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A THEOREM ON CONTINUOUS DEPENDENCE ON A PARAMETER

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1. Notation and introduction. E_n denotes the set of all $x = [x_1, \ldots, x_n]$, with real x_i. By definition, $x \leqq y$ in E_n means that $x_i \leqq y_i$ for each $i = 1, \ldots, n$. We put $|x| = \max\{|x_1|, \ldots, |x_n|\}$, and similarly in $[t, x]$-space of all $[t, x_1, \ldots, x_n]$, with t real. For $a, b \in E_n$, $a = [a_1, \ldots, a_n]$, $b = [b_1, \ldots, b_n]$, $a_i < b_i$, $i = 1, \ldots, n$, (a, b) denotes the set $(a_1, b_1) \times \ldots \times (a_n, b_n)$. For ε real, $\vec{\varepsilon}$ denotes $[\varepsilon, \ldots, \varepsilon] \in E_n$. Measurability notions refer to the Lebesgue measure on the real line; a.e. stands for almost everywhere.

In this paper, we investigate differential equations

$$(\mathscr{E}_\mu) \qquad x' = f(t, x, \mu), \quad x(\tau) = \xi(\mu)$$

depending on a parameter μ in a metrical space, and prove that under some assumptions φ_μ, a solution of $(\mathscr{E}_\mu)$, depends continuously on μ. Here, for each fixed μ, any solution of $(\mathscr{E}_\mu)$ is considered in the sense of Carathéodory, i.e. it is an absolutely continuous function φ_μ on an interval I, with $\varphi'_\mu(t) = f(t, \varphi_\mu(t), \mu)$ holding a.e. on I.

The aim of this note is to prove a generalization of the following theorem.

Theorem 1. *Let D be a region of $[t, x]$-space, $[\tau, \xi] \in D$, $I_\mu = \{\mu \in E_k; |\mu - \mu_0| < c\}$, $c > 0$, $D_\mu = D \times I_\mu$. Let a function $[t, x, \mu] \to f(t, x, \mu)$ on D_μ with values in E_n fulfil the following conditions:*

(1.1) *for fixed $[x, \mu]$, the function $f(., x, \mu)$ is measurable*

(1.2) *for fixed $[t, \mu]$, the function $f(t, ., \mu)$ is continuous*

(1.3) *for fixed t, the function $f(t, ., .)$ is continuous at μ_0*

(1.4) *the differential equation*

$$x' = f(t, x, \mu_0), \quad x(\tau) = \xi$$

has a unique solution φ_0 on the interval $\langle a, b\rangle$, where $\tau \in \langle a, b\rangle$

(1.5) *there exists a summable function* m *on* $\langle a, b\rangle$ *such that* $|f(t, x, \mu)| \leqq m(t)$ *for each* $t \in \langle a, b\rangle$, $[t, x, \mu] \in D_\mu$.

Then there exists a $\delta > 0$ *such that, for any fixed* μ *satisfying* $|\mu - \mu_0| < \delta$, *every solution* φ_μ *of*

$$x' = f(t, x, \mu), \quad x(\tau) = \xi$$

exists over $\langle a, b\rangle$ (*in the sense that if* φ_μ *is a solution of the above equation on an interval* $\langle c, d\rangle \subset \langle a, b\rangle$, *then there exists an extension of* φ_μ *to the whole of* $\langle a, b\rangle$ *as a solution of this equation*), *and as* $\mu \to \mu_0$, $\varphi_\mu \to \varphi_0$ *uniformly over* $\langle a, b\rangle$.

Proof. See [1], Chapter II, theorem 4.2.

2. We show that (1.5) in the preceding theorem may be weakened. First, we prove the following generalization of the Lebesgue convergence theorem.

Theorem 2. *Let* f_m, g_m, h_m, $m = 0, 1, 2, \ldots$, *be measurable functions defined on* $\langle a, b\rangle$, *and let them fulfil the following conditions*:

(2.1) $g_m \leqq f_m \leqq h_m$ *a.e. on* $\langle a, b\rangle$ *for* $m = 1, 2, \ldots$

(2.2) $g_m \to g_0$, $f_m \to f_0$, $h_m \to h_0$ *a.e. on* $\langle a, b\rangle$ *for* $m \to \infty$

(2.3) g_m, h_m, $m = 0, 1, 2, \ldots$, *are integrable on* $\langle a, b\rangle$

(2.4) $\int_a^b g_m \to \int_a^b g_0$, $\int_a^b h_m \to \int_a^b h_0$.

