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ON AN ORDERING OF THE VERTICES OF A GRAPH

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This paper contains some results on the ordering of the set of vertices of a connected graph.

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1.1. Let G be a set. Then $D(G)$ will denote the system of all two-point subsets of G,

$$D(G) = \{X; X = \{x, y\}, x \in G, y \in G, x \neq y\}.$$

For a two-element set $\{x, y\}$ we shall also use the symbol (x, y).

1.2. By *a graph* we shall understand a nonempty set G (the elements of which we shall call the *vertices* of the graph) together with a system $\varrho \subset D(G)$. This graph is then denoted by (G, ϱ). A two-point set $(a, b) \in \varrho$ will be called the *edge* of the graph (G, ϱ) connecting a and b. If $\varrho = D(G)$ we shall call the graph (G, ϱ) *complete*. If card $G = \aleph_0$, we call (G, ϱ) an enumerable graph. If card G is finite, then (G, ϱ) is called a finite graph.

1.3. Let (G_1, ϱ_1), (G_2, ϱ_2) be two graphs, $G_1 \subset G_2$, $\varrho_1 \subset \varrho_2$. Then we call (G_1, ϱ_1) *a subgraph* of (G_2, ϱ_2). If $\varrho_1 = \varrho_2 \cap D(G_1)$, we call (G_1, ϱ_1) *a saturated subgraph* of the graph (G_2, ϱ_2) (in greater detail: a saturated subgraph on the set G_1).

1.4. Let $G = \{a_1, \ldots, a_{n+1}\}$, $n \geqq 1$ integer, be an ordered set ($a_1 \leqq a_2 \ldots \leqq a_{n+1}$) with $n + 1$ elements. Let $\varrho = \{(a_1 a_2), (a_2 a_1), \ldots, (a_n, a_{n+1})\}$. Then the graph (G, ϱ) will be called a *path* connecting a_1 and a_{n+1}. The number n (in [2], p. 137^{12} should be $n - 1$ instead of n) is the length of this path. The path may be denoted in a simpler way by $(a_1, \ldots, a_{n+1})$. Let $n \geqq 2$ and $\varrho_1 = \varrho \cup \{(a_{n+1}, a_1)\}$. Then we call (G, ϱ_1) a *circle* with length $n + 1$. We may also denote it by $(a_1, \ldots, a_{n+1}, a_1)$. Let $G = \{a_1, a_2, \ldots\}$ be an enumerable set ordered in a sequence of type ω. Let $\varrho = \{(a_1, a_2), \ldots, (a_n, a_{n+1}), \ldots\}$. Then we call the graph (G, ϱ) an one-sided infinite path, and denote it by $(a_1, a_2, \ldots)$.

Let (G, ϱ) be a graph, $G_1 = \{a_1, \ldots, a_{n+1}\} \subset G$ a finite ordered subset. Let the path $(a_1, \ldots, a_{n+1})$ be a subgraph in (G, ϱ). Then we say that $(a_1, \ldots, a_{n+1})$ is the path connecting a_1 and a_{n+1} in (G, ϱ).

Similarly we can define circles in (G, ϱ).

1.5. A graph (G, ϱ) is called *connected* if each pair of vertices is connected by a path.

Let (G_1, ϱ_1) be a saturated subgraph of a graph (G, ϱ) such that 1) it is connected, 2) $x \in G - G_1 \Rightarrow (G_1 \cup \{x\}, \varrho \cap D(G_1 \cup \{x\}))$ is not a connected graph. Then (G_1, ϱ_1) is a *connected component* (or merely component) in (G, ϱ). In every connected graph a *metric* can be defined as follows: to every pair of different vertices a and b assign the number $\mu(a, b)$, which is the least length of paths in (G, ϱ) connecting a and b; for $a = b$ put $\mu(a, a) = 0$. If $A \subset G$, $B \subset G$, $A \neq \emptyset \neq B$, then $\mu(A, B) = \min_{x \in A, y \in B} \mu(x, y)$.

1.6. Let (G, ϱ) be a graph, $a \in G$. Let there exist just one $b \in G$ such that $(a, b) \in \varrho$. Then a will be called an *end-vertex* in (G, ϱ). Let (G, ϱ) be a connected graph. Let be $(a, b) \in \varrho$. Let $(G, \varrho - \{(a, b)\})$ be a not connected graph. Then we call (a, b) a *bridge* in (G, ϱ).

1.7. Let (G, ϱ) and (G_1, ϱ_1) be two graphs and let there exist a one-to-one mapping f of G onto G_1 such that for $a, b \in G$, $(a, b) \in \varrho$ is equivalent to $(f(a), f(b)) \in \varrho_1$. Then the graphs (G, ϱ) and (G_1, ϱ_1) will be called *isomorphic* and f an isomorphism between (G, ϱ) and (G_1, ϱ_1).

It is obvious that for the isomorphism f the following assertions hold:

a) The image of a path of length n is a path of length n.

b) The image of a circle of length n is a circle of length n.

c) If (G, ϱ) is connected then (G_1, ϱ_1) is also connected.

d) If (G, ϱ) is connected and the distance of a and b is n, then the distance in (G_1, ϱ_1) of the vertices $f(a)$ and $f(b)$ is also n.

e) The image of a component is a component.

f) The image of an end-vertex is an end-vertex.

g) The image of a bridge is a bridge.

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2.1. Let there be given a enumerable connected graph (G, ϱ). Let us order the set G in a sequence of type ω, $\pi = \{a_1, \ldots, a_n, \ldots\}$ (so for $a \in G$ there exists just one index n such that $a = a_n$). Let us denote the set of all these sequences by $\pi(G)$. For given π, let $p(\pi) = \{\mu(a_1, a_2), \mu(a_2, a_3), \ldots, \ldots\}$. Let $P(G, \varrho) = \{p(\pi); \pi \in \pi(G)\}$. Further on we shall deal with the structure of sequences π and the set $P(G, \varrho)$.

2.2. Let S be a set, $S \subset D(G)$. We shall say that S has a *finite basis* if there exists a finite set $N \subset G$ such that $(a, b) \in S \Rightarrow (a, b) \cap N \neq \emptyset$.

2.3. 1. *Let* (G, ϱ) *be a complete graph. Then* card $P(G, \varrho) = 1$.

2. *Let there exist in* $D(G, \varrho)$ *two systems* S *and* T *such that neither* S *nor* T *have finite basis and* $(a, b) \in S$, $(c, d) \in T \Rightarrow \mu(a, b) \neq \mu(c, d)$.

Then card $P(G, \varrho) = 2^{\aleph_0}$.

3. *If neither* 1 *nor* 2 *then card* $P(G, \varrho) = \aleph_0$.

