadavek (iv') $|a + b| \leqq \max(|a|, |b|)$. Absolutní hodnoty $|.|_1$ a $|.|_2$ se nazývají ekvivalentními, jestliže z $|a|_1 < 1$ plyne $|a|_2 < 1$ pro každé $a \in K$; $|.|_1$ je ekvivalentní s $|.|_2$ právě když existuje $s > 0$ tak, že $|a|_2 = (|a|_1)^s$ pro každé $a \in K$. Nyní platí důležitá Ostrowskiho věta: ***Těleso K s archimedovskou absolutní hodnotou je isomorfní s nějakým podtělesem tělesa komplexních čísel C a jeho absolutní hodnota je ekvivalentní s tou, která je indukována přirozenou absolutní hodnotou na C.***

Z toho plyne, že jakákoliv analýza, rozvinutá nad tělesem s absolutní hodnotou a různá od obvyklé reálné nebo komplexní analýzy, musí být analýza nearchimedovská. Pro větší objasnění uveďme příklad nearchimedovské absolutní hodnoty. Nechť Q je těleso racionálních čísel, p nějaké prvočíslo. Každé $a \in Q$ můžeme psát jednoznačně ve tvaru $a = p^n \alpha/\beta$, kde α a β jsou celá čísla nedělitelná prvočíslem p. Definujeme-li $|a| = p^{-n}$ a $|0| = 0$, dostaneme tzv. p-adickou nearchimedovskou absolutní hodnotu. Platí dokonce následující věta: *Každá archimedovská absolutní hodnota na Q je ekvivalentní s obyčejnou absolutní hodnotou a každá nearchimedovská je ekvivalentní s některou p-adickou absolutní hodnotou.*

V celé knize se předpokládá, že K je komutativní těleso s absolutní hodnotou, která je nearchimedovská a netriviální; dále se požaduje, aby K bylo úplné. Existuje velký rozdíl mezi analýzou nad K a obvyklou analýzou nad reálnými čísly. Tak např. K není uspořádáno. Intervaly se definují formálně stejným způsobem jako $\{x \in K; |x - a| \leqq \varepsilon\}$ resp. $\{x \in K; |x - a| < \varepsilon\}$, jsou však současně otevřené i uzavřené. K rovněž nemusí být lokálně kompaktní.

Druhá kapitola se zabývá klasickou nearchimedovskou analýzou. Konvergentní řady je možno definovati obvyklým způsobem právě tak jako např. spojitost funkcí. Značné potíže však už vznikají při definici analytických funkcí. Derivace funkce $f: K \to K$ je možno definovati jako $f'(x) = \lim_{h \to 0} [f(x + h) - f(x)]/h$, ale potom existuje mnoho nekonstantních funkcí $K \to K$, které mají všude nulovou derivaci.

Třetí kapitola pojednává o vektorových prostorech nad K. Zde je možno vytvořiti teorii lokálně konvexních prostorů, která připomíná teorii nad R. Užívá se této definice: podmnožina A vektorového prostoru E nad K se nazývá konvexí, jestliže $\lambda x + \mu y + \nu z \in A$ pro všechna $x, y, z \in A$; $\lambda, \mu, \nu \in \mathcal{O} = \{a \in K; |a| \leqq 1\}$; $\lambda + \mu + \nu = 1$.

Čtvrtá kapitola obsahuje věty o struktuře nearchimedovských normovaných prostorů, v další kapitole jsou probrány základní vlastnosti lokálně konvexních prostorů nad K včetně teorie duality. Šestá kapitola je úvodem do teorie integrace pro funkce $X \to K$, kde X je topologický lokálně kompaktní prostor. Konečně v poslední kapitole jsou uvedeny některé speciální výsledky a otevřené problémy.

