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ON A CLASS OF LINEAR DIFFERENTIAL EQUATIONS OF ORDER n, $n \geqq 3$

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In the papers [1], [2] the transformation theory of linear differential equations (or equations, as we shall write for short) was worked out. In this paper some applications of that theory are given. Besides the results of that theory the notations introduced in it will be used. Further I will denote some open interval and it will be assumed $n \geqq 3$.

The simplest homogeneous equation of the n-th order is the equation

$$(1)\qquad (L^*(v) \equiv) \frac{\mathrm{d}^n v}{\mathrm{d}\xi^n} = 0\,.$$

By this equation and by the interval I the class C of all equations of the form

$$(2)\qquad (L(y) \equiv) \frac{\mathrm{d}^n y}{\mathrm{d}x^n} + \sum_{k=2}^{n} \binom{n}{k} p_k(x) \frac{\mathrm{d}^{n-k} y}{\mathrm{d}x^{n-k}} = 0\,, \quad p_k(x) \in C_0(I)\,, \quad k = 2, \ldots, n\,,$$

which are locally equivalent to the equation (1), is defined. As we shall see, the equations of this class have similar oscillatory properties as the equation (1). Further the class C will be studied in detail.[1])

The meaning of the equation (2) being locally equivalent to the equation (1) is given by the following definition ([2], Definition 5).

[1]) Originally the equations of the class C were called the equations with zero fundamental invariants. This title is mentioned in the paper [3], which follows up this paper. Since it was shown that in this case there is no need for the notion of fundamental invariants, this notion was omitted and the title of this paper was changed. The equations with zero fundamental invariants are dealt with in the papers [4], [5].

Definition 1. Equation (2) is called *locally equivalent to the equation* (1) if there exists a solution $u(x) \not\equiv 0$ of its accompanying equation of the 2-nd order such that on each interval $I_{2x} \subset I$, where $u(x) \neq 0$, the equivalence $(1)\, I_{1\xi} \sim (2)\, I_{2x}$, $I_{1\xi}$ being an interval, holds.

Recall that under *the accompanying equation of the 2-nd order* (in short the accompanying equation) is meant the equation

$$(3) \qquad \frac{d^2y}{dx^2} + \frac{3}{n+1}\, p_2(x)\, y = 0\,.$$

In the paper [2] the following properties of the equations belonging to the class C were proved:

1. If the equation (2) is of the class C and $r(x)$, $s(x)$ form a fundamental set for the equation (3), then on each interval I^*_{2x} in which $s(x) \neq 0$ the equivalence $(1)\, I^*_{1\xi} \sim (2)\, I^*_{2x}\{\xi(x), c/\sqrt{(|\xi'(x)|^{n-1})}\}$, where $\xi(x) = r(x)/s(x)$, $\xi(I^*_{2x}) = I^*_{1\xi}$ and $c \neq 0$ is a constant, is valid.

2. The coefficients $p_k(x)$ of the equation (2) of the class C fulfil the relation $p_k(x) \in C_{n-k}(I)$, $k = 2, \ldots, n$.

3. For each $f(x) \in C_{n-2}(I)$ there exists one and only one equation (2) of the class C whose coefficient $p_2(x) = f(x)$, $x \in I$. This equation shall be denoted by $I_n(y, f(x)) = 0$. Evidently $I_n(y, 0) = 0$ is identical with the equation $d^ny/dx^n = 0$.

4. Each equation of the class C is self-adjoint on I.

5. Let (3_f), (3_g) be the accompanying equations of the equation $I_n(y, f(x)) = 0$, $I_n(v, g(\xi)) = 0$, respectively. Then it holds: $I_n(v, g(\xi)) = 0\ I_{2\xi} \sim I_n(y, f(x)) = 0\ I_{1x}\{\xi(x), c/\sqrt{(|\xi'(x)|^{n-1})}\} \Leftrightarrow (3_g)\ I_{2\xi} \sim (3_f)\ I_{1x}\{\xi(x), c_1/\sqrt{(|\xi'(x)|)}\}$, $c \neq 0$, $c_1 \neq 0$ are constants.

By its properties 1–5 the class C is similar to the class of equations (3). In particular, the property 5 means that the equivalence of two equations of the class C is found if and only if their accompanying equations are equivalent.

The operator standing on the left side of the equation (1) is decomposable into a product of n equal factors. The same assertion is valid for the operator L on the left side of the equation of the class C. Here the decomposition into regular operators is considered ([1], p. 400).