Proof.

Ad 1. The assertion is obvious.

Ad 2. Let M be a set of all sequences $\{u_n\}$ consisting of zeros and ones (i.e. $u_n = 1$ or 0 for every positive integer n). As S and T do not have finite basis, they are infinite, and we can order G in a sequence $\pi = \{a_1, a_2, \ldots\} \in \pi(G)$ such that $(a_{4n}, a_{4n+1}) \in \in S \cup T$, $(a_{4n}, a_{4n+1}) \in S \Leftrightarrow u_n = 1$, $(a_{4n}, a_{4n+1}) \in T \Leftrightarrow u_n = 0$, where $\{u_n\} \in M$ is any given sequence. For two different sequences of M we obtain two different sequences of $P(G, \varrho)$ (these sequences differ in some member with index divisible by 4). As card $M = 2^{\aleph_0}$ and card $P(G, \varrho) \leqq 2^{\aleph_0}$, we conclude card $P(G, \varrho) = 2^{\aleph_0}$.

Ad 3. For every positive integer let $\tau(d)$ be the set of all those pairs of vertices of (G, ϱ), which have distance d (in the metric μ). First we shall show that under our suppositions there exists at least one d such that $\tau(d)$ does not have finite basis. Let $D \subset \subset D(G, \varrho)$ be a decomposition on G, i.e. $(a, b), (c, d) \in D$, $(a, b) \neq (c, d) \Rightarrow (a, b) \cap \cap (c, d) = \emptyset$, $\bigcup_D (a, b) = G$. Put $D(d) = D \cap \tau(d)$. Assume that all sets $D(d)$ are finite. Then we choose from each of these sets (if possible) one element and the system thus obtained (evidently infinite) may be divided into two infinite disjoint subsystems S and T. Then S and T satisfy the assumption of point 2 of our lemma, which is in contradiction with the assumption of point 3. So there exists a d_1 such that $D(d_1)$ is an infinite set. It follows from the definition of D that $D(d_1)$ does not have finite basis, and, because $D(d_1) \subset \tau(d_1)$, $\tau(d_1)$ also does not have finite basis. The set $D(G, \varrho) - \tau(d_1)$ is nonempty (this is obvious for $d_1 \neq 1$, for $d_1 = 1$ this follows from the assumption that (G, ϱ) is not a complete graph). As $(a, b) \in D(G) - \tau(d_1) \Rightarrow \mu(a, b) \neq d_1$, $(a, b) \in \tau(d_1) \Rightarrow \mu(a, b) = d_1$, therefore the set $D(G) - \tau(d_1)$ must have an finite basis N. Let $\{a_n\} \in \pi(G)$. Then there exists an m_1 such that $n > m_1 \Rightarrow a_n$ non $\in N$, and thus $\mu(a_n, a_{n+1}) = d_1$. Therefore every sequence from $P(G, \varrho)$ consists of d_1 with a finite number of exceptions. Thus card $P(G, \varrho) \leqq \aleph_0$. Furthermore, for every m there exists a sequence $\{a_n\} \in \pi(G)$ such that $\mu(a_1, a_2) = \mu(a_3, a_4) = \ldots = \mu(a_{2m-1}, a_{2m}) = d_1 \neq \mu(a_{2m+1}, a_{2m+2})$, and thus card $P(G, \varrho) = \aleph_0$.

2.4. In the following sections (2.4–2.12) let (G, ϱ) denote an enumerable connected graph, $\{a_n\} \in \pi(G)$. For every positive integer n let $(c_1, \ldots, c_j)$ be a path connecting a_n and a_{n+1} in (G, ϱ) (thus $a_n = c_1$, $a_{n+1} = c_j$), and having length $\mu(a_n, a_{n+1})$ (thus $\mu(a_n, a_{n+1}) = j - 1$). This path will be denoted by C_n. Note that, generally, $\mu(c_{i_1}, c_{i_2}) = i_2 - i_1$ for $1 \leqq i_1 \leqq i_2 \leqq j$. If $1 < i < j$ then we shall say that the vertex c_i is *skipped* over at the n-th step. For i, $1 \leqq i < n$, define $n(i)$ in the following manner: if a_i is skipped over at some m-th step for $m \geqq n$, denote by $n(i)$ the smallest such m.

Otherwise put $n(i) = n$. Denote $\operatorname{Max}_{0 \leqq i < n} (n(i)) + 1$ by $H(n)$. Let $M(n)$ denote the sum $\mu(a_n, a_{n+1}) + \ldots + \mu(a_{H(n)-1}, a_{H(n)})$.

2.5. Let $n \geqq 2$ be a positive integer, d a nonnegative integer. We shall say that the system of positive integers

$$u_1, u_2, \ldots, u_n$$

satisfies all *polygonal inequalities with defect d* if, for $1 \leqq i \leqq n$,

$$u_i \leqq u_1 + \ldots + u_{i-1} + u_{i+1} + \ldots + u_n + d\,.$$

If $d = 0$, we shall say that the given system satisfies all *polygonal inequalities.*

2.6. *Let d be a nonnegative integer. Let $u_{1,1}, \ldots, u_{1,n_1}; u_{2,1}, \ldots, u_{2,n_2}; u_{m,1}, \ldots, u_{m,n_m}$ be m systems of positive integers each of which satisfies all polygonal inequalities with defect d and let $d < m$. Then the system*

$$u_{1,1}, \ldots, u_{1,n_1}, \ldots, u_{m,1}, \ldots, u_{m,n_m}$$

satisfies all polygonal inequalities. In particular, if the systems of positive integers $u_1, \ldots, u_n$; $v_1, \ldots, v_m$ satisfy all polygonal inequalities, then the system $u_1, \ldots, u_n$, $v_1, \ldots, v_m$ satisfies all polygonal inequalities.

The proof is evident.

2.7. *Let n, (G, ϱ), C_n, $H(n)$ be the notions from 2.4, n fixed. Let $s \geqq H(n)$. For a_i, a_k $(i, k \leqq s)$, respectively, let $i \geqq n$ and $k \geqq n$ or $a_i(a_k)$ be skipped over at the i'-th (k'-th) step, where $i' \geqq n$ $(k' \geqq n)$ and $k \geqq n$ $(i \geqq n)$ or a_i be skipped over at the i'-th step and a_k at the k'-th step and $k' \geqq n$, $i' \geqq n$. Then*

$$\mu(a_i, a_k) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{s-1}, a_s)\,. \tag{1}$$

Proof.

a) If $i \geqq n$, $k \geqq n$, (1) is obvious.

b) Let be $i < n$, $k \geqq n$. For example let $n(i) < k$.
Then $\mu(a_i, a_k) \leqq \mu(a_{n(i)}, a_{n(i)+1}) + \ldots + \mu(a_{k-1}, a_k)$, from which (1) follows immediately. The proof for $n(i) \geqq k$ is analogous. So is the case for $k < n$, $i \geqq n$.

c) Let $i, k < n$ and e.g. $n(i) \leqq n(k)$. Then $\mu(a_i, a_k) \leqq \mu(a_{n(i)}, a_{n(i)+1}) + \ldots + \mu(a_{n(k)}, a_{n(k)+1})$, from which (1) follows immediately.