Nearchimedovská analýza se začala rozvíjeti v posledních třiceti letech a nyní existuje asi 150 prací z tohoto oboru. Autor spojil dosažené výsledky, a to velmi zdařile, do recesované knihy. Její text je spíše vyprávěním: sice jsou uváděny definice a věty, ale místo mnoha důkazů jsou odkazy na literaturu a autor se raději věnuje komentářům a srovnávání probírané látky s obvyklou látkou běžné analýzy. Kniha tím velmi získala a je značně přehledná. V předmluvě autor poznamenává, že M. R. Remmert připravuje s M. U. Güntzerem knihu o nearchimedovské analýze v „klasickém" smyslu, proto obsah druhé kapitoly je velmi stručný. Knihu je možno jen doporučiti; přimlouvám se za to, aby odborníci v analýze si ji alespoň prolistovali.

Alois Švec, Praha

N. Bourbaki: VARIÉTÉS DIFFÉRENTIELLES ET ANALYTIQUES. (Fascicule de résultats) Paragraphes 8 à 15. Eléments de mathématique, fasc. XXXVI. Hermann, Paris 1971. Stran 99, cena neudána.

Prvořadým nedostatkem tohoto svazku je, že neobsahuje citace; podle mého mínění by seznam literatury byl málem cennější než sama sbírka definic a výsledků.

V paragrafu osmém (tj. prvním tohoto svazku) je probrán diferenciální počet prvního řádu. Diferencovatelné variety se uvažují nad tělesem K reálných nebo komplexních čísel nebo nad tělesem s nearchimedovskou absolutní hodnotou; předpokládá se, že K má nulovou charakteristiku nebo variety mají lokálně konečnou dimensi. Definuje se tečný fibrovaný prostor variety, vektorové pole, vnější formy a diferenciál. Jsou uvedeny základní vlastnosti komplexních a skorokomplexních variet. Všechny tyto záležitosti jsou dokonale známé v případě $K = R$ a variet konečné dimense; zde jsou však vysloveny v plné obecnosti, důkazy příslušných vět již nejsou běžné (až snad na výklad v Langově knize, pokud ovšem tuto považujeme za zcela běžně známou), takže citace mi opravdu chybí. Tak jest tomu i v dalším textu. Paragraf devátý má název Diferenciální rovnice a rozlistování (nevím, jak překládati „feuilletage"). Zde jsou v podstatě probrány věty o existenci integrálních křivek vektorových polí a rovnice v totálních diferenciálech.

V následujících dvou paragrafech je vybudována teorie míry, definované diferenciální formou, a je probrána Stokesova věta. Zde se předpokládá $K = R$ a varieta konečné dimense. S hranicemi a obecnou formulací Stokesovy věty jsou ovšem již potíže, čtenář je na tomto místě raději odkázán na Cartanův seminář resp. Whitneyovu knihu o integraci.

Paragraf dvanáctý je věnován jetovému aparátu. Zde mi chybí definice prodloužení variety a kanonických forem na ní; zkrátka Ehresmann je poněkud ignorován. Další paragraf se jmenuje bodové distribuce; je věnován komplikacím, které v nestandartním případě nekonečně-dimensionálních variet vznikají při zobecnění Diracových měr. Paragraf čtrnáctý je úvodem do teorie diferenciálních operátorů. Příslušné variety se předpokládají konečně-dimensionální; uvažují se pouze lineární operátory. Většina textu je věnována definici a základním vlastnostem symbolu a Greenova operátoru. Nevím, do které partie matematiky zařazuje Bourbaki existenční věty, teorii úplně integrabilních operátorů, involutivnost atd., velmi je však postrádám právě zde. Poslední velmi krátký paragraf se zabývá varietami diferencovatelných zobrazení.

Předchozí díl (§§ 1—7) vyšel v r. 1967.

Alois Švec, Praha

N. Bourbaki: GROUPES ET ALGÈBRES DE LIE. Chap. I: Algèbres de Lie. Éléments de mathématique, fasc. XXVI. Hermann, Paris 1971. Stran 146, cena neudána.

