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A NOTE ON $\mathscr{K}$-STOCHASTIC OPERATORS

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Dedicated to the memory of Prof. VOJTĚCH JARNÍK

A concept of a $\mathscr{K}$-stochastic operator in a Banach space with a cone $\mathscr{K}$ is introduced and some of its properties are studied. In particular, a characterization is given for a class of $\mathscr{K}$-stochastic operators in a Hilbert space to have symmetric projections corresponding to the principal eigenvalue 1. The characterization leads to the doubly $\mathscr{K}$-stochastic operators.

E. J. BELL and H. G. DAELLENBACH have shown in [1] a necessary condition for the limiting matrix P of an irreducible Markov chain $1 = (t_{jk})$, $1 \leqq j, k \leqq m$ to have a special form. The sufficiency of this condition is obvious [2, p. 167–168].

Theorem A. *Let T be an irreducible stochastic matrix and let $P = \lim (1/N) \sum_{k=1}^{N} T^k$ be the corresponding limiting matrix. Then for P to have the form $P = (p_{jk})$ with*

$$p_{jk} = \frac{1}{m} \tag{1}$$

it is necessary for T to satisfy

$$\sum_{j=1}^{m} t_{jk} = 1 \,. \tag{2}$$

Remark. It is obvious that condition (1) is equivalent to the requirement

$$\sum_{k=1}^{m} p_{jk} = \sum_{k=1}^{m} p_{kj} \tag{3}$$

for T irreducible. However, (3) is a more general condition than (1) since it is meaningful also for the reducible case.

The aim of this note is to show, besides other things, an infinite dimensional analogue of Theorem A. Since a general concept of positivity is used in our formulation the result presented is more general than that of [1] even for the finite dimensional case. In our opinion, the infinite dimensional generalization might be useful in the theory of Markov processes [3].

Let $\mathscr{Y}$ be a real or complex Banach space, $\mathscr{Y}'$ the dual space of continuous linear functionals on $\mathscr{Y}$ with the norm $\|x'\| = \sup\{|x'(x)| : \|x\| \leqq 1\}$ and $[\mathscr{Y}]$ the space of bounded linear transformations of $\mathscr{Y}$ into $\mathscr{Y}$ with the norm $\|T\| = \sup\{\|Tx\| : : x \in \mathscr{Y}, \|x\| \leqq 1\}$. If $\mathscr{Y}$ is real then $\mathscr{X}$ denotes its complexification, i.e. $\mathscr{X} = \mathscr{Y} \oplus \mathrm{i}\mathscr{Y}$ with the norm $\|z\| = \sup\{\|x \cos \vartheta + y \sin \vartheta\| : z = x + \mathrm{i}y,\ x, y \in \mathscr{Y},\ 0 \leqq \vartheta \leqq \leqq 2\pi\}$. We also use the symbol $\mathscr{X}$ for $\mathscr{Y}$ if $\mathscr{Y}$ is complex.

It is assumed that there exists a normal cone $\mathscr{K}$ [4] which generates $\mathscr{Y}$, i.e. $\mathscr{K}$ satisfies (i) $\mathscr{K} + \mathscr{K} \subset \mathscr{K}$, (ii) $\alpha\mathscr{K} \subset \mathscr{K}$ for $\alpha \geqq 0$, (iii) $\mathscr{K} \cap (-\mathscr{K}) = \{0\}$, (iv) $\mathscr{K}$ is closed, (v) there is a $\delta > 0$ such that for all $x, y \in \mathscr{K}$ one has $\|x + y\| \geqq \delta\|x\|$ (vi) $y \in \mathscr{Y}$ can be written as $y = y^+ - y^-$ with $y^+, y^- \in \mathscr{K}$.

Let $T \in [\mathscr{Y}]$ and $\mathscr{Y}$ be real. Then $\tilde{T}$ defined as $\tilde{T}z = Tx + \mathrm{i}Ty$, where $z = x + \mathrm{i}y$, $x, y \in \mathscr{Y}$, called the complex extension of T, clearly belongs to $[\mathscr{X}]$. Let $A \in [\mathscr{X}]$. By $\sigma(A)$ we denote the spectrum of A, i.e. the set of all singularities of the resolvent operator $R(\lambda, A) = (\lambda I - A)^{-1}$, where I is the identity operator. By $r(A)$ is denoted the spectral radius of A, i.e. $r(A) = \sup\{|\lambda| : \lambda \in \sigma(A)\}$. If $\mathscr{Y}$ is real and $T \in [\mathscr{Y}]$, then we set $\sigma(T) = \sigma(\tilde{T})$ and $r(T) = r(\tilde{T})$; furthermore $R(\lambda, T) = (\lambda I - \tilde{T})^{-1}$.

