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CONTINUITY AND DIFFERENTIABILITY PROPERTIES OF NONLINEAR OPERATORS

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1. INTRODUCTION

A number of fixed-point theorems and approximate methods of solutions of nonlinear equations involving continuous, weakly continuous and strongly continuous operators have been recently discovered by methods of the nonlinear functional analysis. Hence it is important to establish some simple conditions under which a mapping F or its derivative $F'(u)$ possess certain continuity properties at some point $u_0 \in X$ or on some subset M of a normed linear space X.

This note is devoted to the study of the above mentioned problems and it is a continuation of our papers [1], [2], [3], [4]. For the recent results concerning the related topics, see the bibliography in [1–4].

2. TERMINOLOGY AND NOTATION

Let X, Y be normed linear spaces, X^*, Y^* their (adjoint) dual spaces. The pairing between the points of X^* or Y^* and the elements of X or Y respectively we denote by $\langle .,. \rangle$. We use the symbols "$\to$", "$\rightharpoonup$" to denote the strong and weak convergence in X, Y. To fix our notation we introduce the following well-known definitions. A mapping $F : X \to Y$ is said to be

a) "closable" if $u_n \to 0$, $F(u_n) \to v$ implies $v = F(0)$;
b) weakly continuous at $u_0 \in X$, if $u_n \rightharpoonup u_0$ implies $F(u_n) \rightharpoonup F(u_0)$;
c) strongly continuous at $u_0 \in X$, if $u_n \rightharpoonup u_0$ implies $F(u_n) \to F(u_0)$;
d) bounded in X, if for each bounded subset $M \subset X$, $F(M)$ is bounded in X;
e) compact in X, if for each bounded set $N \subset X$, $F(N)$ is compact in Y (a subset $M \subset X$ is called compact in X, if from each sequence $(u_n) \in M$ one can select a subsequence (u_{n_k}) so that (u_{n_k}) converges to some point $u_0 \in X$);
f) p – positively homogeneous on X, if $F(tu) = t^p F(u)$ for each $t \geqq 0$ and $u \in X$, $(p > 0)$.

A mapping $F : X \to X^*$ is said to be monotone, if $\langle F(u) - F(v), u - v \rangle \geqq 0$ for each $u, v \in X$.

For the Gâteaux, Fréchet differentials and derivatives, the notions of compactness, strong continuity of the Fréchet derivative and uniform differentiability of mappings see the terminology and notations given in the Vainberg's book [5, Chap. I.]. We need also the concept of the bounded differential which is due to SUCHOMLINOV [6]. This notion can be introduced equivalently as follows: We shall say that a mapping $F : X \to Y$ possesses a bounded differential $dVF(u_0, h)$ at $u_0 \in X$, if

$$F(u_0 + h) - F(u_0) = dVF(u_0, h) + \omega(u_0, h), \quad h \in X,$$

where $\lim_{\|h\| \to 0} \|\omega(u_0, h)\|/\|h\| = 0$, $dVF(u_0, .)$ is bounded in some open neighborhood $V(0)$ of 0 and $dVF(u_0, \alpha h) = \alpha\, dVF(u_0, h)$ for each real α, $h \in X$.

Suppose that there exists a linear Gâteaux differential $DF(u, h)$ in some neighborhood $V(u_0)$ of $u_0 \in X$. Then $DF(u, h)$ is said to be

g) continuous jointly at $(u_0, u_0) \in X \times X$, if $(u_n) \in V(u_0)$, $(h_n) \in X$, $u_n \to u_0$, $h_n \to u_0$ imply

$$DF(u_n, h_n) \to DF(u_0, u_0);$$

h) weakly continuous jointly (strongly continuous jointly) at (u_0, u_0) if $(u_n) \in V(u_0)$, $(h_n) \in X$, $u_n \rightharpoonup u_0$, $h_n \rightharpoonup u_0$ imply $DF(u_n, h_n) \rightharpoonup DF(u_0, u_0)$ $(DF(u_n, h_n) \to \to DF(u_0, u_0))$.

3. CONTINUITY AND DIFFERENTIABILITY OF NONLINEAR OPERATORS

Theorem 1. *Let X, Y be normed linear spaces, F a p-positively homogeneous mapping on X. Suppose one of the following two conditions to be fulfilled*: 1) $F : X \to \to Y$, $\dim Y < \infty$, *F is "closable"*. 2) $F : X \to X^*$ *is monotone on X*, $\dim X < \infty$. *Then F is continuous at 0 and bounded in X.*