An analogous reasoning applies to $n(i) \geqq n(k)$.

2.8. *Let $s \geqq H(n)$, g a positive integer, $n \leqq g < s$. For all i, $n \leqq i < s$, with the eventual exception of $i = g$, let*

$$\mu(a_i, a_{i+1}) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{i-1}, a_i) + \mu(a_{i-1}, a_{i-2}) + \ldots + \mu(a_{s-1}, a_s) + 2\,.$$

Then there exists an $e > s$ such that for all i, $n \leqq i < e$, with the eventual exception of $i = g$, there holds

$$\mu(a_i, a_{i+1}) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{i-1}, a_i) + \mu(a_{i+1}, a_{i+2}) + \ldots + \mu(a_{e-1}, a_e) + 2\,. \tag{2}$$

Proof. Let $C_s = (c_1, \dots, c_j)$. Let us distinguish two cases:

a) If $\{c_2, \dots, c_{j-1}\} \subset \{a_1, \dots, a_s\}$ then, according to 2.7, $\mu(c_2, c_{j-1}) \leqq \mu(a_n, a_{n+1}) + \dots + \mu(a_{s-1}, a_s)$, $\mu(a_s, a_{s+1}) \leqq 2 + \mu(a_n, a_{n+1}) + \dots + \mu(a_{s-1}, a_s)$. It suffices then to put $e = s + 1$.

b) Let $\{c_2, \dots, c_{j-1}\}$ non $\subset \{a_1, \dots, a_s\}$. Let c_m be the first element in C_s such that c_m non $\in \{a_1, \dots, a_{s+1}\}$. Let $c_m = a_t$. Thus $t > s + 1$. According to 2.7 ($c_{m-1} \neq a_{s+1}$) $\mu(a_s, c_{m-1}) \leqq \mu(a_n, a_{n+1}) + \dots + \mu(a_{s-1}, a_s)$. From the fact that μ is a metric it follows that

$$\mu(c_m, a_{s+1}) \leqq \mu(a_{s+1}, a_{s+2}) + \dots + \mu(a_{t-1}, a_t).$$

Therefore

$$\mu(a_s, a_{s+1}) \leqq \mu(a_n, a_{n+1}) + \dots + \mu(a_{s-1}, a_s) + \mu(a_{s+1}, a_{s+2}) + \dots + \mu(a_{t-1}, a_t) + 1.$$

Hence (2) follows immediately for $i = s$ and $e = t$.

For integral i, $s + 1 \leqq i < t$, there holds

$$\mu(a_i, a_{i+1}) \leqq \mu(a_{i+1}, a_{i+2}) + \dots + \mu(a_{t-1}, a_t) + \mu(a_t, a_{s+1}) + \dots + \mu(a_{i-1}, a_i) \leqq \mu(a_{i+1}, a_{i+2}) + \dots + \mu(a_{t-1}, a_t) + \mu(a_s, a_{s+1}) + \dots + \mu(a_{i-1}, a_i).$$

(2) follows with t instead of e. It suffices to put $e = t$.

2.9. *Let all the suppositions of* 2.8. *be satisfied. Then there exists an L such that the system of numbers*

$$\mu(a_n, a_{n+1}), \dots, \mu(a_{L-1}, a_L)$$

satisfies all polygonal inequalities with defect 2.

Proof. It suffices to apply 2.8 m times, where $m \geqq \mu(a_g, a_{g+1})$.

2.10. *Let r, p be positive integers, $r \geqq p$, with the following properties*

1) $p - n - 1 \geqq M(n)$ (for the definition of $M(n)$ see 2.4).

2) $C_i \subset \{a_1, \dots, a_{r-1}\}$ for $n \leqq i < p$.

Then for $n \leqq i < p$,

$$(3) \qquad \mu(a_i, a_{i+1}) \leqq \mu(a_n, a_{n+1}) + \dots + \mu(a_{i-1}, a_i) + \mu(a_{i+1}, a_{i+2}) + \dots + \mu(a_{r-1}, a_r) + 2.$$

Proof. a) Let $n \leqq i < H(n)$. Then

$$\mu(a_i, a_{i+1}) \leqq M(n) \leqq \mu(a_n, a_{n+1}) + \dots + \mu(a_{i-1}, a_i) + \mu(a_{i+1}, a_{i+2}) \dots + \mu(a_{p-1}, a_p),$$

according to assumption 1. From this follows (3).

b) Let $H(n) \leqq i < p$. Let $C_i = (c_1, \dots, c_j)$. Define the function $\varphi(u)$ for $u = 1, \dots, j$ thus:

$$c_u = a_{n_u},\ n_u \geqq n \Rightarrow \varphi(u) = n_u, \quad c_u = a_{n_u},\ n_u < n \Rightarrow \varphi(u) = n(n_u).$$

Then $n \leqq \varphi(u) < r$ for all $u = 1, 2, \ldots, j$. For k, $n \leqq k < H(n)$, the number of those u for which $\varphi(u) = k$ at most equals $\mu(a_k, a_{k+1})$. (Thus card $\varphi^{-1}(k) \leqq$ $\leqq \mu(a_k, a_{k+1})$.) Indeed, if $\varphi(u) = k$, then either $a_{n_u} = a_k$ or a_{n_u} is skipped over at the k-th step. But at the k-th step there are skipped over $\mu(a_k, a_{k+1}) - 1$ vertices. The same estime holds for $k \geqq H(n)$ because then card $\varphi^{-1}(k) \leqq 1 \leqq \mu(a_k, a_{k+1})$. Furthermore card $\varphi^{-1}(i) = 1$, because only $\varphi(1) = i$, which follows immediately from $i \geqq H(n)$. Therefore

$$\mu(a_i, a_{i+1}) = j - 1 = \text{card } \varphi^{-1}(n) + \text{card } \varphi^{-1}(n+1) + \text{card } \varphi^{-1}(i-1) + \\ + \text{card } \varphi^{-1}(i) + \ldots + \text{card } \varphi^{-1}(r-1) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{i-1}, a_i) + \\ + 1 + \mu(a_{i+1}, a_{i+2}) + \ldots + \mu(a_{r-1}, a_r).$$

This implies (3).