Jest celkem obtížné napsati cokoliv k textu knihy. Podání jest přesné, obsah dobře známý a rozhodně nevzrušující, odkazy na literaturu tradičně chybějí. A tak jen názvy kapitol: definice Lieových algeber, obaly Lieových algeber, representace, nilpotentní algebry, řešitelné algebry, polojednoduché algebry, Ado-ova věta. Daleko zajímavějších je 36 stran petitem tištěných příkladů. V těchto příkladech je probráno mnoho teorií, např. teorie kohomologií Lieových algeber G s hodnotami v G-modulu M, klasické typy jednoduchých algeber, atd.

Alois Švec, Praha

DÁLE VYŠLO

František Zítek: VYTVOŘUJÍCÍ FUNKCE, Mladá fronta, Praha 1972, stran 148, cena 11 Kčs.

Toto je už 29. svazek edice Škola mladých matematiků, kterou vydává Ústřední výbor matematické olympiády v nakladatelství Mladá fronta. Knížka je určena především řešitelům matematické olympiády.

Redakce

Časopis pro pěstování matematiky, roč. 97 (1972), Praha

ZPRÁVY

PROF. RNDR. CYRIL PALAJ 60-ROČNÝ

VÁCLAV MEDEK, Bratislava

Dňa 24. augusta t. r. sa dožíva 60 rokov Prof. RNDr. CYRIL PALAJ, vedúci I. Katedry matematiky Prírodovedeckej fakulty Univerzity P. J. Šafárika v Košiciach a ako externista vedie aj Katedru matematiky a deskriptívnej geometrie Drevárskej fakulty Vysokej školy lesníckej a drevárskej vo Zvolene.

Narodil sa v Novej Bani v maloroľníckej rodine. Po stredoškolských štúdiach v Leviciach a Kláštore pod Znievom v časoch hospodárskej krízy dostal miesto výpomocného učiteľa na Obecnej ľudovej škole v Plavých Vozokanoch. Tak sa dostal na učiteľskú dráhu a preto si postupne zvyšoval kvalifikáciu v tomto smere. Maturoval na Učiteľskom ústave v Banskej Bystrici a napokon externe vyštudoval aj Prírodovedeckú fakultu UK v Bratislave, ktorú absolvoval v r. 1942. Hodnosť doktora prírodných vied mu udelila Karlova univerzita v Prahe v r. 1952. Profesorom pre matematiku bol menovaný 1. 8. 1965. Na vysokých školách pôsobí od r. 1951, najmä na VŠLD vo Zvolene a UPJŠ v Košiciach.

Počas svojej 40 ročnej učiteľskej dráhy na všetkých stupňoch škôl získal vynikajúce pedagogické schopnosti, ktoré uplatňuje teraz na prednáškach pre študentov univerzity v Košiciach i techniky vo Zvolene. Študenti ho majú radi pre vysokú kvalitu jeho prednášok, ale najmä pre jeho ľudský prístup k nim. Svoj vzťah k študentom preukázal aj tým, že neváhal venovať mnoho času vypracovaniu rôznych skrípt a dočasných vysokoškolských učebníc.

Hlavným oborom vedeckej činnosti Prof. Palaja je klasická algebraická geometria a lineárna algebra. O vedeckých prácach Prof. Palaja možno povedať, že prinášajú obsahove i metodicky nové výsledky pripúšťajúce ďalší rozvoj. Najmä treba oceniť jeho prínos v teórii priestorových matíc a ich geometrických aplikácií. V tomto smere išiel vlastnými cestami a ukázal, že teória priestorových matíc pripúšťa cenné aplikácie najmä pri vyšetrovaní algebraických útvarov vyšších stupňov, na geometrické príbuznosti a iné matematické disciplíny.

Prof. Palaj vedie dlhé roky seminár z algebraickej geometrie a príbuzných disciplín. Prebúdza tak medzi svojimi mladšími spolupracovníkmi záujem o algebraickú geometriu a dáva im témy pre vedeckú prácu. Pod jeho vedením dosiahli niekoľkí ašpiranti hodnosť kandidáta vied. Predniesol celý rad prednášok doma i v zahraničí o výsledkoch svojej vedeckej práce. Osobitne si treba ceniť jeho, možno tak povedať, osvetovú činnosť medzi učiteľmi ZDŠ a SVŠ na Slovensku. Prednášal takmer na všetkých školeniach poriadaných pre tento účel a svojimi širokými známosťami docielil bohatú účasť kvalitných prednášateľov z celej republiky, takže zabezpečil vždy vysokú úroveň týchto školení.

