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AN INEQUALITY FOR UNIVALENT FUNCTIONS DUE TO DVOŘÁK[1])

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1. In a recent note DVOŘÁK established the following result [1].

Theorem A. *Let $f(z) = z + a_2 z^2 + \ldots$ be analytic and univalent in the unit disc D. Then $f(z)$ satisfies the inequality*

$$\mathrm{Re}\,\sqrt{(f(z)/z)} > \tfrac{1}{2} \tag{1}$$

for $|z| < r'_0$ where r'_0 is the smallest positive root of the equation

$$r \log \frac{1+r}{1-r} = 2\,.$$

A computation shows

$$r'_0 = 0{\cdot}83355\ldots \tag{2}$$

In this note we obtain the exact value of r'_0.

Theorem B. *Let $f(z) = z + a_2 z^2 + \ldots$ be analytic and univalent in the unit disc D. Then $f(z)$ satisfies (1) for $|z| < r_0$ where r_0 is the smallest positive root of the equation*

$$\left[S^{-1}\left(\tfrac{1}{2}\log\frac{1+r}{1-r}\right)\right]^2 + \left[E^{-1}\left(\frac{\sqrt{(1-r^2)}}{4}\log\frac{1+r}{1-r}\right)\right]^2 = \left[\tfrac{1}{2}\log\frac{1+r}{1-r}\right]^2, \tag{3}$$

where $S^{-1}(x)$ and $E^{-1}(x)$ are the inverse of $S(x) = [x/\sin x]$ and $E(x) = xe^{-x}$ respectively. This result is sharp. A computation shows

$$r_0 = 0{\cdot}83559\ldots \tag{4}$$

Proof. It is easy to see that the condition (1) is equivalent to the inequality

$$|\sqrt{(z/f(z))} - 1| < 1\,. \tag{5}$$

Now GRUNSKY has shown that for normalized univalent functions in the unit disc we must have the sharp inequality

$$|\log(f(z)/z) + \log(1 - |z|^2)| \leqq \log\frac{1+|z|}{1-|z|} \tag{6}$$

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for all z in D [3; p. 113]. From (6) we obtain

$$\left|\log \sqrt{(z/f(z))} - \tfrac{1}{2}\log(1 - |z|^2)\right| \leqq \tfrac{1}{2}\log\frac{1+|z|}{1-|z|}. \tag{7}$$

We now set $w = \log\sqrt{(z/f(z))}$, $A = \frac{1}{2}\log(1 - |z|^2)$, $B = \frac{1}{2}\log[(1+|z|)/(1-|z|)]$ in (5) and (7) to obtain

$$|e^w - 1| < 1 \tag{8}$$

and

$$|w - A| < B\,, \tag{9}$$

respectively.

We are now going to show how A and B must be related in order that the inequality (8) should hold subject to the condition (9). We set $W = e^w = Re^{i\Theta}$ in (8) and (9) to obtain

$$R < 2\cos\Theta \tag{10}$$

and

$$(\log R - A)^2 + \Theta^2 < B^2\,, \tag{11}$$

respectively. The relations (10) and (11) define domains in the $W-$ plane that correspond to the domains defined by (8) and (9) in the $w-$ plane. If $|z| = r$ is small, it is clear that the domain (11) lies in the domain (10). As $|z| = r$ increases, the boundary of (11) eventually makes contact with that of (10) *before* r reaches 1.

Let us consider this first point of contact. At such a point we must have

$$\log R = \log(2\cos\Theta) = A + \sqrt{(B^2 - \Theta^2)} \tag{12}$$

and

$$\frac{dR}{d\Theta} = -2\sin\Theta = \frac{-\Theta}{\sqrt{(B^2-\Theta^2)}}\, e^{A+\sqrt{(B^2-\Theta^2)}}\,. \tag{13}$$

If we eliminate Θ from (12) and (13), then we obtain

$$\tfrac{1}{2}Be^A = \sqrt{(B^2 - \Theta^2)}\, e^{-\sqrt{(B^2-\Theta^2)}}\,. \tag{14}$$

Now (13) and (14) yield

$$\frac{\Theta}{\sin\Theta} = B\,. \tag{15}$$

If we let $E^{-1}(x)$ and $S^{-1}(x)$ denote the inverse of $E(x) = xe^{-x}$ and $S(x) = x/\sin x$,