2.11. *There exists a number s, $s > n$, such that the system of numbers $\mu(a_n, a_{n+1})$, $\ldots$, $\mu(a_{s-1}, a_s)$ satisfies all polygonal inequalities with defect* 2.

Proof. Let p be the number from 2.10. Let now r be a number such that $r \geqq p$ and that:

$$i < p,\ a_k \text{ is skipped over at the } i\text{-th step} \Rightarrow k < r.$$

Two cases can occur:

1. If a_k is skipped over at the i-th step, $i < r$, then also $k < r$.

Then the assumptions from 2.10 are satisfied with r instead of p. Then for $n \leqq i < r$ we have $\mu(a_i, a_{i+1}) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{i-1}, a_i) + \mu(a_{i+1}, a_{i+2}) + \ldots + \mu(a_{r-1}, a_r) + 2$ and it suffices to put $s = r$.

2. Let there exist an a_k such that $k \geqq r$ and a_k is skipped over at the i-th step, $i < r$. Let us denote by g the first index i for which there exist such a_k. Evidently $g \geqq p$. Let $C_g = (c_1, \ldots, c_j)$. Let $n \leqq i < g$. Then according to 2.10 (put g for p)

$$\mu(a_i, a_{i+1}) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{i-1}, a_i) + \mu(a_{i+1}, a_{i+2}) + \ldots + \\ + \mu(a_{r-1}, a_r) + 2,$$

and thus

$$\mu(a_i, a_{i+1}) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{i-1}, a_i) + \mu(a_{i+1}, a_{i+2}) + \ldots + \\ + \mu(a_{k-1}, a_k) + 2.$$

Let $g + 1 \leqq i < k$. Then (as μ is a metric)

$$\mu(a_i, a_{i+1}) \leqq \mu(a_{i+1}, a_{i+2}) + \ldots + \mu(a_{k-1}, a_k) + \mu(a_k, a_{g+1}) + \ldots + \\ + \mu(a_{i-1}, a_i) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{i-1}, a_i) + \mu(a_{i+1}, a_{i+2}) + \ldots + \\ + \mu(a_{k-1}, a_k).$$

Thus the relation

$$\mu(a_i, a_{i+1}) \leqq \mu(a_n, a_{n+1}) + \ldots + \mu(a_{i-1}, a_i) + \mu(a_{i+1}, a_{i+2}) + \ldots + \\ + \mu(a_{k-1}, a_k) + 2$$

holds for all i, $n \leqq i < k$, with the eventual exception of $i = g$. Also $k \geqq r \geqq p \geqq \geqq M(n) + n + 1 > H(n)$. Then it suffices to choose the L described in 2.9 and put $s = L$.

2.12. *There exist infinite infinetely many k, $k > n$, such that the system of numbers $\mu(a_n, a_{n+1}), \ldots, \mu(a_{k-1}, a_k)$ satisfies all polygonal inequalities.*

The proof follows imediately from 2.6 and 2.11.

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3.1. Let (G, ϱ) be a finite graph. Let a non $\in G$, $G_1 = G \cup \{a\}$, $b \in G$, $\varrho_1 = \varrho \cup \cup \{(a, b)\}$. The graph (G_1, ϱ_1) is called an *α-prolongation* of the graph (G, ϱ).

Let $n > 1$ be a number greater then the length of an arbitrary circle in (G, ϱ). Let be $a_1, \ldots, a_n$ non $\in G$, $G_1 = G \cup \{a_1, \ldots, a_n\}$, $b, c \in G$, $b \neq c$, $\mu(b, c) < n$, $\varrho_1 = \varrho \cup \cup \{(b, a_1), (a_1, a_2), \ldots, (a_n, c)\}$. Then we call (G_1, ϱ_1) *a β-prolongation* of (G, ϱ) with norm n.

Let $n > 2$ be a number such that is greater than the length of an arbitrary circle in (G, ϱ), $a_1, \ldots, a_n$ non $\in G$, $b \in G$, $G_1 = G \cup \{a_1, \ldots, a_n\}$, $\varrho_1 = \varrho \cup \{(b, a_1), \ldots, (a_n, a_1)\}$. Then (G_1, ϱ_1) will be called a *γ-prolongation* of (G, ϱ) with norm n.

Sometimes we shall also use the notation "$(G, \varrho) \rightarrow (G_1, \varrho_1)$ is a ξ-prolongation" with $\xi = \alpha, \beta, \gamma$.

3.2. Let (G, ϱ) be a connected subgraph in a connected finite graph (G_1, ϱ_1). For all points $a, b \in G$, let the distance a from b in (G, ϱ) be the same as the distance of a from b in (G_1, ϱ_1). Then we shall say that (G, ϱ) is *metrically embedded* in (G_1, ϱ_1).

3.3. It is easy to see that the following statements hold:

1. If (G, ϱ) is a connected finite graph and (G_1, ϱ_1) its ξ-prolongation ($\xi = \alpha, \beta, \gamma$), then (G_1, ϱ_1) is also connected.

2. Let (G_1, ϱ_1) be *a* ξ-prolongation ($\xi = \alpha, \beta, \gamma$) of a connected (G, ϱ). Then (G, ϱ) is metrically embedded in (G_1, ϱ_1).

3. Let (G_1, ϱ_1) be a β- or γ-prolongation of the graph (G, ϱ) with norm n. Let (L, λ) be a subgraph in (G_1, ϱ_1), which is a circle and L non $\subset G$. Then the length of the circle (L, λ) is at least n.

4. Let (G, ϱ) be a finite connected graph and (G_1, ϱ_1) its γ-prolongation with norm n. With the notation of 3.1, (b, a_1) is a bridge in (G_1, ϱ_1), a_1 lies on the circle $(a_1, \ldots, a_n, a_1)$.

3.4. Let (G_1, ϱ_1) be a graph with only one vertex. Define by induction the sequence of graphs $\{(G_n, \varrho_n)\}$ thus: (G_{n+1}, ϱ_{n+1}) originates from (G_n, ϱ_n) by ξ-prolongation ($\xi = \alpha, \beta, \gamma$) and it holds:

If $(G_n, \varrho_n) \rightarrow (G_{n+1}, \varrho_{n+1})$ is an α- or γ-prolongation and $i > n + 1$, then $x \in G_n$, $y \in (G_i - G_{i-1}) \Rightarrow (x, y)$ non $\in \varrho_i$. Let us call such a sequence an *ω-sequence*. If we put $G = \bigcup_{n=1}^{\infty} G_n$ and $\varrho = \bigcup_{n=1}^{\infty} \varrho_n$ we shall say that (G, ϱ) is the *ω-limit* of the sequence $\{(G_n, \varrho_n)\}$.

