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FREDHOLM ALTERNATIVE FOR NONLINEAR OPERATORS AND APPLICATIONS TO PARTIAL DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

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1. Introduction. The problem of solving a nonlinear boundary value problem or an integral equation can be reduced often to the following abstract one: find a solution u of $Tu = f$, where T is a mapping from a real, reflexive Banach space B to its dual B^*.

Example 1. Let Ω be a bounded domain with Lipschitz boundary $\partial\Omega$ and let $a_i(x, \xi_0, \xi_1, \ldots, \xi_n)$, $i = 0, 1, \ldots, n$, be continuous functions in $\bar{\Omega} \times R_{n+1}$, satisfying growth conditions

$$|a_i(x, \xi)| \leqq c(1 + |\xi|)^{m-1}, \tag{1.1}$$

where $1 < m < \infty$. Let $f_i \in L_{m'}(\Omega)$, $1/m' + 1/m = 1$, $i = 0, \ldots, n$. By $W_m^{(1)}(\Omega)$ we denote the well-known Sobolev space of real L_m functions whose first derivatives are also L_m functions. $W_m^{(1)}(\Omega)$ is a Banach space with the norm $\|u\|_{W_m^{(1)}} = (\int_\Omega (|u|^m + \sum_{i=1}^n |\partial u/\partial x_i|^m) \, dx)^{1/m}$ and is separable. $W_m^{(1)}(\Omega)$ is also reflexive as the closed subspace of $[L_m]^{n+1}$. Let $\mathring{W}_m^{(1)}(\Omega)$ be the closure of $D(\Omega)$, the space of infinitely differentiable functions with compact support, in the space $W_m^{(1)}(\Omega)$. We have to find $u \in \mathring{W}_m^{(1)}(\Omega)$ such that for any $v \in \mathring{W}_m^{(1)}(\Omega)$

$$\int_\Omega \left(\sum_{i=1}^n \frac{\partial v}{\partial x_i} a_i \left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right) + v a_0 \left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n} \right) \right) dx = \tag{1.2}$$
$$= \int_\Omega v f_0 \, dx - \int_\Omega \sum_{i=1}^n \frac{\partial v}{\partial x_i} f_i \, dx .$$

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The function u is called weak solution of the differential equation

$$(1.3)\quad -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(a_i\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)\right) + a\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = f + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

in Ω, satisfying on the boundary the condition $u = 0$.

Denoting by (w^*, u) the pairing between B^* and B, we can define an operator T: $: B \to B^*$, putting

$$(Tu, v) \overset{\text{df}}{=} \int_\Omega \left(\sum_{i=1}^{n} a_i \left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) \frac{\partial v}{\partial x_i} + a_0\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) v \right) \mathrm{d}x \, .$$

Because $\int_\Omega f_0 v \,\mathrm{d}x - \int_\Omega \sum_{i=1}^{n} f_i(\partial v/\partial x_i) \,\mathrm{d}x = (f, v)$, the equation (1.2) is reduced to the problem of solving the equation $Tu = f$.

Example 2. Let us consider the Hammerstein's integral equation

$$(1.4)\qquad u(x) - \lambda \int_M K(x, y) f(y, u(y)) \,\mathrm{d}y = w(x)\,,$$

where the solution is supposed in $L_2(M)$, M being a compact subset of R_n, $w \in L_2(M)$, $f(y, u)$ is a continuous function on $M \times R_1$, satisfying the growth condition $|f(y, u)| \leqq c(1 + |u|)$. We suppose $\int_M \int_M K^2(x, y) \,\mathrm{d}x \,\mathrm{d}y < \infty$. If $(Tu)(x) \overset{\text{df}}{=} u(x) - \lambda \int_M K(x, y) f(y, u(y)) \,\mathrm{d}y$, then $T: L_2(M) \to L_2(M)$ and the problem is reduced to the solution of $Tu = w$.

2. Borsuk type theorem. A mapping T is said to be bounded if the image of bounded set is bounded and it is said to be demicontinuous, if from $u_n \to u$ (strong convergence) follows $Tu_n \rightharpoonup Tu$ (weak convergence).