Let λ_0 be an isolated singularity of $R(\lambda, T)$, where $T \in [\mathscr{X}]$. Then [10 p. 305]

$$R(\lambda, T) = \sum_{k=0}^{\infty} A_k(\lambda - \lambda_0)^k + \sum_{k=1}^{\infty} B_k(\lambda - \lambda_0)^{-k},$$

where $A_{k-1} \in [\mathscr{X}]$ and $B_k \in [\mathscr{X}]$, $k = 1, 2, \ldots$ and

$$B_1 = \frac{1}{2\pi i} \int_{C_0} R(\lambda, T)\, \mathrm{d}\lambda, \quad B_{k+1} = (T - \lambda_0 I)\, B_k, \quad k = 1, 2, \ldots$$

with $C_0 = \{\lambda : |\lambda - \lambda_0| = \varrho_0, \varrho_0 > 0\}$ and such that $K_0 \cap \sigma(T) = \{\lambda_0\}$, where $K_0 = \{\lambda : |\lambda - \lambda_0| \leqq \varrho_0\}$. If there is an index q for which $B_q \neq \Theta$ and $B_{q+1} = \Theta$ where Θ denotes the zero-operator, then λ_0 is called a pole and q its multiplicity.

Let us denote by $\mathscr{K}'$ the dual cone to $\mathscr{K}$, i.e.

$$\mathscr{K}' = \{x' \in \mathscr{Y}' : \langle x, x' \rangle \geqq 0 \quad \text{for all} \quad x \in \mathscr{K}\},$$

where $\langle x, x' \rangle$ denotes the value $x'(x)$.

We call a vector $\hat{x} \in \mathscr{K}$ quasiinterior if $\langle \hat{x}, x' \rangle \neq 0$ for all $x' \in \mathscr{K}'$, $x' \neq 0$.

A $\mathscr{K}$-positive linear functional $\hat{x}'$ (i.e. $\langle x, \hat{x}' \rangle \geqq 0$ for $x \in \mathscr{K}$) is called strictly positive if $\langle x, \hat{x}' \rangle \neq 0$ whenever $x \in \mathscr{K}$, $x \neq 0$.

An operator $T \in [\mathscr{Y}]$ is called $\mathscr{K}$-positive, or shortly positive, whenever $T\mathscr{K} \subset \mathscr{K}$.

A positive operator $T \in [\mathscr{Y}]$ is called $\mathscr{K}$-irreducible, or simply irreducible, if for every couple $x \in \mathscr{K}$, $x \neq 0$, $x' \in \mathscr{K}'$, $x' \neq 0$, there exists a positive integer $p = p(x, x')$ such that $\langle T^p x, x' \rangle \neq 0$. This concept was introduced by I. Sawashima and T was called originally semi-non support in [8].

A set $\mathscr{H}' \subset \mathscr{K}'$ is called $\mathscr{K}$-total if the relations

$$\langle x, x' \rangle \geqq 0 \quad \text{for all} \quad x' \in \mathscr{H}'$$

imply that $x \in \mathscr{K}$.

A $\mathscr{K}$-positive operator $T \in [\mathscr{Y}]$ is called $\mathscr{K}$-stochastic (with e and $\mathscr{H}'$) if there is a quasiinterior element $e \in \mathscr{K}$ and a $\mathscr{K}$-total set $\mathscr{H}'$ such that

$$\langle Te, x' \rangle = \langle e, x' \rangle = 1 \quad \text{for all} \quad x' \in \mathscr{H}' .$$

Let $T \in [\mathscr{Y}]$. We denote by T' the transposed operator, i.e. $T'y' = x'$ if and only if $\langle Tx, y' \rangle = \langle x, x' \rangle$ for all $x \in \mathscr{Y}$.