3.5. The following simple statements hold:

1. For every n, (G_n, ϱ_n) is a connected finite graph. (G, ϱ) is a connected enumerable graph.

2. If $(G_n, \varrho_n) \to (G_{n+1}, \varrho_{n+1})$ is an α-prolongation, then $(G_{n+1}, \varrho_{n+1}) \to (G_{n+2}, \varrho_{n+2})$ is an α- or γ-prolongation (it follows immediately from the condition for ω-sequence).

3. If $(G_n, \varrho_n) \to (G_{n+1}, \varrho_{n+1})$ is an α-prolongation, and if $G_{n+1} - G_n = \{b\}$ then b does not lie on any circle in (G, ϱ). If $(a, b) \subset \varrho_{n+1}$ then (a, b) is a bridge in (G, ϱ). (This follows from 2.)

4. Let $G_1 = \{a\}$. Then a is the only end-vertex in (G, ϱ) (this follows from 3).

5. For all n, (G_n, ϱ_n) is metrically embedded in (G, ϱ) (this follows from 3.3.2).

6. Let (L, λ) be a circle with length e which is a subgraph in (G, ϱ). Let m be the first index such that (L, λ) is a subgraph of (G_m, ϱ_m). Then $(G_{m-1}, \varrho_{m-1}) \to (G_m, \varrho_m)$ is a β- or γ-prolongation with norm at most e. (This follows from the definition of prolongation and the property of norms.)

7. Let (a, b) be a bridge in (G, ϱ). Then there exists an n such that $(G_n, \varrho_n) \to$ $\to (G_{n+1}, \varrho_{n+1})$ is an α- or γ-prolongation; if it is an α-prolongation then $a \in G_n$ and $b \in G_{n+1} - G_n$ (possibly after exchanging a and b); if it is a γ-prolongation it is necessary to exchange a and b in the definition of γ-prolongation. Further more, if $c \in G_n$, $d \in G - G_n$ and $(c, d) \neq (a, b)$, then (c, d) non $\in \varrho$.

3.6. *Let $\{(G_n, \varrho_n)\}$, $\{(G'_n, \varrho'_n)\}$ be two ω-sequences, (G, ϱ), (G', ϱ') their ω-limits. Let f be an isomorphic mapping of the graph (G, ϱ) onto (G', ϱ'), f_n the partial mapping of f on the set G_n. Then*

1. *the prolongations $(G_n, \varrho_n) \to (G_{n+1}, \varrho_{n+1})$ and $(G'_n, \varrho'_n) \to (G'_{n+1}, \varrho'_{n+1})$ are of the same kind and, for β- and γ-prolongation, have the same norms.*

2. *f_n is an isomorphic mapping of (G_n, ϱ_n) onto (G'_n, ϱ'_n).*

Proof. Let $G_1 = \{a\}$, $G'_1 = \{a'\}$. According to 3.5, 4, a and a' are the only end-vertices in (G, ϱ) resp. (G', ϱ'). Thus $a' = f(a)$.

Let n be a positive integer. Suppose that f_n is an isomorphic mapping (G_n, ϱ_n) on (G'_n, ϱ'_n).

a) Let $(G_n, \varrho_n) \to (G_{n+1}, \varrho_{n+1})$ be an α-prolongation. Let $G_{n+1} - G_n = \{b\}$, then (a, b) is a bridge in (G, ϱ) (3.5.3). So $(f(a), f(b))$ is a bridge in (G', ϱ') (1.7g). As $f(a) \in G'_n, f(b) \notin G'_n$ and $(f(a), f(b)) \in \varrho'$, according to 3.5 7 there must be $(G'_n, \varrho'_n) \to$ $\to (G'_{n+1}, \varrho'_{n+1})$ an α- or γ-prolongation and $f(b) \in G'_{n+1}$, for α-prolongations $f(b) \in$ $\in G'_{n+1} - G'_n$, for γ-prolongations $f(b) = a_1$ (a_1 from the definition on γ-prolongation). According to 3.5 3 b does not lie on any circle in (G, ϱ). So $f(b)$ does not lie on any circle in (G', ϱ'). But if $(G'_n, \varrho'_n) \to (G'_{n+1}, \varrho'_{n+1})$ were a γ-prolongation, then a_1 would lie on a circle (3.3.4), a contradiction. So $(G'_n, \varrho'_n) \to (G'_{n+1}, \varrho_{n+1})$ is an α-prolongation and f_n is an isomorphism between (G_{n+1}, ϱ_{n+1}) and $(G'_{n+1}, \varrho'_{n+1})$.

b) Let $(G_n, \varrho_n) \to (G_{n+1}, \varrho_{+1})$ be a γ-prolongation. Analogously as in a) we get that $(G'_n, \varrho'_n) \to (G'_{n+1}, \varrho'_{n+1})$ is a γ-prolongation.

Let m and m' be the corresponding norms. Let $a \in G_{n+1} - G_n$. Then a lies on a circle of length m. So $f(a)$ lies in (G', ϱ') on a circle with length m. According to the definition of norm and 3.3 3, all the circles with any vertex in $G - G_{n+1}$ are of length greater than m. This means that $f(a)$ lies on a circle of the smallest length with vertices in $G' - G'_n$. But (again according to 3.3 3) this is the circle with the set of vertices $G'_{n+1} - G'_n$. Hence it follows that $f(G_{n+1}) = G'_{n+1}$, $m = m'$, and the fact that f_n is an isomorphic mapping of (G_{n+1}, ϱ_{n+1}) onto $(G'_{n+1}, \varrho'_{n+1})$.

c) If $(G_n, \varrho_n) \to (G_{n+1}, \varrho_{n+1})$ is a β-prolongation, then it follows from a) and b) that $(G'_n, \varrho'_n) \to (G'_{n+1}, \varrho'_{u+1})$ is also a β-prolongation. As in b) we find that the norms are the same in both cases, $f(G_{n+1}) = G'_{n+1}$ and f_n is an isomorphic mapping of (G_{n+1}, ϱ_{n+1}) onto $(G'_{n+1}, \varrho'_{n+1})$.

4

4.1. Let $d_1, d_2, \ldots, d_n, \ldots$ be an infinite sequence of positive integers such that for every index n there exists an m with $m > n$ and such that the system $d_n, d_{n+1}, \ldots, d_m$ satisfies all polygonal inequalities.

From 2.6 it follows that then there are infinitely many such numbers. Let $d_n \neq 1$. We can show that then there are also infinitely many m such that the system of numbers $d_n - 1, \ldots, d_m$ satisfies all polygonal inequalities. It suffices to take systems $d_n, \ldots, d_m; d_{m+1}, \ldots, d_p$ satisfying all polygonal inequalities. Then the system

$$d_n - 1, \ldots, d_m, d_{m+1}, \ldots, d_p$$

satisfies all polygonal inequalities.