Mnoho úsilia a času venoval budovaniu a vedeniu jemu zverených katedier. Nikdy sa neodťahoval ani od iných funkcií na školách a bol poverovaný vážnymi úlohami. Všetky úlohy vždy zodpovedne splnil a tak je medzi spolupracovníkmi veľmi vážený.

Nevyhýbal sa ani verejnej činnosti. Širokú činnosť vyvíjal ako funkcionár Krajského výboru Socialistickej akadémie. No, najviac si vážime jeho činnosť v JČSMF. Od založenia pobočky Jednoty vo Zvolene je jej predsedom a táto pobočka je jednou z najaktívnejších. Je členom predsedníctiev JČSMF i JSMF, členom ÚV JČSMF a členom Hlavného výboru JSMF. Podielal sa prakticky na všetkých väčších podujatiach Jednoty v posledných rokoch a preto na jubilejnom zjazde v r. 1962 získal čestný titul „Zaslúžilý člen JČSMF".

Napokon by som chcel zdôrazniť vysoké ľudské kvality Prof. Palaja. Teší sa vysokej úcte nielen medzi matematikmi, ale i medzi svojimi terajšími i bývalými žiakmi, ktorých sú už tisíce. Jeho obetavosť pri plnení množstva úloh vyplývajúcich zo všetkých jeho funkcií mu už čiastočne naštrbila zdravie, no pracuje oduševnene ďalej. K jeho význačnému životnému jubileu mu srdečne blahoželáme a prajeme veľa síl do ďalšej tvorivej práce a v nie poslednom rade i veľa osobnej pohody a šťastia, ktoré sa vždy usiloval dopriať iným.

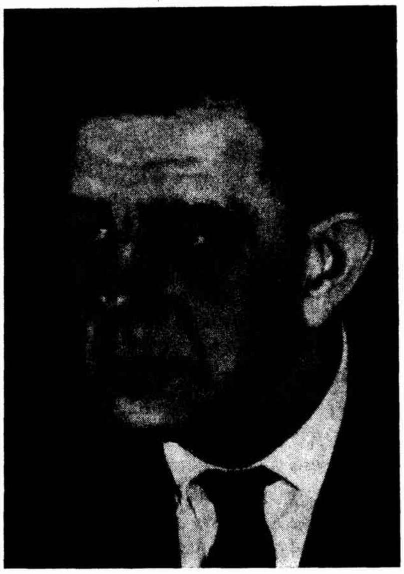
ZEMŘEL PROF. RNDR. KAREL HRUŠA

MILAN KOMAN, Praha

Za katedrou Karlovy university v Praze stálo již mnoho vynikajících českých matematiků — vědců i vysokoškolských pedagogů. Jen málo z nich však zasvětilo téměř veškerou svou celoživotní činnost teorii vyučování matematice jako vědecké disciplíně. Mezi nimi byl snad jediný, který se zabýval touto disciplínou v celé šíři, prvními ročníky základní školy počínaje a posledními semestry vysoké školy konče. Ojedinělé byly zejména jeho bohaté znalosti problematiky teorie vyučování matematice v elementárních ročnících. Ano, byl to profesor dr. KAREL HRUŠA. Dnes — bohužel — již jen byl. Zemřel 16. listopadu 1971 v Praze.

Profesor Hruša se narodil 7. července 1905 v Mnichově Hradišti. Po maturitě studoval na přírodovědecké fakultě Karlovy university v Praze. V roce 1929 obhájil dizertační práci: *O racionálních kvartikách rovinných* a získal doktorát přírodních věd. Po absolvování vysokoškolských studií působil téměř 20 let jako středoškolský profesor. Po 2. světové válce, v roce 1946 nastupuje dráhu vysokoškolského učitele na nově založené pedagogické fakultě UK.