Theorem 1. *Let $T: B \to B^*$, where B is a reflexive space, be a bounded, demicontinuous mapping. Let $T_t(u) = T(u) - t\,T(-u)$ for $0 \leqq t \leqq 1$. Let for $0 \leqq t \leqq 1$, the condition (S) be satisfied:*

$$(2.1)\qquad \text{if} \quad u_n \rightharpoonup u \quad \text{and} \quad (T_t(u_n) - T_t(u), u_n - u) \to 0\,,$$

then $u_n \to u$, and for $f \in B^*$ the condition

$$(2.2)\qquad T_t u - (1 - t) f \neq 0 \quad \text{for} \quad \|u\| = R > 0\,, \quad 0 \leqq t \leqq 1\,.$$

Then there exists a solution of $Tu = f$.

Let us remark first that the above solution is unique if, for example, the operator T is strictly monotone: $u \neq v \Rightarrow (Tu - Tv, u - v) > 0$.

Theorems as above are based on the concept of monotone operators, and there is a large amount of literature on this subject, compare, for example, M. I. VIŠIK [11],

F. E. BROWDER [1], J. LERAY, J. L. LIONS [6], G. J. MINTY [7]. The concept using Borsuk's theorem was recently used in the paper of D. G. DE FIGUEIREDO, CH. P. GUPTA [3] and elsewhere.

The main ideas of the proof of Theorem 1: First, if $B = R_n$, then the degree $(T_1(u), B(0, R), 0)$ is an odd integer by Borsuk's theorem, hence by homotopy, this is true for $T(u) - f$, hence, there exists $\|u\| < R$ such that $Tu = f$. If $F \subset B$ is a finite dimensional subspace of B and ψ_F is the injection of $T \to B$, ψ_F^* being its dual mapping, then for $T_F \stackrel{\mathrm{df}}{=} \psi_F^* T \psi_F$, it can be proved by contradiction existence of a F such that if $F' \supset F$, then $T_{F'}(u) - tT_{F'}(-u) - (1-t)\,\psi_{F'}^* f \neq 0$ for $\|u\| = R$, $u \in F'$, $0 \leqq t \leqq 1$. Hence for every $F' \supset F$, there exists $u_{F'} \in F'$ such that $T_{F'} u_{F'} = \psi_{F'}^* f$. Let us put $M_{F'} = \{u_{F''} \mid F'' \supset F'\}$. The set of $M_{F'}$ has finite intersection property. If $\overline{M}_{F'}$ is the closure in the weak topology, then $\bigcap_{F'} \overline{M}_{F'} \ni u$. If $w, u \in F'$ for F' such chosen, then there exists $u_n \in M_{F'}$, $u_n \rightharpoonup u$ and because of $\lim_{n\to\infty} (Tu_n - Tu, u_n - u) = \lim_{n\to\infty} (Tu_n, u_n - u) = \lim_{n\to\infty} (f, u_n - u) = 0$, ($(Tu_n, u_n - u) = (f, u_n - u)$ follows from the definition of $T_{F'}$.) the condition (2.1) implies $u_n \to u$, what, in virtue of the demicontinuity of T, gives the result. We have clearly:

Consequence 1. *If the operator T is coercive:*

$$\lim_{\|u\|\to\infty} \frac{(Tu, u)}{\|u\|} = \infty\,, \quad \textit{then} \quad T(B) = B^*\,.$$

This is because $(T_t u, u) \geqq c(\|u\|)\,\|u\|$, with $c(s) \to \infty$ for $s \to \infty$.

Consequence 2. *If the conditions of theorem* 1 *are satisfied and T is odd: $T(-u) = -T(u)$ and if T is weakly coercive: $\lim_{\|u\|\to\infty} \|Tu\| = \infty$, then $T(B) = B^*$.*

Let us consider the following class of operators: first if for $\varkappa > 0$ and every $t > 0$: $A(tu) = t^\varkappa A(u)$, then A is called $\varkappa$-homogeneous.