Proof. First of all, $F(0) = 0$. Suppose that F is not continuous at 0. Then there exists a sequence $(v_n) \in X$, $v_n \to 0$ and $\varepsilon_0 > 0$ so that $\|F(v_n)\| \geqq \varepsilon_0$. Set

$$u_n = \frac{1}{\|F(v_n)\|^{1/p}}\, v_n, \quad (n = 1, 2, \ldots).$$

Then $\|u_n\| \leqq (1/\varepsilon_0^{1/p}) \|v_n\| \to 0$ whenever $n \to \infty$ and

$$\|F(u_n)\| = \left\| F\left(\frac{1}{\|F(v_n)\|^{1/p}}\, v_n\right)\right\| = \left(\frac{1}{\|F(v_n)\|^{1/p}}\right)^p . \|F(v_n)\| = 1$$

for each n $(n = 1, 2, \ldots)$. Denote $K = \{y \in Y : \|y\| \leqq 1\}$, $K^* = \{\omega^* \in X^* : \|\omega^*\| \leqq \leqq 1\}$. As $\dim Y < \infty$ and $\dim X = \dim X^*$ in the case 2), the Riesz's theorem implies that K, K^* are compact in Y, X^*, respectively. Hence there exists a subsequence $(F(u_{n_k}))$ of $(F(u_n))$ so that $F(u_{n_k}) \to y$ as $k \to \infty$ and $y \in Y$, $y \in X^*$, respectively. Assume 1), then $u_{n_k} \to 0$, $F(u_{n_k}) \to y$ imply $y = F(0) = 0$, a contradiction to $\|y\| = 1$. Assuming 2), we have $\langle F(u) - F(u_{n_k}), u - u_{n_k}\rangle \geqq 0$, $u \in X$. Passing to the limit in this inequality, we obtain $\langle F(u) - y, u\rangle \geqq 0$ for all $u \in X$. Set $u = tv$, $t > 0$, $v \in X$. Then $\langle F(tv) - y, v\rangle \geqq 0$, $v \in X$. Since

$$\lim_{t\to 0_+} \|F(tu)\| = \lim_{t\to 0_+} t^p \|F(u)\| = 0 ,$$

we conclude that $\langle y, v\rangle = 0$ for every $v \in X$. Therefore $y = 0$, a contradiction to $\|y\| = 1$. Hence F is continuous at 0 in the both cases 1), 2). Thus for given $\varepsilon = 1$ there exists $\delta_0 > 0$ so that $\|u\| \leqq \delta_0 \Rightarrow \|F(u)\| < 1$. Let $D_R(0) = \{u \in X : \|u\| \leqq R\}$ be an arbitrary closed ball in X. Then there exists an integer n_0 so that $R/n_0 \leqq \delta_0$. For $u \in D_R(0)$ it holds

$$\|F(u)\| = \left\|F\left(\frac{u}{n_0} . n_0\right)\right\| = n_0^p \left\|F\left(\frac{u}{n_0}\right)\right\| < n_0^p .$$

Hence F is bounded in X. This completes the proof.

Theorem 2. *Let X, Y be linear normed spaces, $F : X \to Y$ a p – positively homogeneous operator. Let one of the following three conditions be fulfilled*: (a) *There exists an open subset $G \subset X$, $0 \in G$, so that* $\sup_{u\in G} \|F(u)\| < +\infty$. (b) *There exists a Baire subset $M \subset X$ of the second category in X such that* $\sup_{u\in M} \|F(u)\| < +\infty$ *and*

$$\|F(u - v)\| \leqq f(\max(\|u\|, \|v\|)) \max(\|F(u)\|, \|F(v)\|) \tag{1}$$

for each $u, v \in M$, where a real function $f(r)$ is defined on $J = [0, +\infty]$ and is bounded on each subinterval $[0, a]$ of J. (c) $(u_n) \in X$, $u \in X$, $u_n \to u \Rightarrow \|F(u)\| \leqq \leqq \overline{\lim}_{n\to\infty} \|F(u_n)\|$, *$X$ is of the second category in itself and F satisfies* (1) *on X.*

Then F is continuous at 0 and bounded in X.

Proof. First of all we prove (a). Assume $\varepsilon_0 > 0$ is such that $\|u\| < \varepsilon_0 \Rightarrow u \in G$. Suppose F is not continuous at 0. Then there exists a sequence $(u_n) \in X$, $u_n \to 0$ such that $\|F(u_n)\| \geqq m > 0$. Set

$$v_n = \frac{\varepsilon_0}{2\|u_n\|} u_n , \quad n = 1, 2, \ldots$$

Then $v_n \in G$ and

$$\|F(v_n)\| = \left(\frac{\varepsilon_0}{2\|u_n\|}\right)^p \|F(u_n)\| \geq \left(\frac{\varepsilon_0}{2\|u_n\|}\right)^p m .$$