Celoživotní dílo profesora Hruši ovlivnili zejména dva vynikající čeští matematikové. V době universitních studií i během jeho působení na středních školách to byl akademik B. Bydžovský. Jeho druhým, neméně významným učitelem byl akademik E. Čech, pod jehož vedením začínal v roce 1946 na pedagogické fakultě svou dráhu vysokoškolského učitele.



Pod Čechovým vedením vyrůstá brzy z výborného středoškolského pedagoga Hruši také výborný vysokoškolský pedagog. Po Čechově odchodu z pedagogické fakulty v roce 1951 se stává dr. Hruša sám jedním z hlavních budovatelů vysokoškolské soustavy vzdělání učitelů — matematiků základních a středních škol. Podílí se na vypracování nových učebních plánů a osnov, píše vysokoškolské učebnice, připravuje originální přednášky pro budoucí učitele z algebry, aritmetiky, matematické analýzy, didaktiky matematiky ap. Přes nesmírné zatížení se však neomezuje jen na práci vysokoškolského učitele. Všemi svými silami se snaží povznést všude, kde jen může, i úroveň středoškolské matematiky. Píše řadu učebnic pro střední školy a neúnavně přednáší snad po celé republice pro učitele škol všech stupňů.

V roce 1953 je jmenován docentem na Vysoké škole pedagogické. Roku 1964 je obnovena pedagogická fakulta UK. Docent Hruša se stává vedoucím její katedry matematiky. Brzy nato je jmenován universitním profesorem. Přibývá i dalších funkcí. Jako na celém kulturním světě, tak také u nás přichází na pořad modernizace středoškolského vyučování matematice. Prof. Hruša jako jeden z čelných pracovníků v oboru teorie vyučování matematice se stává vedoucím Kabinetu pro modernizaci vyučování matematice ČSAV v Praze. Je jmenován členem komisí pro udělování vědeckých hodností a doktorátů přírodních věd z teorie vyučování matematice. Pracuje v redakcích různých vědeckých a metodických časopisů atd.

Za svou celoživotní činnost byl prof. Hruša vyznamenán zlatým odznakem pedagogické fakulty UK k 25. výročí založení této fakulty a pamětní medailí Karlovy university. Medaile mu však byla — bohužel — udělena až po jeho smrti — in memoriam.

Osobnost prof. Hruši jako vysokoškolského matematika — učitele je snad nejlépe patrná z jeho knih a učebnic. Během své dlouholeté učitelské praxe zdůrazňoval svým čtenářům a žákům, že pro učitele matematiky není nejdůležitější šíře vědomostí, ale důkladná znalost a pochopení základních pojmů a především metod a myšlenkových postupů, jichž se v matematice používá. V oboru teorie vyučování matematice se vždy snažil postavit školské teorie na pevný vědecký základ. Jednalo se zejména o rozvoj pojmu čísla. Jeho kniha [2] byla vlastně první českou teoretickou učebnicí aritmetiky. Příznačné pro Hrušovy práce v tomto směru je, že se snažil vždy vypracovat teorie tak, aby se co nejvíce přiblížily školskému modelu. Ať už jde o rozšiřování číselných oborů (viz [2] a [7]) nebo o dělitelnost (viz [2], [4]).

Dílo prof. Hruši zůstane základem pro pokračovatele zejména v rozvíjení teorií vyučování matematice na školách všech stupňů.

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[9] *Dlouhý Zb. - Hruša K. - Kust J. - Rohlíček J. - Taišl J. - Zieris J.*: Úvod do matematické analýzy, Praha, SPN 1965, 9—77, 383—417 s., 2. vydání 1970.

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TŘETÍ PRAŽSKÉ TOPOLOGICKÉ SYMPOSIUM 1971

Po dvou úspěšných pražských topologických symposiích v letech 1961 a 1966 bylo uspořádáno třetí symposium o obecné topologii a jejích vztazích k moderní analýze a algebře. Konalo se v Praze ve dnech od 30. srpna do 3. září 1971: Organizace symposia byla svěřena přípravnému výboru ve složení J. Novák (předseda), Z. Frolík, J. Hejcman, M. Hušek, M. Katětov, V. Koutník, V. Pták, M. Sekanina a Št. Schwarz. Československá akademie věd, Slovenská akademie věd, Karlova universita a Jednota československých matematiků a fyziků pozvaly jako hosty symposia vynikající zahraniční odborníky v topologii a jejích aplikacích. Mezinárodní matematická unie poskytla několika pozvaným hostům ze vzdálených zemí finanční podporu na cestovné.