Theorem 1. *Assume there exists a solution $s(x) \neq 0$, $x \in I$, of the equation* (3). *Then the equation* (2) *is of the class C if and only if the operator $[s(x)]^{2n} L$ is decomposable on I into a symbolic product of n equal factors*

$$(4) \qquad [s(x)]^{2n} L = L_1 \ldots L_1 L_1$$

defined by the relation

$$(5) \qquad L_1(y) = [s(x)]^2\, y' - (n-1)\, s'(x)\, s(x)\, y\,.$$

Proof. Let the equation (2) be of the class C. Then, by the property 1, (1) $I^*_{1\xi} \sim$ (2) $I\{\xi(x), t(x)\}$. From the decomposition of the operator L into the product of n equal operators L^*_1, $L^*_1(v) = \mathrm{d}v/\mathrm{d}\xi$ the decomposition

$$\frac{1}{[\xi'(x)]^n} L = L_1 \dots L_1 L_1 \tag{6}$$

into the product of n equal operators L_1 of the first order with the coefficient at their first derivative equal to $1/\xi'(x)$, follows by Corollary to Theorem 9, [1]. Here $L^*_1(v) = 0\ I^*_{1\xi} \sim L_1(y) = 0\ I\{\xi(x), t(x)\}$. From these two conditions and from the fact that $t(x) = c/\sqrt{(|\xi'(x)|^{n-1})}$ we obtain that

$$L_1(y) = \frac{1}{\xi'(x)} \left(y' + \frac{n-1}{2} \frac{\xi''(x)}{\xi'(x)} y \right). \tag{7}$$

If in the function $\xi(x) = r(x)/s(x)$ such a solution $r(x)$ of the equation (3) is chosen that the Wronskian of the functions $r(x)$, $s(x)$ is equal to -1, then (4) follows from (6) and (5) from (7).

Conversely, let the decomposition (4) on I exist, where the operator L_1 is given by (5). If we put $\xi'(x) = 1/[s(x)]^2$, $t(x) = [s(x)]^{n-1}$, $x \in I$, then $L^*_1(v) = 0\ I^*_{1\xi} \sim L_1(y) = 0\ I\{\xi(x), t(x)\}$. Further $\xi(x), t(x) \in C_n(I)$ and $1/([s(x)]^2\ \xi'(x)) = 1$. From this, by virtue of Corollary to Theorem 10, [1], follows (1) $I^*_{1\xi} \sim$ (2) $I\{\xi(x), t(x)\}$, and the theorem is proved.

Remark 1. Theorem 1 also follows from Theorem 4.4 and Remark 4.5a) in [4], p. 180.

If the non-regular operators are considered, then we get from Theorem 1

Corollary. *The equation* (2) *belongs to the class* C *if and only if, for an arbitrary solution* $s(x) \not\equiv 0$ *of the equation* (3) *there exists on* I *a decomposition of the operator* $[s(x)]^{2n} L$ *into the symbolic product* (4) *of* n *equal factors given by the relation* (5).

Theorem 2. *Let* $r(x)$, $s(x)$ *form on* I *a fundamental set for the equation* (3). *Then the equation* (2) *is of the class* C *if and only if the functions*

$$[r(x)]^{n-k}\,[s(x)]^{k-1}\,, \quad k = 1, \dots, n \tag{8}$$

form a fundamental set on I *for this equation.*

Proof. Let the equation (2) be of the class C and let $I^*_{2x} \subset I$ be an arbitrary interval on which $s(x) \neq 0$ and whose each endpoint is either a zero of $s(x)$ or an endpoint of I. Then, by virtue of property 1, the functions

$$\frac{[\xi(x)]^{n-k}}{\sqrt{(|\xi'(x)|^{n-1})}}\,, \quad k = 1, \dots, n\,, \tag{9}$$

form a fundamental set for (2) on I^*_{2x}. Here $\xi(x) = r(x)/s(x)$. From this we obtain that the functions (8) also form a fundamental set for the equation (2) on I^*_{2x} as well as on the union of all intervals I^*_{2x}. From the continuity of the coefficients of the equation (2) and from the fact that $r(x), s(x) \in C_n(I)$ it follows that the functions (8) satisfy the equation (2) at the common endpoints of the intervals I^*_{2x}, too.

Let the functions (8) form a fundamental set for the equation (2) on I and let I^*_{2x} have the same meaning as before. We put $\xi(x) = r(x)/s(x)$, $t(x) = 1/\sqrt{(|\xi'(x)|^{n-1})}$. Evidently $\xi(x)$, $t(x)$ have all properties of the carrier of equivalence. Since $y_k(x)$ are equal, up to a multiplicative constant, to the functions (9), the fundamental set $v_k(\xi) = \xi^{n-k}$, $k = 1, \ldots, n$ for the equation (1) is transformed into a basis of solutions of the equation (2) on I_{2x}. From this it follows (1) $I^*_{1\xi} \sim$ (2) I^*_{2x}, and thus the equation (2) is shown to be of the class C.

Remark 2. Theorem 2 also follows from Lemma 4.1 in [4], p. 179.