It is $u_n \to 0$, $\lim_{n\to\infty} (\varepsilon_0^p/(2\|u_n\|)^p) = +\infty$ and therefore $\sup_{n=1,2,\ldots} \|F(v_n)\| = +\infty$, a contradiction to $v_n \in G$. Hence F is continuous at 0. Assume (b). According to the well-known theorem [7, Chap. 3] the set W of all differences $w = u - v$, where $u, v \in M$, is a neighbourhood of 0. Using (1), we see that F is bounded on some open neighbourhood of 0. This fact together with the assertion (a) imply that F is continuous at 0. Assuming (c), let $X_n = \{u \in X : \|F(u)\| \leqq n\}$. Then X_n are closed and $X = \bigcup_{n=1}^{\infty} X_n$. By the Baire Category Theorem at least one of X_n, say X_{n_0} contains an open ball $D \neq \emptyset$. Since X is of the second category in itself and D is open, D is a Baire subset of the second category in X. Moreover, $\sup_{u\in D} \|F(u)\| \leqq n_0$. Now it suffices to apply (b). Hence F is continuous at 0 in all the cases (a), (b), (c). This property together with the p – positive homogeneity imply the boundedness of F in X. Theorem is proved.

Let us remark that we need not require the assumption of the p – positive homogeneity of F for the boundedness of F in (c). Compare with the proof of Theorem 1 [3]. Theorem 2 extends the Banach's results [8], see also [9], which concern the continuity properties of linear operations.

Theorem 3. *Let X, Y be normed linear spaces, $F : M \to Y$, $M \subset X$ a bounded subset of X, $K : X \to Y$ a linear compact mapping in X such that $\|F(u) - F(v) - K(u - v)\| \leqq \alpha\|u - v\|$, $(\alpha > 0)$ for each $u, v \in M$. Suppose there is a constant $\gamma > 0$ such that $\|F(u) - F(v)\| \geqq \gamma\|u - v\|$, $u, v \in M$. If $\gamma > \alpha$, then F is strongly continuous on M.*

Proof. For $u, v \in M$ we have

$$\|F(u) - F(v)\| \leqq \alpha\|u - v\| + \|K(u - v)\| \leqq$$
$$\leqq \frac{\alpha}{\gamma} \|F(u) - F(v)\| + \|K(u - v)\| .$$

Hence

$$\|F(u) - F(v)\| \leqq \left(1 - \frac{\alpha}{\gamma}\right)^{-1} \|K(u - v)\| , \quad u, v \in M .$$

Suppose u_0 is an arbitrary point of M and $(u_n) \in M$, $u_n \rightharpoonup u_0$. As K is compact and linear, $Ku_n \rightharpoonup Ku_0$. Since $(Ku_n) \in K(M)$ and the weak convergence is equivalent with the strong one on a compact set [5, chapt. I.], $Ku_n \to Ku_0$. Hence K is strongly continuous in X and in view of the last inequality $F(u_n) \to F(u_0)$. This concludes the proof.

The following theorem is a completion and generalization of Proposition 1 [10].

Theorem 4. *Let X, Y be normed linear spaces, $F : X \to Y$ a mapping having a linear Gâteaux differential $DF(u, h)$ on some convex neighborhood $V(u_0)$ of $u_0 \in X$.*

If $DF(u, .)$ is continuous jointly, weakly continuous jointly, strongly continuous jointly at (u_0, u_0), then F is continuous, weakly continuous, strongly continuous at u_0, respectively.

Corollary 1. *Suppose that $F : M \to Y$ is a compact mapping on a convex bounded set $M \subset X$ and that F possesses a linear Gâteaux differential $DF(u, .)$ on M. If $DF(u, h)$ is weakly continuous jointly at the points of the diagonal ΔM of M ($\Delta M = \{(u, u) : u \in M\}$), then F is strongly continuous on M.*

Corollary 2. *Let $M \subset X$ be an open convex bounded set, $F : M \to Y$ a uniformly Fréchet – differentiable mapping on M. Suppose $F'(u)\, h$ is strongly continuous jointly at the points of the diagonal ΔM of M. If $F'(u)$ is compact on M, then $F'(u)$ is strongly continuous on M.*

Proof. Use Theorem 4 and the arguments similar to those in [5, Thm. 4.5].