Do programu třetího pražského topologického symposia byly zařazeny hodinové přednášky a dvacetiminutová sdělení z nejnovějších směrů v topologii, které se rozvinuly nebo vznikly v době po druhém symposiu, tj. v období posledních asi 5 let. Jedním takovým směrem je teorie tvarů. Dalším novým směrem je nekonečně dimensionální topologie, která vzbudila značnou pozornost a zájem matematiků. Třetím směrem byla teorie kompaktních prostorů, která obohatila obecnou topologii řešením velmi těžkých problémů a řadou překvapivých výsledků. Z dalších oblastí preferovaných na symposiu budiž uvedeno zobecnění metrických prostorů a topologické metody v teorii míry. Velká pozornost byla věnována též aplikacím topologie zejména v algebře a ve funkcionální analýze. Z těchto oborů byla přednesena řada důležitých přednášek a sdělení zahraničních i našich vědců. Pro úplnost je uveden přehled hlavních přednášek:

R. D. Anderson: Some open questions in infinite-dimensional topology
M. Ja. Antonovskij: Несимметрические близости, равномерности и разрывные метрики
A. V. Arhangelskii: On cardinal invariants
S. P. Arya: Sum theorems for topological spaces
B. Banaschewski: On profinite universal algebras
K. Borsuk: Some remarks concerning the theory of shape in arbitrary metrizable spaces
Z. Frolík: Topological methods in measure theory and the theory of measurable spaces
J. de Groot: On the topological characterization of manifolds
H. Herrlich: A generalization of perfect maps
E. Hewitt: Harmonic analysis and topology
F. B. Jones: The utility of empty inverse limits
M. Katětov: On descriptive classification of functions
K. Kuratowski: A general approach to the theory of set-valued mappings
S. Mardešić: A survey of the shape theory of compacta
E. Michael: On two theorems of V. V. Filipov
J. Nagata: A survey of the theory of generalized metric spaces
A. Pietsch: Ideals of operators on Banach spaces and nuclear locally convex spaces
V. Pták: Banach algebras with involution
A. K. Steiner: On the lattice of topologies
J. C. Taylor: The Martin compactification in axiomatic potential theory
J. E. West: Identifying Hilbert cubes: General methods and their application to hyperspaces by Schori and West
A. V. Zarelua: On infinite-dimensional spaces

Vedle 22 pozvaných hostů, jejichž jména jsou uvedena v přehledu hlavních přednášek, předneslo vědecká sdělení ještě 88 účastníků symposia. Dvacetiminutová sdělení probíhala ve dvou souběžných sekcích s výjimkou jednoho dne, kdy byla přednesena sdělení ve třech sekcích. Celkem navštívilo třetí topologické symposium 158 matematiků, z toho 107 ze zahraničí a 51 z Československa. Vedle toho přijelo do Prahy ještě 28 doprovázejících osob, pro něž byl uspořádán zvláštní program.

Při hodnocení vědecké úrovně třetího topologického symposia lze konstatovat, že význam pražských topologických symposií stále vzrůstá. Je to nejen proto, že se těchto symposií aktivně účastní přední topologové světa, ale také tím, že pražská symposia poutají zájem stále většího počtu mladých vědců, jejichž práce vzbuzují zájem matematické veřejnosti. Velký význam mělo třetí pražské topologické symposium též pro nejmladší naše i zahraniční účastníky a studenty, kteří měli možnost získat přehled o současném stavu nejdůležitějších směrů v obecné topologii a podněty ke své vlastní vědecké činnosti.