In what follows, we shall take the same basis $r(x)$, $s(x)$ of solutions of the equation (3). Then from the Theorem 2 it follows that the general solution of the equation (2) belonging to the class C can be written in the form

$$y(x) = \sum_{k=1}^{n} c_k [r(x)]^{n-k} [s(x)]^{k-1}, \tag{10}$$

$c_1, \ldots, c_n$ are constants. To this expression the polynomial

$$P_y(\varrho) = \sum_{k=1}^{n} c_k \varrho^{n-k} \tag{11}$$

of the degree at most $n - 1$ may be associated in a one-to-one manner. This polynomial will be called *the auxiliary polynomial*. It will be said to be *dominating* if it is exactly of the degree $n - 1$. The set of all solutions of the equation (2) belonging to the class C is isomorphic to the set of all polynomials which are of the degree at most $n - 1$. From now on, we shall not consider the trivial solution of the equation (2) and its corresponding polynomial.

From the relation between the function (10) and the polynomial (11) it follows that this function can be similarly factorated as its auxiliary polynomial.

Lemma 1. *The auxiliary polynomial $P_y(\varrho)$ is of the degree $n - l$, $1 \leq l \leq n$ and can be factorated into the factors*

$$P_y(\varrho) = \sum_{k=l}^{n} c_k \varrho^{n-k} = c_l(\varrho - \varrho_1) \ldots (\varrho - \varrho_{n-l}), \quad c_l \neq 0, \tag{12}$$

if and only if the function y given by the relation (10) can be written in the form

$$y(x) = c_l [s(x)]^{l-1} . [r(x) - \varrho_1 s(x)] \ldots [r(x) - \varrho_{n-l} s(x)] . \tag{13}$$

Proof. If (12) is valid, then $y(x) = [s(x)]^{l-1} \sum_{k=l}^{n} c_k [r(x)]^{n-k} [s(x)]^{k-l}$. Further, on the intervals I_{2x}^* where $s(x) \neq 0$ we can write

$$y(x) = [s(x)]^{n-1} \sum_{k=l}^{n} c_k (r(x)/s(x))^{n-k} =$$
$$= [s(x)]^{n-1} \{c_l[(r(x)/s(x)) - \varrho_1] \dots [(r(x)/s(x)) - \varrho_{n-l}]\} =$$
$$= c_l [s(x)]^{l-1} [r(x) - \varrho_1 s(x)] \dots [r(x) - \varrho_{n-l} s(x)] .$$

From the continuity of the functions on both sides of the equation (13) this equality is valid at the zeros of $s(x)$, too.

The converse implication can be proved on the interval I_{2x}^*.

Corollary. *The product of exactly $n - 1$ solutions of the equation* (3) *is a solution of the equation* (2) *belonging to the class C.*

Proof. The product $y^*(x)$ of exactly $n - 1$ solutions of the equation (3) can be written in the form (13), where $1 \leqq l \leqq n$, $c_l \neq 0$, $\varrho_1, \dots, \varrho_{n-l}$ (if $l = n$, $y^*(x) = c_n[s(x)]^{n-1}$) are some numbers. Let $\sum_{k=l}^{n} c_k \varrho^{n-k}$ be a polynomial with $\varrho_1, \dots, \varrho_{n-l}$ being its roots. By Lemma 1, the equality $y^*(x) = \sum_{k=l}^{n} c_k [r(x)]^{n-k} [s(x)]^{k-1}$ holds, which implies, with respect to Theorem 2, the assertion.

Remark 3. Corollary also follows from Corollary to Lemma 5, [5], p. 31.

From the Lemma also follows

Theorem 3. *The auxiliary polynomial* (12) *of the solution $y(x)$ of the equation* (2) *belonging to the class C has a decomposition*

$$P_y(\varrho) = c_l(\varrho - \varrho_1) \dots (\varrho - \varrho_m)(\varrho^2 + \alpha_1 \varrho + \beta_1) \dots (\varrho^2 + \alpha_q \varrho + \beta_q) \tag{12'}$$

where $\varrho_1, \dots, \varrho_m, \alpha_1, \beta_1, \dots, \alpha_q, \beta_q$ are real numbers, not necessarily different and $\beta_1 - \alpha_1^2/4 > 0, \dots, \beta_q - \alpha_q^2/4 > 0$, $m + 2q = n - l$ if and only if the function y can be written in the form

$$y(x) = c_l[s(x)]^{l-1} [r(x) - \varrho_1 s(x)] \dots [r(x) - \varrho_m s(x)] \, . \tag{13'}$$
$$. \left\{\left[r(x) + \frac{\alpha_1}{2} s(x)\right]^2 + (\beta_1 - \alpha_1^2/4)[s(x)]^2\right\} \dots$$
$$\dots \left\{\left[r(x) + \frac{\alpha_q}{2} s(x)\right]^2 + (\beta_q - \alpha_q^2/4)[s(x)]^2\right\} .$$

Proof. In (12′) the expressions $\varrho^2 + \alpha_w \varrho + \beta_w$, $w = 1, \dots, q$ are obtained by multiplying the factors with the complex conjugate roots $\sigma_w \pm i\tau_w$. With them the factors of the form $r(x) - (\sigma_w + i\tau_w) s(x)$, $r(x) - (\sigma_w - i\tau_w) s(x)$ are associated, which by multiplying each other give $[r(x)]^2 + \alpha_w r(x) s(x) + \beta_w [s(x)]^2 = [r(x) + (\alpha_w/2) s(x)]^2 + (\beta_w - \alpha_w^2/4) [s(x)]^2$.