Then $\int_a^b f_m \to \int_a^b f_0$.

Proof. From Fatou's lemma we get $\int_a^b (f_0 - g_0) = \int_a^b \lim (f_m - g_m) \leqq$ $\leqq \liminf \int_a^b (f_m - g_m) = \liminf \int_a^b f_m - \int_a^b g_0$; hence $\int_a^b f_0 \leqq \liminf \int_a^b f_m$. Passing to opposite functions, we obtain $\int_a^b f_0 \geqq \limsup \int_a^b f_m$.

Remark. The proof remains true for more general spaces and integrals.

3. We pass to a theorem on continuous dependence on a parameter, first on product sets in $[t, x]$-space. In what follows, Π denotes a metrical space of parameters with distance $|\mu - \nu|$, $\mu_0 \in \Pi$, τ is a real number, $\alpha > 0$. Further, for any $\vartheta > 0$, $\Pi(\vartheta) = \{\mu \in \Pi;\ |\mu - \mu_0| < \vartheta\}$.

Theorem 3. *Let* G *be a region in* E_n, ξ *be a mapping of* Π *into* G *continuous* at μ_0; *put* $\xi_0 = \xi(\mu_0)$. *Let* $[t, x, \mu] \to f(t, x, \mu)$ *be a function defined on* $\langle \tau, \tau + \alpha\rangle \times G \times \Pi$ *with values in* E_n *such that*

(3.1) *for each* $[x, \mu] \in G \times \Pi$, *the function* $f(., x, \mu)$ *is measurable on* $\langle \tau, \tau + \alpha\rangle$

(3.2) *for each* $[t, \mu] \in \langle \tau, \tau + \alpha\rangle \times \Pi$, *the function* $f(t, ., \mu)$ *is continuous on* G

(3.3) *for each* $\mu \in \Pi$, *there exist integrable functions* $m(., \mu)$, $M(., \mu)$ *on* $\langle \tau, \tau + \alpha\rangle$ *such that* $[t, x, \mu] \in \langle \tau, \tau + \alpha\rangle \times G \times \Pi$ *implies* $m(t, \mu) \leqq f(t, x, \mu) \leqq M(t, \mu)$

(3.4) *for almost all* $t \in \langle \tau, \tau + \alpha \rangle$, $m(t, .)$, $M(t, .)$ *are continuous at* μ_0

(3.5) *the systems* $\{L(., \mu)\}$, $\{U(., \mu)\}$, $\mu \in \Pi$, *where* $L(t, \mu) = \int_\tau^t m(., \mu)$, $U(t, \mu) = \int_\tau^t M(., \mu)$ *are equicontinuous on* $\langle \tau, \tau + \alpha \rangle$

(3.6) *for each* $t \in \langle \tau, \tau + \alpha \rangle$, $L(t, .)$, $U(t, .)$ *are continuous at* μ_0

(3.7) *for each* $t \in \langle \tau, \tau + \alpha \rangle$ *and each* $x \in G$, *the function* $f(t, ., .)$ *is continuous at* $[x, \mu_0]$

(3.8) *the equation*

$$(\mathscr{E}_{\mu_0}) \qquad x' = f(t, x, \mu_0), \quad x(\tau) = \xi_0$$

has a solution φ_0 *on* $\langle \tau, \tau + \alpha \rangle$, *which is unique in the following sense: if* ψ *is a solution of* $(\mathscr{E}_{\mu_0})$ *on* $\langle \tau, \tau + \beta \rangle$, $0 < \beta \leqq \alpha$, *then* $\psi(t) = \varphi_0(t)$ *for each* $t \in \langle \tau, \tau + \beta \rangle$.

Then given $\varepsilon > 0$, *there exists* $\delta > 0$ *such that for any* $\mu \in \Pi(\delta)$, *every solution* φ_μ *of* $(\mathscr{E}_\mu)$ *exists over* $\langle \tau, \tau + \alpha \rangle$, *and* $|\varphi_\mu(t) - \varphi_0(t)| < \varepsilon$ *for each* $t \in \langle \tau, \tau + \alpha \rangle$.