Rovněž společenský program symposia vyzněl kladně. Do značné míry k tomu přispělo přijetí předních badatelů v topologii u předsedy Československé akademie věd J. Kožešníka, přátelská beseda na ministerstvu školství a závěrečný večer na rozloučenou.

O vědecké úrovni a společenském úspěchu svědčí hlasy zahraničních vědců. Akademik K. Kuratowski ve svém závěrečném vystoupení řekl: „*S radostí konstatuji, že podle všeobecného mínění bylo toto symposium velkým úspěchem. Bylo to skutečně setkání vynikajících vědců, setkání velkého počtu mladých velmi aktivních matematiků; zúčastnilo se ho 110 zahraničních matematiků z různých zemí a 51 českých a slovenských matematiků; nemluvě o 23 dámách, které doprovázely své manžely nebo otce a přispěly svou přítomností k atraktivnosti setkání.*

Konference nám dala příležitost vyslechnout velké množství referátů obsahujících nové výsledky. Neméně důležité bylo i to, že jsme se mohli setkat s matematiky z různých zemí a navázat nové kontakty, které přinášejí podněty pro další výzkum.

Za to vše vděčíme svým hostitelům: jejich nevšedním organisačním schopnostem a jejich laskavému pohostinství. V tomto krásném městě a v této úžasné atmosféře, kterou vytvořili, mohl člověk skutečně cítit nezapomenutelnou tradici Eduarda Čecha, zakladatele československé topologické školy.

Dovolte mi, abych vyslovil jménem zahraničních účastníků nejsrdečnější díky našim hostitelům, zejména profesorům Novákovi a Frolíkovi za jejich nesmírné úsilí, jehož výsledkem je tento skvělý úspěch třetího pražského topologického symposia."

Další hodnocení je obsaženo v dopise adresovaném předsedovi ČSAV akademiku J. Kožešníkovi:

„*Redakční a poradní rada nového časopisu „General Topology and its Applications", které se sešly na pražském symposiu dne 1. září, dovolují si vyslovit uznání Československé akademii věd za vysoce úspěšnou organisaci a uspořádání třetího pražského topologického symposia. Všichni účastníci si uvědomují, že tři pražská symposia byla nesmírně důležitá, neboť poskytla podněty pro lepší výzkum v obecné topologii a pomohla jak svými zasedáními tak i svými Sborníky zaměřit výzkumnou činnost na nejvýznamější problémy v topologii. Pražské symposium se jasně stalo ústředním a nejdůležitějším symposiem v obecné topologii. Stává se vskutku „tradicí", které se zúčastňuje velký počet nejlepších odborníků v obecné topologii na světě*".

Vysoký počet vynikajících sdělení a referátů přednesených v tomto roce svědčí o tom, že topologové na celém světě stále více uznávají význam symposia. Zdá se, že tento vysoký počet účastníků nynějších symposií, z něhož vyplývá i přetížení jejich programu, si vynucuje úvahu, zda by příští symposia nemohla být trochu delší, dá-li se to zařídit. Je opravdu zásluhou profesorů Nováka, Katětova a dalších členů organisačního výboru, že programy symposií dosáhly dnešních rozměrů a důležitosti. Československá akademie věd může být oprávněně hrdá na první tři pražská symposia a usilovat o to, aby její příští symposia byla ještě důležitější v topologickém světě. Tato pražská symposia poskytují určitě významnou pomoc mladým matematikům a studentům, zvláště z Československa a sousedních zemí, pro jejich vedení a orientaci. Hrají také důležitou úlohu ve zvyšování postavení československé matematiky ve světě.

Průběh třetího symposia potvrzuje, že pražská symposia o obecné topologii významně přispívají k rozvoji této mladé matematické disciplíny a stávají se středem zájmu topologů. Proto všichni účastníci s radostí přijali oznámení, že se r. 1976 bude konat v Praze čtvrté topologické symposium.