By a converse procedure the second part of the theorem will be proved.

With help of the mentioned Corollaries to Theorem 2 we shall prove some theorems concerning the zero-points of the solutions of the equation (2) belonging to the class C. Here the definition of the type for the equation (3) introduced in [6], p. 231, and the following definition will be used.

Definition 2. The equation (2) *is of the type* k $(\geqq 1)$ on the interval $I_1 \subset I$ if each of its solutions has at most k zeros on I_1 and there exists a solution of this equation having exactly k zeros on I_1. Here each zero-point is counted as many times as its multiplicity indicates.

Theorem 4. *The equation* (2) *of the class C is of the type $k(n - 1)$ on the interval $I_1 \subset I$ if and only if the equation* (3) *is of the type k on I_1.*

Proof. If (3) is of the type k on I_1, then every solution of the equation (2) of the class C being of the form (13′), by the Theorem 3, has at most $k(n - 1)$ zero-points on I_1 and at the same time there exists a solution of this equation having exactly $k(n - 1)$ zero-points on I_1. Thus a one-to-one correspondence between the type of the equation (3) and that of the equation (2) is given.

Corollary. *If the equation* (2) *of the class C is non-oscillatory on the interval I_1 (that means if it is of the type $n - 1$ on I_1), then there exists a non-vanishing on I_1 solution of this equation.*

From the Theorem 3 immediately follows

Theorem 5. *Let the equation* (2) *of the class C be of odd order. Then each solution of this equation, whose auxiliary polynomial is dominating and having only complex conjugate roots, is non-vanishing on I.*

Theorem 6. *Let the equation* (2) *of the class C be of even order. Then it has a non-vanishing solution on I if and only if it is non-oscillatory on I.*

Proof. Let the solution $y(x)$ of this equation have no zero-point on I. If its auxiliary polynomial is dominating, then this polynomial is of odd degree and has at least one root ϱ_1. In the decomposition (13′) a non-vanishing on I factor $r(x) - \varrho_1 s(x)$ corresponds to it. Therefore, both the equation (3) and the equation (2) of the class C are non-oscillatory on I. If the auxiliary polynomial of the solution $y(x)$ is not dominating, then in the decomposition (13′) the factor $[s(x)]^{l-1}$, $l \geqq 2$ without zeros on I occurs and the assertion is valid again.

The second part of the Theorem was stated in Corollary to Theorem 4.

Remark 4. Theorem 5 strengthens the Remark 7 in [5], p. 32. Theorem 6 completes Theorem 1 in [5], p. 31.

Let now $a \in I$ be an arbitrary point. Consider the set $M(a)$ of all solutions $y(x)$ of the equation (2) of the class C having a zero at a. The zero-points of the solution $y(x) \in M(a)$ situated on the right (on the left) from the point a, provided they exist, can be enumerated. On the right let they be the points

$$a \leqq a_1 \leqq a_2 \leqq \ldots \leqq a_{n-1} \leqq a_n \leqq \ldots \leqq a_{2(n-1)} \leqq a_{2n-1} \leqq \ldots \leqq a_{3(n-1)} \leqq \ldots$$

and on the left

$$\ldots \leqq a_{-3(n-1)} \leqq \ldots \leqq a_{-(2n-1)} \leqq a_{-2(n-1)} \leqq \ldots \leqq a_{-n} \leqq a_{-(n-1)} \leqq \ldots \leqq$$
$$\leqq a_{-2} \leqq a_{-1} \leqq a\,.$$

Each point is counted as many times as its multiplicity indicates. If, e.g., the point a is a k-tuple zero-point of the solution $y(x)$, then $a = a_1 = \ldots = a_{k-1} < a_k$ (if there exists the last point) and $a_{-k} < a_{-(k-1)} = \ldots = a_{-1} = a$ (if the point a_{-k} exists).

Definition 3. Let $y(x) \in M(a)$, and let k be a natural number. Then the point $a_{k(n-1)}(a_{-k(n-1)})$ will be called *the k-th successive (the k-th preceding) conjugate point to the point a of the solution* $y(x)$. Further the k-th successive (the k-th preceding) conjugate point to the point a of the equation (2) belonging to the class C will be defined as *the lower bound* of the k-th successive (as *the upper bound* of the k-th preceding) conjugate points to the point a of all solutions $y(x) \in M(a)$.