Proof. We say that the theorem holds on $\langle \tau, \tau + \beta \rangle$, if from the assumptions follows the assertion on $\langle \tau, \tau + \beta \rangle$ instead of $\langle \tau, \tau + \alpha \rangle$. First, we show the existence of such a $\beta > 0$. From (3.5) and continuity of ξ at μ_0 we infer that there exists $\beta > 0$ such that $\xi(\mu) + L(t, \mu) \in G$, $\xi(\mu) + U(t, \mu) \in G$ for each $t \in \langle \tau, \tau + \beta \rangle$ and, as we may suppose, each $\mu \in \Pi$; from the proof (with a slight modification) of the existence theorem for Carathéodory differential equation ([1], Chapter II, theorem 1.1) we get that all solutions φ_μ of $(\mathscr{E}_\mu)$, $\mu \in \Pi$, exist over $\langle \tau, \tau + \beta \rangle$. From (3.3) and (3.5) we infer that $\{\varphi_\mu\}$, $\mu \in \Pi$, are uniformly bounded and equicontinuous on $\langle \tau, \tau + \beta \rangle$. Let $\mu_k \to \mu_0$ for $k \to \infty$. Let φ_k be a solution of $(\mathscr{E}_{\mu_k})$ on $\langle \tau, \tau + \beta \rangle$, for $k = 1, 2, \ldots$ We prove that $\varphi_k \to \varphi_0$ uniformly on $\langle \tau, \tau + \beta \rangle$. By Ascoli's lemma, there exist $k_1, k_2, \ldots$ and ψ such that φ_{k_l} converge to ψ uniformly on $\langle \tau, \tau + \beta \rangle$, for $l \to \infty$. According to (3.7), $f(t, \varphi_{k_l}(t), \mu_{k_l})$ converge to $f(t, \psi(t), \mu_0)$ on $\langle \tau, \tau + \beta \rangle$ for $l \to \infty$ so that $\psi(t) = \lim \varphi_{k_l}(t) = \lim [\xi(\mu_{k_l}) + \int_\tau^t f(s, \varphi_{k_l}(s), \mu_{k_l}) = \xi_0 + \int_\tau^t f(s, \psi(s), \mu_0)$, by (3.4), (3.6) and theorem 2. We see that ψ is a solution of $(\mathscr{E}_{\mu_0})$ on $\langle \tau, \tau + \beta \rangle$; hence $\varphi_0 = \psi$ on $\langle \tau, \tau + \beta \rangle$, as it follows from (3.8). Pointwise convergence of $\{\varphi_k\}$ we prove by contradiction, and uniform convergence follows now from equicontinuity; this concludes the proof, as is easy to see.

We pass to the global assertion. Let

$$t_0 = \sup \{t \in \langle \tau, \tau + \alpha \rangle; \text{ the theorem holds on } \langle \tau, t \rangle\}.$$

Suppose first that $t_0 < \tau + \alpha$. We show that there exist $\Gamma > 0$ and $\vartheta_0 > 0$ such that, for each $\mu \in \Pi(\vartheta_0)$, every solution φ_μ of $(\mathscr{E}_\mu)$ exists over $\langle \tau, t_0 + \Gamma \rangle$. As $\varphi_0(t_0) \in G$, we have $\Delta > 0$ such that the cube with edge 8Δ and centre $\varphi_0(t_0)$ lies in G. By (3.5), we can choose $\eta > 0$ such that $t_1, t_2 \in \langle \tau, \tau + \alpha \rangle$, $|t_1 - t_2| \leqq \eta$, $\mu \in \Pi$ implies $|L(t_1, \mu) - L(t_2, \mu)| \leqq \Delta$, $|U(t_1, \mu) - U(t_2, \mu)| \leqq \Delta$. Thence $|\varphi_0(t_0 - \eta) - \varphi_0(t_0)| \leqq \Delta$, and for each $\mu \in \Pi$, the estimations $|L(t_0 - \eta, \mu) - L(t_0, \mu)| \leqq \Delta$, $|U(t_0 - \eta, \mu) -$