4.2. Let G be an enumerable set. Let us order it in a sequence $\{b_1, b_2, \ldots\}$. Put $a_1 = b_1$, $G_1 = \{a_1\}$, $\varrho_1 = \emptyset$.

We shall now define the ω-sequence $\{(G_n, \varrho_n)\}$ by induction in the following way:

Let n be a positive integer and suppose that there are defined graphs $(G_1, \varrho_1), \ldots, (G_n, \varrho_n)$ and that for a certain m there have been selected in G_n m elements $a_1, \ldots, a_m$ such that in (G_n, ϱ_n)

$$\mu(a_1, a_2) = d_1, \ldots, \mu(a_{m-1}, a_m) = d_{m-1}$$

and

$$\{b_1, \ldots, b_n\} \subset \{a_1, \ldots, a_m\}\,.$$

a) Let $\{a_1, \ldots, a_m\} = G_n$.

a_1) Let $d_m = 1$. Let p be the smallest index such that b_p non $\in G_n$. Obviously $p \geqq n + 1$. Then put $G_{n+1} = G_n \cup \{b_p\}$, $\varrho_{n+1} = \varrho \cup \{(a_m, b_p)\}$. Thus (G_{n+1}, ϱ_{n+1}) is an α-prolongation of the graph (G, ϱ). Put $a_{m+1} = b_p$. The suppositions of induction now hold for $n + 1$ (for m we put $m + 1$).

a$_2$) Let $d_m \neq 1$. Choose r such that

1. the system $d_m - 1, \ldots, d_r$ fulfills all polygonal inequalities.

2. $s = d_m - 1 + d_{m+1} + \ldots + d_r$ is greater than the length of the greatest circle in (G_n, ϱ_n), $s > 2$.

Let p again be the smallest index such that b_p non $\in G_n$. Now choose in $G - G_n$ s points (b_p among them) $c_1, \ldots, c_s$ and put $G_{n+1} = G_n \cup \{c_1, \ldots, c_s\}$, $\varrho_{n+1} = \varrho_n \cup \cup \{(a_n, c_1), (c_1, c_2), \ldots, (c_s, c_1)\}$. Then (G_{n+1}, ϱ_{n+1}) is a γ-prolongation (G_n, ϱ_n) with norm s. Put $a_{m+1} = c_{d_m}$, $a_{m+2} = c_{d_m+d_{m+1}}$, $\ldots$, $a_r = c_{s-d_r+1}$, $a_{r+1} = c_1$. In consequence of 1 in (G_{n+1}, ϱ_{n+1}), it holds that

$$\mu(a_m, a_{m+1}) = d_m,\ \mu(a_{m+1}, a_{m+2}) = d_{m+1},\ \ldots,\ \mu(a_r, a_{r+1}) = d_r\,.$$

b) Let $\{a_1, \ldots, a_n\} \neq G_n$. Let t be the first index such that $b_t \in G_n$, b_t non $\in \{a_1, \ldots, a_m\}$, p again the first index such that b_p non $\in G_n$. Choose r so that the system $d_m, \ldots, d_r$ satisfies all polygonal inequalities,

$$s = d_m + \ldots + d_r > \mu(a_m, b_t) + 1 \tag{4}$$

and that $s - 1$ is greater than the length of the greatest circle in (G_n, ϱ_n). Choose $s - 1$ points $c_1, \ldots, s_{s-1}$ from $G - G_n$ (b_p among them) and define

$$G_{n+1} = G_n \cup \{c_1, \ldots, c_{s-1}\},\ \varrho_{n+1} = \varrho_n \cup \{(a_m, c_1), (c_1, c_2), \ldots, (c_{s-1}, b_t)\}\,.$$

Then (G_{n+1}, ϱ_{n+1}) is a β-prolongation of the graph (G_n, ϱ_n). Put $a_{m+1} = c_{d_m}$, $a_{m+2} = c_{d_m+d_{m+1}}$, $\ldots$, $a_{r+1} = b_t$. According to (4),

$$\mu(a_m, a_{m+1}) = d_m,\ \mu(a_{m+1}, a_{m+2}) = d_{m+1},\ \ldots,\ \mu(a_r, a_{r+1}) = d_r\,.$$

Thus the assumptions are again satisfied (m in place of $r + 1$).

From the construction mentioned above (in both cases a) and b)) it is evident that if, for an n, case a occurs then $(G_n, \varrho_n) \to (G_{n+1}, \varrho_{n+1})$ is an α- or γ-prolongation; and if $i > n + 1$, then for $a \in G_n$ and $b \in G_i - G_{i-1}$ we have (a, b) non $\in \varrho_i$. Thus $\{(G_n, \varrho_n)\}$ forms an ω-sequence, $\bigcup G_n = G$. Put $\varrho = \bigcup \varrho_n$. The sequence $\{a_n\}$ defined in the construction belongs to $\pi(G)$ and $p(\{a_n\}) = \{d_1, d_2, d_3, \ldots\}$. In consequence of this result and lemma 2.12,

4.3. *To the sequence of positive integers $\{d_1, d_2, \ldots\}$ there exists an enumerable connected graph (G, ϱ) and $\pi \in \pi(G)$ such that $p(\pi) = \{d_1, d_2, \ldots\}$ if and only if for every n there is an $m > n$ such that the system*

$$d_n, \ldots, d_m$$

satisfies all polygonal inequalities.

4.4. We shall now prove that to the given sequence from 4.1 there belong $2^{\aleph_0}$ of the graphs (mutually non-isomorphic) which are spoken about in 4.3.

a) First suppose that the considered sequence $\{d_1, d_2, \ldots\}$ has the property that there exists an N so that for $n > N$ there is $d_n = 1$. Then apparently in the construction 4.2 for a certain N_1 it holds that $n \geqq N_1 \Rightarrow \{(G_n, \varrho_n) \to (G_{n+1}, \varrho_{n+1})$ is an α-prolongation$\}$. A saturated subgraph on the set $G - G_{N_1}$ in (G, ϱ) is then a onesided infinite path $(a_j, a_{j+1}, \ldots)$ (for a suitable j). Let us choose the sequence

(5) $$n_1, n_2, \ldots, n_i, \ldots$$

of integers not less than 2 and construct a graph (G, ϱ^*), where $\varrho^* = \varrho \cup \{(a_j, a_{j+n_1}) . . (a_{j+n_1}, a_{j+n_1+n_2}), \ldots\}$. To two different sequences (5) there belong two non-isomorphic graphs (G, ϱ^*). The cardinality of the set of sequences (5) is $2^{\aleph_0}$, so that there are $2^{\aleph_0}$ such graphs (G, ϱ^*). Simultaneously $\{a_n\} \in \pi(G)$ and $\{d_n\} = p(\{a_n\}) \in P(G, \varrho^*)$ for arbitrary (5).

b) Assume that the index N described in a) does not exist. Then in the construction 4.2, for infinitely many n, $(G_n, \varrho_n) \to (G_{n+1}, \varrho_{n+1})$ is a β- or γ-prolongation. As for every such prolongation we have $\aleph_0$ possibilities of choice of the norm of prolongation, there follows from 3.6 that it is possible, by the construction 4.2, to construct $2^{\aleph_0}$ of the mutually non-isomorphic graphs mentioned in 4.3.