If there is no solution $y(x) \in M(a)$ having the k-th successive (the k-th preceding) conjugate point to the point a, then we shall say that there does not exist the k-th successive (the k-th preceding) conjugate point to the point a of the equation (2) belonging to the class C.

Theorem 7. *Let the equation* (2) *be of the class C, let k be a natural number, and let* $y_0(x)$ *be the solution of the considered equation having* $(n - 1)$*-tuple zero-point at the point* $a \in I$. *Then the k-th successive (the k-th preceding) conjugate point to the point a of the equation* (2) *exists if and only if there exists the k-th successive (the k-th preceding) conjugate point to the point a of the solution* $y(x)$. *If both the points exist, then they are equal to each other.*

Proof. The theorem will be proved only for the successive conjugate points. The case of the preceding conjugate points can be dealt with similarly. Each solution $y(x) \in M(a)$ contains a factor $[\widetilde{s(x)}]^k$, $1 \leqq k \leqq n - 1$, $\widetilde{s(x)}$ being a solution of the equation (3) with a simple zero a, in its decomposition (13′). Here $[\widetilde{s(x)}]^{n-1}$ appears if and only if $y(x)$ are linearly dependent with $y_0(x)$. Therefore the k-th successive conjugate point a_k to the point a of the solution $y_0(x)$ is identical with the k-th successive conjugate point to the point a of the solution $\widetilde{s(x)}$. Since the solutions $y(x) \in M(a)$ are not multiples of $y_0(x)$, the product of at most $n - 1$ solutions of the accompanying equation may appear in the decomposition (13′). From this, by virtue of the Separation Theorem, the statement of the theorem follows.

From the proof it can be easily seen the following

Corollary. *The conjugate points of the equation* (2) *of the class C are identical with the conjugate points of its accompanying equation* (3).

From the proof of the last theorem it also follows that except the solution $y_0(x)$, all solutions $y(x) \in M(a)$ being products of $n - 1$ solutions of the equation (3) have the same conjugate points as the equation (2). Further, we see that all zeros of the solution $y_0(x)$ are $(n - 1)$-tuple, whereby for even n the values of the function $y_0^{(n-1)}(x)$ change their sign at these points while for n odd they are of the same sign.

From the Separation Theorem, using Theorem 3, we get that the behaviour of the zero-points of the other solutions $y(x)$ of the equation (2) belonging to the class C is the same between two successive zeros of the solution $y_0(x)$, that is, if a, b and c, d are two pairs of the neighbouring zero-points of the solution $y_0(x)$, then $y(x)$ has the same number of zeros on (a, b) as on (c, d), whereby the order of the multiplicity of the zeros is the same on both the intervals. In a similar manner we obtain this generalized separation theorem.

Theorem 8. *Let the equation* (2) *be of the class C and let $y_0(x)$ be its arbitrary solution with $(n - 1)$-tuple zeros. Then in the case of even n every solution $y(x)$ of the equation* (2) *that is not a multiple of $y_0(x)$ must vanish between any two successive zeros of $y_0(x)$. If n is odd, this statement holds for any solution $y(x)$ of the equation* (2) *with at least one zero-point on I.*

Let us consider the n-th order equation with constant coefficients defined for n even by the symbolic product

$$(14_e)\quad \left[\frac{d^2}{d\xi^2} + (n-1)^2\right]\left[\frac{d^2}{d\xi^2} + (n-3)^2\right]\dots\left[\frac{d^2}{d\xi^2} + 1^2\right]v = 0\,, \quad \xi \in (-\infty, \infty)\,,$$

and for n odd by the relation

$$(14_o)\quad \left[\frac{d^2}{d\xi^2} + (n-1)^2\right]\left[\frac{d^2}{d\xi^2} + (n-3)^2\right]\dots\left[\frac{d^2}{d\xi^2} + 2^2\right]\frac{d}{d\xi}\,v = 0\,, \xi \in (-\infty, \infty)\,.$$

The general solution of the equation (14_e) is

$$(15_e)\qquad v(\xi) = \sum_{k=1}^{n/2}\left[c_k \cos(2k-1)\,\xi + d_k \sin(2k-1)\,\xi\right],$$

c_k and d_k are constants, while that of (14_o) is

$$(15_o)\qquad v(\xi) = c_0 + \sum_{k=1}^{(n-1)/2}\left[c_k \cos 2k\xi + d_k \sin 2k\xi\right],$$

c_k and d_k are constants. From the definition of the equations (14_e), (14_o) it follows that they are of the form

$$(2') \qquad \frac{d^n v}{d\xi^n} + \sum_{k=2}^{n} \binom{n}{k} q_k(\xi) \frac{d^{n-k} v}{d\xi^{n-k}} = 0 .$$