Let $\mathscr{H} \subset \mathscr{K}$ have the following property: From $\langle x, x' \rangle \geqq 0$ for all $x \in \mathscr{H}$ it follows that $x' \in \mathscr{K}'$. We call T, where $T \in [\mathscr{Y}]$ is a $\mathscr{K}$-positive operator, $\mathscr{K}'$-stochastic (with e' and $\mathscr{H}$), if there is a strictly positive element $e' \in \mathscr{K}'$ such that

$$\langle x, T'e' \rangle = \langle x, e' \rangle = 1 \quad \text{for all} \quad x \in \mathscr{H} .$$

If T is $\mathscr{K}$-stochastic and $\mathscr{K}'$-stochastic simultaneously, then T is called $(\mathscr{K}, \mathscr{K}')$-stochastic. In particular, if $\mathscr{Y}$ is a Hilbert space, and $\mathscr{K} = \mathscr{K}'$, and $e = e'$, then a $(\mathscr{K}, \mathscr{K}')$-stochastic operator T is called doubly $\mathscr{K}$-stochastic.

Let $\mathscr{Y}$ be a Hilbert space with the inner product $(.,.)$. Let $T \in [\mathscr{Y}]$. Then T^* denotes the adjoint or hermitean adjoint operator to T, i.e. $T^*y = x$ if and only if $(x, z) = (y, Tz)$ for all $z \in \mathscr{Y}$.

To be able to formulate the promised analogue of the result due to Bell and Daellenbach mentioned above we consider a special class of $\mathscr{K}$-stochastic operators. Denote by $\mathfrak{P}$ the class of $\mathscr{K}$-positive operators in $[\mathscr{Y}]$ whose spectral radii are poles of the corresponding resolvent operators, i.e.

$$\mathfrak{P} = \{T \in [\mathscr{Y}] : T\mathscr{K} \subset \mathscr{K},\ r(T) \text{ is a pole of } R(\lambda, T)\} .$$

In what follows we use the Laurent expansion of the resolvent operator with $\lambda_0 = r(T)$ exclusively and $B_1, B_2, \dots$ are elements of the spectral decomposition of T belonging to $r(T)$.

Proposition 1. *Let $T \in \mathfrak{P}$ and let q be the multiplicity of $r(T)$ as a pole of $R(\lambda, T)$. Then B_q is a $\mathscr{K}$-positive operator.*

Proof. For λ real and $\lambda > r(T)$ one has

$$(\lambda - r(T))^q R(\lambda, T) = B_q + \sum_{k=1}^{q-1} (\lambda - r(T))^q B_k + \sum_{k=0}^{\infty} (\lambda - r(T))^{q+k} A_k ,$$

hence

$$B_q = \lim_{\substack{\lambda \to r(T) \\ \lambda > r(T)}} (\lambda - r(T))^q R(\lambda, T)$$

and the assertion is a consequence of the $\mathscr{K}$-positivity of $R(\lambda, T)$ for $\lambda > r(T)$ and the closedness of $\mathscr{K}$.

Proposition 2. *Let $T \in \mathfrak{P}$ be a $\mathscr{K}$-stochastic operator. Then the element e is an eigenvector of T corresponding to the eigenvalue $r(T) = 1$ and this eigenvalue is a simple pole of $R(\lambda, T)$, i.e. $B_2 = \Theta$.*

Proof. It follows from the relations $\langle Te, x' \rangle = \langle e, x' \rangle = 1$ for all $x' \in \mathscr{K}'$ that $Te - e$ and $e - Te$ are both in $\mathscr{K}$. By virtue of (iii) in the definition of a cone we obtain that $Te - e = 0$ and thus the value 1 is an eigenvalue of T with the eigenvector e. We shall show that $1 = r(T)$. Since $r(T)$ is a pole of $R(\lambda, T)$, $r(T)$ is an eigenvalue of T. By virtue of our assumption $B_q \neq \Theta$, where q is the multiplicity of $r(T)$ as a pole of $R(\lambda, T)$, and, according to Proposition 1, B_q is a $\mathscr{K}$-positive operator. Since e is a quasiinterior element of $\mathscr{K}$ we see that there is a $y' \in \mathscr{K}'$, for which

$$0 < \langle e, B_q' y' \rangle = \langle B_q e, y' \rangle$$

and consequently $B_q e \neq 0$. In case $r(T) \neq 1$ we would have [10, p. 299] $B_1 e = 0$. It follows that $B_1 e = e$ and $r(T) = 1$. It remains to show that $q = 1$. Let us assume that $q > 1$. Then

$$0 < \langle B_q e, y' \rangle = \langle B_{q-1}(Te - e), y' \rangle = 0\,.$$

This contradiction shows that the assumption $q \neq 1$ was false and hence $q = 1$; Proposition 2 is proved.