Josef Novák, Praha

ZPRÁVA O USTAVENÍ MATEMATICKÉ VĚDECKÉ SEKCE JEDNOTY ČESKÝCH MATEMATIKŮ A FYZIKŮ

Po roční práci přípravného výboru matematické vědecké sekce se 21. února 1972 konalo v Praze její ustavující shromáždění za účasti 114 členů a 10 hostů. Shromáždění schválilo prozatímní organizační řád a v tajných volbách zvolilo za členy výboru B. Budinského, V. Havla, A. Kufnera, J. Kurzweila, I. Marka, Z. Nádeníka, J. Nagye, B. Nováka, E. Novákovou, M. Sekaninu, T. Sturma, P. Vopěnku a M. Zlámala. Výbor zvolil svým předsedou J. Nagye a tajemnicí E. Novákovou (oba z katedry matematiky fakulty elektrotechnické ČVUT, Technická 1902, Praha 6). V rezoluci se shromáždění vyslovilo pro přátelskou a aktivní spolupráci s pobočkami a ostatními sekcemi JČSMF, pro sjednocování tvůrčího úsilí matematiků z vysokých škol, ústavů i praxe a pro šíření stanovisek angažovaných matematiků, která by vedla v širší veřejnosti k propagaci a správnému chápání matematiky, její úlohy a jejího významu. Členové matematické vědecké sekce uložili výboru, aby rozvíjel činnost hlavně v těchto směrech: a) zřizování, podpora a koordinace odborných skupin a komisí MVS; b) systematické informování členů MVS o aktuálních problémech matematického života v ČSSR i v zahraničí; c) využívání tiskového střediska i dalších zařízení JČSMF pro potřeby MVS; d) iniciativní úsilí o příznivé ovlivňování všech akcí, týkajících se matematiky; e) další rozvíjení záměrů, započatých přípravným výborem MVS.

Podrobnější zprávy o ustavujícím shromáždění budou uveřejněny ve členském časopisu „Pokroky matematiky, fyziky a astronomie" *17* (1972) a v „Informacích matematické vědecké sekce JČMF".

Zbyněk Nádeník, Praha

OBHAJOBY A DISERTAČNÍ PRÁCE DOKTORŮ A KANDIDÁTŮ VĚD

Před *komisí pro obhajobu doktorských disertačních prací* obhájil dne 17. února 1972 doc. RNDr. LADISLAV KOUBEK, CSc. práci na téma: „Algoritmus překladače z jazyka ALGOL 60".

Před *komisemi pro obhajoby kandidátských disertačních prací* obhájili dne 14. ledna 1972 JAROSLAV PECHANEC práci na téma: „Representace předsvazků uzávěrových prostorů", dne 26. ledna 1972 JOZEF KAČÚR práci na téma: „O existencii slabého riešenia nelineárnych parciálnych diferenciálnych rovníc eliptického typu", SVATOPLUK FUČÍK práci na téma: „Řešení nelineárních operátorových rovnic" a JIŘÍ SOUČEK práci na téma: „Prostory funkcí na Ω, jejichž k-té derivace jsou míry definované na $\bar{\Omega}$", dne 21. února 1972 JAROSLAV SMÍTAL práci na téma: „O postupnostiach funkcií s Darbouxovou vlastnosťou", dne 23. února 1972 ROMAN FRIČ práci na téma: „Sekvenční struktury a jejich aplikace v teorii pravděpodobnosti", dne 28. února 1972 PETR PŘIKRYL práci na téma: „Optimal universal approximation of Fourier coefficients in spaces of continuous periodic functions", dne 2. března 1972 LADISLAV BERAN práci na téma: „Submodulární svazky a jejich aplikace" a LADISLAV NEBESKÝ práci na téma: „Algebraické vlastnosti stromů" a dne 9. března 1972 PAVEL ČIHÁK práci na téma: „Dvojnásobně stochastické matice a komparabilita měr".

Redakce

OZNÁMENÍ

Mezinárodní matematické centrum S. Banacha ve Varšavě uspořádá od ledna do června 1973 semestr o problémech základů matematiky. Tematikou semestru bude teorie modelů, rekursivní teorie a některé otázky aplikace logiky.

Účast bude možná na pozvání. Informace poskytne Matematický ústav ČSAV, Praha 1, Žitná 25.