With respect to the equalities

$$\sum_{k=1}^{n/2} (2k-1)^2 = \binom{n+1}{3}, \ n \text{ is even}, \quad \sum_{k=1}^{(n-1)/2} (2k)^2 = \binom{n+1}{3}, \ n \text{ is odd},$$

we obtain $q_2(\xi) = (n+1)/3$. Thus the accompanying equation of the equations (14_e), (14_o) is the equation

$$(3') \qquad \frac{d^2 v}{d\xi^2} + v = 0 .$$

Further it is true that the equations (14_e), (14_o) are of the class C on the interval $(-\infty, \infty)$. In fact, because of the trigonometrical identity

$$(\cos \xi)^{n-k} (\sin \xi)^{k-1} = \operatorname{Re} \left\{ \frac{1}{2^{n-1}} \frac{1}{i^{k-1}} \sum_{l=0}^{n-k} \sum_{m=0}^{k-1} (-1)^m \binom{n-k}{l} \binom{k-1}{m} e^{i(n-1-2m-2l)\xi} \right\},$$
$$k = 1, \ldots, n$$

we have that with the functions (15_e), (15_o) the functions

$$(16) \qquad (\cos \xi)^{n-k} (\sin \xi)^{k-1}, \quad k = 1, \ldots, n$$

are the solutions of (14_e), (14_o), too. Conversely, by making linear combinations from the system (16) we get the functions $(\cos \xi)^{n-2l-k} (\sin \xi)^{k-1}$, $k = 1, \ldots, n-2l$ for $l = 0, 1, \ldots, (n/2) - 1$, if n is even and $l = 0, 1, \ldots, (n-1)/2$ if n is odd, respectively. From these functions, by means of the relation

$$\cos (n-1-2l)\xi + i \sin (n-1-2l)\xi =$$
$$= \sum_{k=1}^{n-2l} \binom{n-1-2l}{k-1} i^{k-1} (\cos \xi)^{n-2l-k} (\sin \xi)^{k-1}$$

the functions (15_e), (15_o) can be obtained. Thus the functions (16) form a fundamental set for the equation (14_e), (14_o), respectively. By Theorem 2 these equations belong to the class C on $(-\infty, \infty)$. From this, in virtue of the property 5 of the equations belonging to the class C and of the equivalence $(3')\, I_1 \sim (3)\, I$, I_1 is an interval, it follows

Theorem 9. *If the equation (2) is of the class C and n is even (n is odd), then there exists an interval I_1 such that $(14_e)\, I_1 \sim (2)\, I$ $((14_o)\, I_1 \sim (2)\, I)$.*

With regard to Definition 1 and Theorem 9 the equation (1) and the equation (14_e), (14_o) will be called *a typical non-oscillatory* and *oscillatory equation of the class C*, respectively.

Consider the nonhomogeneous equation

$$L(y) = p_{n+1}(x)\,, \quad p_{n+1}(x) \in C_0(I)\,, \tag{17}$$

whose corresponding homogeneous equation (2) is of the class C.

Assume the equation (2) is non-oscillatory on I and $p_{n+1}(x)$ has exactly k zeros on this interval, $0 \leqq k < +\infty$. Then $(1)\, I_1 \sim (2)\, I\{\xi(x), t(x)\}$, I_1 is an interval. If $x(\xi)$ is the inverse function to $\xi(x)$, $t_1(\xi) = 1/t[x(\xi)]$ and

$$\frac{d^n v}{d\xi^n} = [x'(\xi)]^n\, t_1(\xi)\, p_{n+1}[x(\xi)]\,, \quad \xi \in I_1\,, \tag{18}$$

then in virtue of Theorems 1 a 8, [1], $(17)\, I \sim (18)\, I_1\{x(\xi), t_1(\xi)\}$ holds. The right side of (18) has exactly k zero-points. Using Rolle's Theorem successively we get that every solution of this equation has at most $k + n$ zeros in I_1. The same is true for the solutions of the equation (17) on I. This completes the proof of

Theorem 10. *Let the corresponding homogeneous equation of the equation* (17) *be of the class C and let be non-oscillatory on I. Let $p_{n+1}(x)$ have exactly k zeros in I. Then every solution of the equation* (17) *has at most $k + n$ zeros.*

Consider now the case that the equation (2) being of the class C is oscillatory on I, that is, there exists at least one its solution having infinitely many zeros in I. Then, from Theorem 3 it follows that for n even every solution of the equation (2) has infinitely many zeros, while in the case of n odd this is true for each solution having at least one zero-point.