Similarly one proves the following dual assertion.

Proposition 3. *Let $T \in \mathfrak{P}$ be a $\mathscr{K}'$-stochastic operator. Then the element e' is an eigenfunctional of T' corresponding to the eigenvalue $r(T) = 1$ and this eigenvalue is a simple pole of $R(\lambda, T)$.*

We say that an operator $T \in [\mathscr{Y}]$ has property (S) if every $\lambda \in \sigma(T)$ for which $|\lambda| = r(T)$ is a pole of the resolvent operator $R(\lambda, T)$.

Proposition 4. *Let $T \in \mathfrak{P}$ have property (S) and let T be either a $\mathscr{K}$-stochastic operator or a $\mathscr{K}'$-stochastic operator. Then*

$$\lim \left\| \frac{1}{N} \sum_{k=1}^{N} T^k - B_1 \right\| = 0\,.$$

Proof. According to the generalized Pringsheim theorem concerning the power series with $\mathscr{K}$-positive elements [9] all singularities of $R(\lambda, T)$ on the circumference $|\lambda| = 1$ are simple poles. The result is then a consequence of the Cauchy theorem on residui of $R(\lambda, T)$ (see [6]).

We remark that a $\mathscr{K}$-irreducible operator $T \in \mathfrak{P}$ always has property (S) if $\mathscr{Y}$ is not only a Banach space but a Banach lattice (see [7], [5]).

Theorem B. *Let $\mathscr{Y}$ be a real Hilbert space and let $T \in [\mathscr{Y}]$, $T\mathscr{K} \subset \mathscr{K}$, $T^*\mathscr{K} \subset \mathscr{K}$, be a $\mathscr{K}$-stochastic operator (with e and $\mathscr{H}'$) such that its spectral radius is an isolated pole of the resolvent operator $R(\lambda, T)$. Then*

$$B_1^* e = B_1 e$$

if and only if T^* is $\mathscr{K}$-stochastic (with e and $\mathscr{H}'$).

Proof. The sufficiency of the condition is obvious. For if

$$\langle T^* e, x' \rangle = \langle e, x' \rangle = 1 \quad \text{for all} \quad x' \in \mathscr{H}',$$

then $T^* e = e$, whence $e = B_1^* e$.

The necessity follows easily from the relations

$$\langle T^* e, x' \rangle = \langle T^* B_1^* e, x' \rangle = \langle B_1^* e, x' \rangle = \langle e, x' \rangle = 1$$

valid for all $x' \in \mathscr{H}'$. Thus, Theorem B is proved.

In the particular case of irreducible $\mathscr{K}$-stochastic operators we have

Theorem C. *Let $\mathscr{Y}$ be a real Hilbert space, and let $T \in [\mathscr{Y}]$, $T\mathscr{K} \subset \mathscr{K}$, $T^*\mathscr{K} \subset \subset \mathscr{K}$, be an irreducible $\mathscr{K}$-stochastic operator (with e and $\mathscr{H}'$) such that its spectral radius is an isolated pole of the resolvent $R(\lambda, T)$. Then*

$$B_1^* = B_1$$

if and only if T^* is $\mathscr{K}$-stochastic (with e and $\mathscr{H}'$).

Proof. It is enough to show that $B_1^* e = B_1 e$ implies $B_1^* = B_1$. We have that

$$B_1 x = \Lambda(x)\, e\,,$$

where $\Lambda(x)$ is a bounded linear functional on $\mathscr{Y}$ which is nonnegative on $\mathscr{K}$. According to the Riesz representation theorem [10, p. 245] there is an element $v \in \mathscr{K}$ for which

$$\Lambda(x) = (x, v)\,.$$

Moreover, v is orthogonal to the set

$$\mathfrak{X} = \{x \in \mathscr{Y} : \Lambda(x) = 0\} = \{x \in \mathscr{Y} : B_1 x = 0\}\,.$$

Thus $v = B_1^* v$ and we have

$$B_1 x = (x, B_1^* v)\, e = (B_1 x, B_1^* v)\, e\,.$$

Since the range of B_1^* is one dimensional [8] we have that $B_1^* v = \alpha B_1^* e$ for some $\alpha > 0$.