$- U(t_0, \mu)| \leqq \Delta$ hold. Now choose $\vartheta_0 > 0$ such that $\mu \in \Pi(\vartheta_0)$ implies $|\varphi_\mu(t_0 - \eta) - \varphi_0(t_0 - \eta)| \leqq \Delta$. Hence we have $|\varphi_\mu(t_0 - \eta) - \varphi_0(t_0)| \leqq 2\Delta$ for each $\mu \in \Pi(\vartheta_0)$, and it follows from the equation

$$x(t) = \varphi_\mu(t_0 - \eta) + \int_{t_0 - \eta}^{t} f(s, x(s), \mu) \tag{3.9}$$

that all solutions φ_μ, $\mu \in \Pi(\vartheta_0)$, may be extended over $\langle \tau, t_0 \rangle$, with $|\varphi_\mu(t_0) - \varphi_0(t_0)| \leqq \leqq 3\Delta$. Indeed, to prove this, it suffices to consider the inequality $\varphi_\mu(t_0 - \eta) + \int_{t_0 - \eta}^{t} M(., \mu) \leqq \varphi_0(t_0) + \overrightarrow{3\Delta}$, valid for $t \in \langle t_0 - \eta, t_0 \rangle$, $\mu \in \Pi(\vartheta_0)$, a similar estimation from below, and the existence theorem. Now, the possibility of an extension to $\langle \tau, t_0 + \Gamma \rangle$, for a suitable $\Gamma > 0$, follows from (3.5).

Suppose for simplicity that all solutions of $(\mathscr{E}_\mu)$ exist over $\langle \tau, t_0 + \Gamma \rangle$ for each $\mu \in \Pi$. We prove that the theorem holds on $\langle \tau, t_1 \rangle$, where $t_1 > t_0$. First, we prove that it is true on $\langle \tau, t_0 \rangle$. Let φ_μ be a solution of $(\mathscr{E}_\mu)$ on $\langle \tau, t_0 + \Gamma \rangle$. By the assumption, $\lim_{\mu \to \mu_0} \varphi_\mu(t) = \varphi_0(t)$ for each $t \in \langle \tau, t_0)$. Supposing $\lim_{\mu \to \mu_0} \varphi_\mu(t_0) = \varphi_0(t_0)$ is false, we can evidently choose a sequence $\{\mu_k\}$ converging to μ_0 such that $\lim_{k \to \infty} \varphi_{\mu_k}(t_0) = \kappa \neq \varphi_0(t_0)$. However, using equicontinuity, we get from

$$|\varphi_0(t_0) - \varphi_{\mu_k}(t_0)| \leqq |\varphi_0(t_0) - \varphi_0(t)| + |\varphi_0(t) - \varphi_{\mu_k}(t)| + |\varphi_{\mu_k}(t) - \varphi_{\mu_k}(t_0)| \tag{3.10}$$

a contradiction. Hence $\lim_{\mu \to \mu_0} \varphi_\mu(t_0) = \varphi_0(t_0)$, and by equicontinuity of the solutions φ_μ, $\{\varphi_\mu\}$ converges to φ_0 uniformly on $\langle \tau, t_0 \rangle$.

Now consider the equations

$$x(t) = \varphi_\mu(t_0) + \int_{t_0}^{t} f(s, x(s), \mu), \quad \mu \in \Pi \tag{3.11}$$

on $\langle t_0, t_0 + \Gamma \rangle$. Using the first part of this proof, as we clearly may, we get the validity of the theorem on $\langle \tau, t_1 \rangle$, $t_1 > t_0$; a contradiction. The proof of the case $t_0 = \tau + \alpha$ is included in the above considerations.

4. Before stating the main theorem of this article, we introduce the following notations. For $D \subset [t, x]$-space and $x \in E_n$, let $D^{[.,x]} = \{t \in E_1; [t, x] \in D\}$, and similarly for $D^{[t,.]}$. Further let $\mathrm{proj}_t D = \cup\{D^{[.,x]}; x \in E_n\}$, and similarly for $\mathrm{proj}_x D$.

Theorem 4. *Let D be a region of $[t, x]$-space, $\langle a, b \rangle \subset \mathrm{proj}_t D$, $\tau \in \langle a, b \rangle$, Π a metrical space of parameters, $\mu_0 \in \Pi$. Let ξ be a mapping of Π into $\mathrm{proj}_x D$ continuous at μ_0; put $\xi_0 = \xi(\mu_0)$ and suppose that $[\tau, \xi_0] \in D$. Let $[t, x, \mu] \to f(t, x, \mu)$ be a function on $D \times \Pi$ with values in E_n such that*

(4.1) *for each $x \in \mathrm{proj}_x D$, $\mu \in \Pi$, $f(., x, \mu)$ is measurable on $D^{[.,x]}$*

(4.2) *for each $t \in \mathrm{proj}_t D$, $\mu \in \Pi$, $f(t, ., \mu)$ is continuous on $D^{[t,.]}$*