Suppose n is even and the function $p_{n+1}(x)$ is given by the relation

$$p_{n+1}(x) = f(x)\, u(x) \tag{19}$$

where $f(x) \in C_0(I)$, $f(x) \neq 0$, $x \in I$, and $u(x)$ is a solution of (2) with $(n-1)$-tuple zeros. If we denote

$$\left[\frac{d^2}{d\xi^2} + (n-1)^2\right]\left[\frac{d^2}{d\xi^2} + (n-3)^2\right]\cdots\left[\frac{d^2}{d\xi^2} + 1^2\right] v = g(\xi)\, s(\xi)\,, \tag{20}$$

where $g(\xi) = [x'(\xi)]^n f[x(\xi)]$, $s(\xi) = t_1(\xi)\, u[x(\xi)]$, $\xi \in I_1$, then from the Theorem 9 we get $(17)\, I \sim (20)\, I_1\{x(\xi), t_1(\xi)\}$. Here $x(\xi)$ is the inverse function to $\xi(x)$, $t_1(\xi) = 1/t[x(\xi)]$, whereby $(14_e)\, I_1 \sim (2)\, I\{\xi(x), t(x)\}$. With regard to the meaning of the functions $x(\xi)$, $t_1(\xi)$ the function $s(\xi)$ represents a solution of (14_e) having infinitely many $(n-1)$-tuple zero-points.

The oscillatory properties of the solutions of the equation (20) will be dealt with on the basis of a Comparison Theorem which is a generalization of the classical Sturm's Theorem. For this purpose let us consider two equations of the $2n$-th order

$$(21)\qquad u^{(2n)} + a_2 u^{(2n-2)} + a_4 u^{(2n-4)} + \dots + a_{2n-2} u'' + \varphi_{2n}(x)\, u = \varphi_{2n+1}(x)$$

$$(22)\qquad v^{(2n)} + a_2 v^{(2n-2)} + a_4 v^{(2n-4)} + \dots + a_{2n-2} v'' + \psi_{2n}(x)\, v = \psi_{2n+1}(x)\,,$$

in which $a_2, \dots, a_{2n-2}$ are some constants, $\varphi_{2n}(x), \varphi_{2n+1}(x), \psi_{2n}(x), \psi_{2n+1}(x) \in C_0(I_1)$, I_1 is an interval, $u^{(l)} = \mathrm{d}^l u/\mathrm{d}x^l$, $v^{(l)} = \mathrm{d}^l v/\mathrm{d}x^l$, $l = 1, \dots, 2n$. Let $u(x)$ be a solution of the former equation and let $v(x)$ be a solution of the latter one. By simple combining these equations we get for $x \in I_1$

$$u(x)\, v^{(2n)}(x) - u^{(2n)}(x)\, v(x) + a_2[u(x)\, v^{(2n-2)}(x) - u^{(2n-2)}(x)\, v(x)] + \dots +$$
$$+ a_{2n-2}[u(x)\, v''(x) - u''(x)\, v(x)] + [\psi_{2n}(x) - \varphi_{2n}(x)]\, u(x)\, v(x) =$$
$$= \psi_{2n+1}(x)\, u(x) - \varphi_{2n+1}(x)\, v(x)\,.$$

Integrating this equality on the interval $\langle x_1, x_2 \rangle \subset I_1$, we come to the relation

$$[\sum_{k=0}^{n-1} (-1)^k \{u^{(k)}(x)\, v^{(2n-1-k)}(x) - u^{(2n-1-k)}(x)\, v^{(k)}(x)\}]_{x_1}^{x_2} +$$
$$+ a_2[\sum_{k=0}^{n-2} (-1)^k \{u^{(k)}(x)\, v^{(2n-3-k)}(x) - u^{(2n-3-k)}(x)\, v^{(k)}(x)\}]_{x_1}^{x_2} + \dots +$$
$$+ a_{2n-2}[u(x)\, v'(x) - u'(x)\, v(x)]_{x_1}^{x_2} + \int_{x_1}^{x_2} [\psi_{2n}(x) - \varphi_{2n}(x)]\, u(x)\, v(x)\, \mathrm{d}x =$$
$$= \int_{x_1}^{x_2} [\psi_{2n+1}(x)\, u(x) - \varphi_{2n+1}(x)\, v(x)]\, \mathrm{d}x\,.$$

Under the assumption that the solution $u(x)$ of the equation (21) fulfils the conditions

$$(23)\qquad \begin{aligned} u(x_1) &= u'(x_1) = \dots = u^{(2n-2)}(x_1) = 0\,, \quad u^{(2n-1)}(x_1) \neq 0 \\ u(x_2) &= u'(x_2) = \dots = u^{(2n-2)}(x_2) = 0\,, \quad u^{(2n-1)}(x_2) \neq 0 \end{aligned}$$

the last equality reduces to the form

$$(24)\qquad -u^{(2n-1)}(x_2)\, v(x_2) + u^{(2n-1)}(x_1)\, v(x_1) + \int_{x_1}^{x_2} [\psi_{2n}(x) - \varphi_{2n}(x)]\, u(x)$$
$$v(x)\, \mathrm{d}x = \int_{x_1}^{x_2} [\psi_{2n+1}(x)\, u(x) - \varphi_{2n+1}(x)\, v(x)]\, \mathrm{d}x\,.$$