Theorem 5. *Let $G \subset X$ be a convex bounded subset of X, $F : G \to Y$ a mapping such that F possesses the Fréchet derivative $F'(u)$ and the second linear Gâteaux differential $D^2F(u, h, k)$ on G. Assume $D^2F(u, h, k)$ is strongly continuous jointly in (u, k) at the points of the diagonal ΔG of G for each (but fixed) $h \in X$. If $F'(u)$ is compact on G, then $F'(u)$ is strongly continuous on G.*

Proof. Let $u_0 \in G$ be arbitrary (but fixed), $(u_n) \in G$ so that $u_n \rightharpoonup u_0$, $h \in X$. By the mean-value theorem for any $e_n^* \in Y^*$, $\|e_n^*\| = 1$, $(n = 1, 2, \ldots)$, we have

$$\begin{aligned}\langle F'(u_n)\, h - F'(u_0)\, h, e_n^*\rangle &= \langle D^2F(u_0 + \tau_n(u_n - u_0), h, u_n - u_0), e_n^*\rangle = \\ &= \langle D^2F(u_0 + \tau_n(u_n - u_0), h, u_n), e_n^*\rangle - \\ &- \langle D^2F(u_0 + \tau_n(u_n - u_0), h, u_0), e_n^*\rangle \leqq \\ &\leqq \|D^2F(u_0 + \tau_n(u_n - u_0), h, u_n) - D^2F(u_0, h, u_0)\| + \\ &+ \|D^2F(u_0, h, u_0) - D^2F(u_0 + \tau_n(u_n - u_0), h, u_0)\|,\end{aligned}$$

where $\tau_n = \tau_n(e_n^*) \in (0, 1)$. As $u_n \rightharpoonup u_0$ and $u_0 + \tau_n(u_n - u_0) \rightharpoonup u_0$, the both terms on the right hand side of the last inequality tend to 0. By the Hahn-Banach theorem we can choose $e_{n(0)}^* \in Y^*$ with $\|e_{n(0)}^*\| = 1$, $(n = 1, 2, \ldots)$ so that

$$\langle F'(u_n)\, h - F'(u_0)\, h, e_{n(0)}^*\rangle = \|F'(u_n)\, h - F'(u_0)\, h\|.$$

Hence $F'(u_n)\, h \to F'(u_0)\, h$ for each $h \in X$ whenever $n \to \infty$. As $(F'(u_n)) \in F'(G) = \{F'(u) : u \in G\}$ and $F'(G)$ is a compact set in the space $(X \to Y)$ of all linear continuous operators from X into Y, $F'(u_n) \to F'(u_0)$ as $n \to \infty$ in the norm of $(X \to Y)$ by Lemma 4.2 [5]. This completes the proof.

Theorem 6. *Let X, Y be normed linear spaces, $F : X \to Y$ a p – positively homogeneous operator on X. If F possesses a bounded differential $dVF(u_0, h)$ at some $u_0 \in X$, then F has $dVF(u, h)$ on the set $\{tu_0, t > 0\}$ and $t^p \omega(u_0, h) = \omega(tu_0, th)$ for each $t > 0$ and $h \in X$.*

Proof. First of all, we prove the following fact: if F possesses the Gâteaux differential $VF(u_0, h)$ at u_0, then $VF(tu_0, h)$ exists and $VF(tu_0, h) = t^{p-1} VF(u_0, h)$ for each $t > 0$ and $h \in X$. Indeed,

$$VF(tu_0, h) = \lim_{\alpha \to 0} \frac{1}{\alpha} [F(tu_0 + \alpha h) - F(tu_0)] =$$

$$= \lim_{\alpha \to 0} (1/t) [F(t(u_0 + (\alpha/t) h)) - F(tu_0)] (t/\alpha) =$$

$$= t^{p-1} \lim_{\alpha' \to 0} \frac{1}{\alpha'} [F(u_0 + \alpha' h) - F(u_0)] = t^{p-1} VF(u_0, h), \quad \alpha' = \alpha/t.$$

Now it is easy to see that $dVF(tu_0, h)$ exists for each $t > 0$ and $h \in X$ and that

$$dVF(tu_0, h) = t^{p-1} dVF(u_0, h). \tag{2}$$

For each $h \in X$ we have

$$F(t(u_0 + h)) - F(tu_0) = dVF(tu_0, th) + \omega(tu_0, th).$$

This equality, the p – positive homogeneity of F and (2) give

$$t^p(F(u_0 + h) - F(u_0)) = t^p \, dVF(u_0, h) + \omega(tu_0, th), \quad h \in X, \quad (t > 0).$$

By the hypothesis

$$F(u_0 + h) - F(u_0) = dVF(u_0, h) + \omega(u_0, h). \quad h \in X.$$

Our assertion follows immediately from the last two equalities. This concludes the proof.

4. SOME REMARKS

i) The following assertion is a simple consequence of Thm. 2(b). Suppose that X, Y are normed linear spaces, $F : X \to Y$ a p – positively homogeneous operator on X. Assume there exists a subset $M \subset X$ of the second category in X, a mapping $G : M \to Y$ having the Baire property in M (i.e. there exists a subset $A \subset M$ of the first category in M so that the restriction $G/(M - A)$ of G to $M - A$ is continuous) so that $u \in M \Rightarrow \|F(u)\| \leqq \|G(u)\|$. If F satisfies the inequality (1) for each $u, v \in M$, then F is continuous at 0 and bounded in X.