4.5. (Corollary.) *To every sequence p of positive integers smaller than a given number k there exists an enumerable connected graph (G, ϱ) such that there is an ordering $\pi \in \pi(G)$ for which $p(\pi) = p$.*

Proof. This follows immediately from 4.3, since obviously p satisfies the assumptions on the sequence $\{d_1, d_2, \ldots\}$.

4.6. In the next theorem the graphs mentioned in 4.5 for $k \geqq 4$ will be characterised.

Let k be a positive integer, $k \geqq 4$. A necessary and sufficient condition for the set of vertices of a connected enumerable graph (G, ϱ) to be ordered in a sequence

(6) $$a_1, a_2, \ldots, a_n, \ldots$$

such that there holds for every n

(7) $$\mu(a_n, a_{n+1}) \leqq k$$

is the validity of the implication

(8) $$A \cup B \cup C = G, \text{ with } C \text{ finite and } A, B \text{ infinite} \Rightarrow \mu(A, B) \leqq k .$$

Proof. *Necessity.* Suppose that G is ordered in a sequence (6) for which (7) holds. Let A, B be infinite subsets from G, C finite and $A \cup B \cup C = G$. There exists an index m such that for $n > m$, $a_n \in A \cup B$. As A, B are infinite sets, then there exists an index $p > m$ such that $a_p \in A$, $a_{p+1} \in B$. As $\mu(a_p, a_{p+1}) \leqq k$ then also $\mu(A, B) \leqq k$.

Sufficiency. Order G in a sequence $b_1, b_2, \ldots, b_n, \ldots$. Let $(K, \varrho \cap D(K))$ be a finite connected saturated subgraph in (G, ϱ) containing b_1 and b_2 among its vertices. Let $(G - K, D(G - K) \cap \varrho)$ decompose into components $L_1, L_2, \ldots, L_n, \ldots$. To each of these components assign a vertex $c_n \in K$ such that $\mu(c_n, L_n) = 1$. A component L_n will be called a *component of the first kind* if it is infinite or finite and there exist infinitely many other finite components L_j such that $c_n = c_j$. Components not of the first kind will be called *components of the second kind*. There are only finitely many components of the second kind (K is a finite set); let these be $L_{i_1}, \ldots, L_{i_k}$. Let $(G_1, \varrho_1) = (K \cup L_{i_1} \cup \ldots \cup L_{i_k}, \varrho \cap D(K \cup L_{i_1} \cup \ldots \cup L_{i_k}))$. The components in $(G - G_1, \varrho \cap D(G - G_1))$ (with respect to the graph (G_1, ϱ_1) where the vertices c_j have the same significance as for (G, ϱ)) are components of the first kind only.[1]) We shall choose one of them and denote it by L_{j_1}. Order the vertices of the graph (G_1, ϱ_1) in a sequence

$$a_1, \ldots, a_{m_1},$$

where $a_{m_1} = c_{j_1}$ in such a manner that (7) holds. Such a sequence exists according to Lemma 3 in [2].[2]) Let $n \geqq 1$ and suppose that there is defined a connected saturated finite subgraph in (G, ϱ) (G_n, ϱ_n) with the following properties:

1. To each component L_j from $(G - G_n, \varrho \cap D(G - G_n))$ there is assigned a vertex $c_j \in G_n$ in such a way that all the components are components of the first kind.

2. The set G_n is ordered in a sequence $a_1, \ldots, a_m$ such that (7) holds and $\mu(a_m, c_{j_n}) \leqq 1$ for a certain component L_{j_n}.

3. $b_1, b_2, \ldots, b_n \in G_n$.

Let ν be the smallest index with b_ν non $\in G_\nu$. Then $\nu \geqq n + 1$. Let b_ν lie in a component L, to which a vertex c is assigned in G_n.

a) Let $c = c_{j_n}$ and L be finite. Then order L in a sequence $a_{m+1}, \ldots, a_p$ such that $\mu(a_{m+1}, c_{j_n}) \leqq 2$ and $\mu(a_p, c_{j_n}) = 1$ and that (7) holds.

We put then

$$G_{n+1} = G_n \cup L, \quad \varrho_{n+1} = \varrho \cap D(G_{n+1}).$$

The set G_{n+1} is ordered in a sequence $a_1, \ldots, a_m, \ldots, a_p$. The set of components in $(G - G_{n+1}, \varrho \cap D(G - G_{n+1}))$ differs from the set of components in $(G - G_n, \varrho \cap D(G - G_n)$ only by L. As L is finite, c is assigned to infinitely many components in $(G - G_n, \varrho \cap D(G - G_n))$ so that the induction suppositions 1, 2, 3 are satisfied (with p instead of m; the assignement of vertices to components remains the same).

b) Let $c = c_{j_n}$ and L be infinite. There exists an $a' \in L$ such that $\mu(a', c) = 1$ and an $a \in L$ for which $a \neq a'$ and $\mu(a, c) = 1$ or 2. Furthermore, there exists in L a subgraph (S, σ) of the first kind, which contains a, a' and b_ν. Let us order its vertices in a sequence $a_{m+1}, \ldots, a_p$ such that (7) holds, $a_{m+1} = a$ or a' and a_p is assigned to some component from $(L - S, \varrho \cap D(L - S))$. We put $G_{n+1} = G_n \cup S$, $\varrho_{n+1} = \varrho \cap D(G_{n+1})$ and G_{n+1} is to be ordered in a sequence

$$a_1, \ldots, a_m, a_{m+1}, \ldots, a_p, \tag{9}$$

[1]) A connected finite subgraph will be termed of the first kind if all of its components are of the first kind after suitable choice of vertices c_j. It may be observed in the just described construction that in a connected graph there exists a subgraph of the first kind containing a prescribed finite set of vertices.