From it we get the following Comparison Theorem:

Theorem 11. *Let the following conditions on the interval* $\langle x_1, x_2\rangle$ *be satisfied:*

1. $\varphi_{2n}(x) \leqq \psi_{2n}(x)$.
2. $\varphi_{2n+1}(x) = 0$.
3. *Let* $u(x)$ *be a solution of* (21) *satisfying the conditions* (23).
4. *Let* $\operatorname{sgn} u^{(2n-1)}(x_1) \neq \operatorname{sgn} u^{(2n-1)}(x_2)$ *and if* $\varphi_{2n}(x) \not\equiv \psi_{2n}(x)$ *on* $\langle x_1, x_2\rangle$, *let* $u(x) \neq 0$, $x \in (x_1, x_2)$.
5. *Let* $\psi_{2n+1}(x) = f(x)\,u(x)$, $f(x) \in C_0\langle x_1, x_2\rangle$.

Then, if $f(x) \geqq 0\ (\leqq 0)$ *on* $\langle x_1, x_2\rangle$ *and if there exists a solution* $v(x)$ *of* (22) *such that* $v(x) \neq 0$, $x \in (x_1, x_2)$, $\operatorname{sgn} v(x) \neq \operatorname{sgn} u^{(2n-1)}(x_1)$ $(\operatorname{sgn} v(x) = \operatorname{sgn} u^{(2n-1)}(x_1))$, $x \in (x_1, x_2)$, *then the equation* (22) *is identical with the equation* (21) *and* $v(x_1) = v(x_2) = 0$.

Particularly, if $f(x) \equiv 0$ *on* $\langle x_1, x_2\rangle$ *and if there exists a non-vanishing on* (x_1, x_2) *solution* $v(x)$ *of the equation* (22), *then the equation* (22) *is identical with the equation* (21) *and* $v(x_1) = v(x_2) = 0$.

Proof. Consider the case $f(x) \geqq 0$ on $\langle x_1, x_2\rangle$. Then the right side of (24) being equal to $\int_{x_1}^{x_2} f(x)\,[u(x)]^2\,\mathrm{d}x$ is not negative. If $\varphi_{2n}(x) \not\equiv \psi_{2n}(x)$, then $u(x) > 0\ (<0)$ on (x_1, x_2). Simultaneously $u^{(2n-1)}(x_1) > 0\ (<0)$ and $u^{(2n-1)}(x_2) < 0\ (>0)$. Suppose there exists a solution $v(x)$ of (22) such that $v(x) < 0\ (>0)$ on (x_1, x_2). This implies that in (24) the first term is $\leqq 0$ and the second one <0. From the obtained contradiction the equality $\varphi_{2n}(x) = \psi_{2n}(x)$ on $\langle x_1, x_2\rangle$ follows. Then in (24) the first term must be equal to 0. This arises if and only if $v(x_1) = v(x_2) = 0$. Simultaneously it must be $\int_{x_1}^{x_2} f(x)\,[u(x)]^2\,\mathrm{d}x = 0$. This gives $f(x) = 0$ on $\langle x_1, x_2\rangle$, q.e.d.

The case $f(x) \leqq 0$ can be treated similarly.

By similar consideration Theorem 11′ will be proved.

Theorem 11′. *Let the assumptions* 1–5 *of Theorem* 11 *be satisfied. Then, if* $f(x) > 0\ (<0)$ *on* (x_1, x_2), *there does not exist the solution* $v(x)$ *of the equation* (22) *such that* $v(x)\,u^{(2n-1)}(x_1) \leqq 0\ (\geqq 0)$, $x \in \langle x_1, x_2\rangle$.

Corollary. *Let the following conditions be satisfied on* I_1:

1. $\varphi_{2n}(x) \leqq \psi_{2n}(x)$.
2. $\varphi_{2n+1}(x) = 0$.
3. *Let* $u(x)$ *be a solution of the equation* (21) *with* $(2n-1)$*-tuple zeros* x_m, $m = 1, 2, \ldots$, *on* I_1.
4. *Let for each* m $\operatorname{sgn} u^{(2n-1)}(x_m) \neq \operatorname{sgn} u^{(2n-1)}(x_{m+1})$ *and if* $\varphi_{2n}(x) \not\equiv \psi_{2n}(x)$, $x \in \langle x_m, x_{m+1}\rangle$, *then let* $u(x) \neq 0$, $x \in (x_m, x_{m+1})$.
5. *Let* $\psi_{2n+1}(x) = f(x)\,u(x)$, $f(x) \in C_0(I_1)$, $f(x) \neq 0$, $x \in I_1$. *Then every solution* $v(x)$ *of the equation* (22) *changes its sign in the interval* (x_m, x_{m+2}) *at least once,* $m = 1, 2, \ldots$