An operator S is asymptotically zero if for $\varkappa > 0$ $\lim_{\|u\|\to\infty} \|Su\|/\|u\|^\varkappa = 0$.

We have the following Fredholm alternative:

Theorem 2. *Let $T = A + S$, where A is demicontinuous, $\varkappa$-homogeneous, satisfies the condition* (S) *(i.e. if $u_n \rightharpoonup$ and $(A(u_n) - A(u), u_n - u) \to 0$, then $u_n \to u$), S is demicontinuous, asymptotically zero (with the same $\varkappa$ as for A) and T is bounded odd and satisfies the condition* (S). *Then the range of T is all of B^* if $Au = 0 \Rightarrow u = 0$. In this case, for every solution,*

$$\|u\| \leqq c(1 + \|f\|^{1/\varkappa})\,. \tag{2.3}$$

If (2.3) *is true for every solution, then $Au = 0 \Rightarrow u = 0$.*

Theorems of this type are recent. It seems the first paper is due to S. I. POCHOŽAJEV [10] and to the author [8]. For further results, compare F. E. BROWDER [2] and the forthcoming paper of J. NEČAS [9]; compare also M. KUČERA [5].

Proof of Theorem 2:

(i) If (2.3) is true and there exists $u_0 \neq 0$ such that $Au_0 = 0$, then for $u = tu_0$:

$$\|u_0\| \leqq c\left(\frac{1}{t} + \frac{\|S(u_0 t)\|^{1/\varkappa}}{t\|u_0\|}\right)\|u_0\| \to 0$$

which is a contradiction.

(ii) Let $Au = 0 \Rightarrow u = 0$. Then (2.3) is true: if not, there exists a sequence $\|u_n\| \to \to \infty$ such that

$$(2.4) \qquad \|u_n\|^\varkappa > n(1 + \|Tu_n\|) \quad \text{and putting} \quad v_n = \frac{u_n}{\|u_n\|},$$

we can suppose $v_n \rightharpoonup v$ and we obtain from (2.4) $Av_n \to 0$ and using (S) condition: $v_n \to v$, hence $\|v\| = 1$ and $Av = 0$ which is a contradiction.

(iii) (2.3) implies (2.2), and (2.1) is satisfied because $T_t(u) = (1 + t)\,T(u)$.

3. Back to the applications. Let us remark first that it is only a question of introducing enough of indices to treat general systems instead of one partial differential equation as we will do; there is no essential difference.

I) We consider first the problem:

$$(3.1) \qquad -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \lambda a_0\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) = f_0(x) + \sum_{i=1}^{n}\frac{\partial f_i}{\partial x_i}$$

with $a_{ij} \in L_\infty(\Omega)$, $\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geqq c|\xi|^2$. Let us suppose

$$(3.2) \qquad \left|\frac{1}{t}\,a_0(x, t\xi) - \sum_{i=0}^{n} b_i(x)\,\xi_i\right| \leqq c(t)\,[(\sum_{i=0}^{n}\xi_i^2)^{1/2} + 1]$$

with $c(t) \to 0$ for $t \to \infty$, $b_i \in L_\infty(\Omega)$. The condition (3.2) implies immediately that $Ru \overset{\text{df}}{=} a_0(x, u, \partial u/\partial x_1, \ldots, \partial u/\partial x_n) - \sum_{i=1}^{n} b_i(x)\,\partial u/\partial x_i - b_0(x)\,u$ satisfies the condition $\lim_{\|u\|\to\infty} \|Ru\|_{L_2}/\|u\|_{W_2^{(1)}} = 0$. Supposing $a_0(x, -\xi) = -a_0(x, \xi)$ and defining

$$(Au, v) \overset{\text{df}}{=} \int_\Omega \sum_{i,j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i}\,\mathrm{d}x\,, \quad (Su, v) \overset{\text{df}}{=} -\lambda\int_\Omega a_0\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right) v\,\mathrm{d}x\,,$$

we obtain in virtue of the fact the imbedding $W_2^{(1)}(\Omega) \to L_2(\Omega)$ is completely continuous, that Su is a completely continuous operator from $\mathring{W}_2^{(1)} \to (\mathring{W}_2^{(1)})^*$. Because A

and S above defined satisfy with $\varkappa = 1$ the conditions of the theorem 2, this altogether gives by theorem 2 this result:

For every $f_i \in L_2(\Omega)$, $i = 0, \ldots, n$, there exists a solution of (3.1) with $u = 0$ on $\partial\Omega$ and for every solution, we have $\|u\|_{W_2^{(1)}} \leqq c(1 + \|f_0\|_{L_2} + \sum_{i=1}^{n} \|f_i\|_{L_2})$ if and only if λ is not an eigenvalue for the linear problem

$$- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right) - \lambda \sum_{i=1}^{n}\left(b_i(x)\frac{\partial u}{\partial x_i} + b_0(x)\,u\right) = 0\,, \quad u \in \mathring{W}_2^{(1)}(\Omega)\,.$$

II) If we consider a nonlinear problem with $m \neq 2$, then we suppose:

$$\left|\frac{a_i(x, t\xi)}{t^{m-1}} - A_i(x, \xi)\right| \leqq c_i(t)\,(1 + |\xi|^{m-1})\,, \quad i = 0, \ldots, n\,, \tag{3.3}$$

where $A_i(x, \xi)$ satisfy the conditions (1.1) and $c_i(t) \to 0$ for $t \to \infty$. Let $A_i(x, \xi)$ and $a_i(x, \xi)$ be odd in ξ and $A_i(x, t\xi) = t^{m-1}A_i(x, \xi)$, $t > 0$. We shall suppose for $a_i(x, \xi)$ and $A_i(x, \xi)$ the conditions (we write them only for A_i): if $[\xi_1, \ldots, \xi_n] \neq [\xi'_1, \ldots, \xi'_n]$ then

$$\sum_{i=1}^{n} (A_i(x, \xi_0, \xi_1, \ldots, \xi_n) - A_i(x, \xi_0, \xi'_1, \ldots, \xi'_n))\,(\xi_i - \xi'_i) > 0 \tag{3.4}$$

and

$$\sum_{i=1}^{n} A_i(x, \xi_0, \xi_1, \ldots, \xi_n)\,\xi_i \geqq c_1 \sum_{i=1}^{n} |\xi_i|^m - c_2|\xi_0|^m\,. \tag{3.5}$$

For to apply theorem 2, we can easily verify (for details compare J. Leray, J. L. Lions [6]) the hypothesis eventually with the exception of the condition (S): for to see this, let $u_k \rightharpoonup u$ in $\mathring{W}_m^{(1)}(\Omega)$. We have first by the complete continuity of the imbedding $W_m^{(1)}(\Omega) \to L_m(\Omega)$: $u_k \to u$ in $L_m(\Omega)$. Choosing a subsequence, if necessary, still noted u_k, we have $u_k(x) \to u(x)$ almost everywhere. By hypothesis,

$$\lim_{k\to\infty} \int_\Omega \sum_{i=1}^{n}\left(a_i\left(x, u_k, \frac{\partial u_k}{\partial x_1}, \ldots, \frac{\partial u_k}{\partial x_n}\right) - a_i\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)\right)$$
$$\left(\frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i}\right) \mathrm{d}x + \int_\Omega \left(a_0\left(x, u_k, \frac{\partial u_k}{\partial x_1}, \ldots, \frac{\partial u_k}{\partial x_n}\right) - a_0\left(x, u, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)\right)$$
$$(u_k - u)\,\mathrm{d}x = 0\,.$$

The second member tends to zero, hence also the first, but in virtue of (3.4) putting

$$f_k(x) = \sum_{i=1}^{n}\left(a_i\left(x, u_k, \frac{\partial u_k}{\partial x_1}, \ldots, \frac{\partial u_k}{\partial x_n}\right) - a_i\left(x, u_k, \frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_n}\right)\left(\frac{\partial u_k}{\partial x_i} - \frac{\partial u}{\partial x_i}\right)\right),$$