[2]) This lemma reads as follows: Let (G, ϱ) be a finite connected graph, $a, b \in G$, $a \neq b$. Then a set G may be ordered in a sequence $a_1, \ldots, a_n$ (n = card G), where $a_1 = a$, $a_n = b$, $\mu(a_i, a_{i+1}) \leqq 3$ for $i = 1, \ldots, n - 1$.

for which (7) holds ($\mu(a_m, a_{m+1}) \leqq 3$). The suppositions of induction are satisfied (the component L is replaced by components from $(L - S, \varrho \cap D(L - S))$). If an element c is assigned to some other component $L' \neq L$, then let this assignement be preserved.

c) Let $c \neq c_{j_n}$. Let us order the vertices from (G_n, ϱ_n) to which there is assigned at least one component from $(G - G_n, \varrho \cap D(G - G_n))$ in a sequence $c^1, \ldots, c^r$ such that $c_{j_n} = c^1$, $c = c^r$. Let the point c^h be assigned to components $L_1^h, \ldots, L_p^h, \ldots$ Let $K^h = \bigcup_p L_p^h$. Define a graph on the set of all K^h in following way: K^h and $K^{h'}$ ($h \neq h'$) will be connected by an edge precisely if in $K^{h'}$ there exist infinitely many vertices, which have a distance $\leqq k$ from K^h and simultaneously in $K^{h'}$ there exist infinitely many vertices which have a distance $\leqq k$ from K^h. We shall show that the graph thus defined is connected. Assume the contrary. Then we can decompose the system of all K^h into two disjoint subsystems $K^{h_1}, \ldots, K^{h_s}$ and $K^{h_{s+1}}, \ldots, K^{h_r}$ such that no two K^h from various systems are connected by an edge in defined graph. Put $A_1 = K^{h_1} \cup \cup \ldots \cup K^{h_s}$, $B_1 = K^{h_{s+1}} \cup \ldots \cup K^{h_r}$. Both A_1 and B_1 are infinite sets, G_n is finite and $A_1 \cup B_1 \cup G_n = G$. By the assumptions of our theorem, $\mu(A_1, B_1) \leqq k$. Consequently there exist $a' \in A_1$ and $b' \in B_1$ such that $\mu(a', b') \leqq k$. Let $A_2 = A_1 - \{a'\}$, $B_2 = = B_1 - \{b'\}$. The assumptions of our theorem are again satisfied for the sets A_2, B_2, $G_n \cup \{a', b'\}$. Thus $\mu(A_2, B_2) \leqq k$ and there exist vertices $a'' \in B_2$, $b'' \in B_2$ such that $\mu(a'', b'') \leqq k$. Analogously, one may define by induction vertices $a^{(n)}$, $b^{(n)}$ such that $\mu(a^{(n)}, b^{(n)}) \leqq k$. The points $a^{(n)}$ are distinct and belong to some of the sets $K^{h_1}, \ldots, K^{h_s}$, and then the points $b^{(n)}$ are distinct and belong to some of the sets $K^{h_{s+1}}, \ldots, K^{h_r}$. Therefore there exist infinite sequences $a^{(n_1)}, \ldots, a^{(n_m)}, \ldots$ and $b^{(n_1)}, \ldots, b^{(n_m)}, \ldots$ such that all points from the first sequence belong to the same set K^h and all points from the second sequence belong to the same set $K^{h'}$. Then K^h and $K^{h'}$ are connected by an edge, which is the contradiction. Thus the graph on the set of K^h is connected.

We shall now return to the definition of the graph (G_{n+1}, ϱ_{n+1}). Let $K^{h_1}, \ldots, K^{h_t}$ be a path connecting K^1 and K^r (thus $K^{h_1} = K^1$, $K^{h_t} = K^r$).

Assume $\mu(a, c_{j_n}) = 1$ for $a \in K^1$. Let a belong to the component L. Let $a' \in K'$, $a' \neq a$, with $\mu(a', K^{h_2}) \leqq k$. Let a' belong to the component L'.

a) Let $L = L'$.

a_1) If L is finite, we can order its vertices in a sequence

$$\alpha_1, \ldots, \alpha_s \tag{10}$$

such that (7) holds and

$$\alpha_1 = a \quad \text{and} \quad \alpha_s = a' . \tag{11}$$

a_2) Let L be infinite. Construct in it a subgraph of the first kind which contains a' and a and order its vertices in a sequence (10) for which (11) holds.

b) Let $L \neq L'$.

b_{11}) If L is finite, we may order it in a sequence (10) for which $\alpha_1 = a$ and $\mu(\alpha_s, a) \leqq \leqq 1$.

b_{12}) Let L be infinite. Then construct in L a subgraph of the first kind which contains a and order its vertices in a sequence (10), for which there hold the relations mentioned in b_{11}).

b_{21}) Let L' be finite. Then order it in a sequence

(12) $$\alpha_{s+1}, \ldots, \alpha_n$$

such that

(13) $$\mu(\alpha_{s+1}, c_{j_n}) = 1 \text{ or } 2\,, \quad \alpha_n = a'\,.$$

b_{22}) Let L' be infinite. Then construct in L' a subgraph of the first kind which contains a' and a point α_{s+1}, for which $\mu(\alpha_{s+1}, c_{j_n}) = 1$ or 2, $a' \neq \alpha_{s+1}$. Order the vertices of this graph in a sequence (12).

In all cases in b)

$$\alpha_1, \ldots, \alpha_s, \alpha_{s+1}, \ldots, \alpha_n$$

satisfies (7) ($\mu(\alpha_s, \alpha_{s+1}) \leqq 4$).

c) Let be $1 < i < t$ and let there already be defined a sequence

(14) $$\alpha_1, \ldots, \alpha_v$$

such that $\mu(\alpha_v, K^i) \leqq k$. Assume $a \in K^i$, $\mu(a, \alpha_v) \leqq k$. Then we may proceed as in a) or b) with the exception that instead of a vertex c_{j_n} we consider the point c^i which is assigned to the component from K^i and a has the meaning just defined.[3])

d) Next let $i = t$ and assume we have a sequence (14). Let a again be a vertex in K^r $(= K^{h_t})$ with $\mu(a, \alpha_v) \leqq k$. Let the vertex a belong to a component $L (L \subset K^r)$.

d_1) Let $b_v \in L$.

d_{11}) If L is finite, there exists in K^r a further finite component L'. Order L in a sequence

(15) $$\alpha_{v+1}, \ldots, \alpha_w$$

such that (7) holds, $\alpha_{v+1} = a$ and $\mu(\alpha_w, c) \leqq 2$ (the point c is assigned to components from K^r). Order the component L' in a sequence

(16) $$\alpha_{w+1}, \ldots, \alpha_z$$

for which (7) holds and $\mu(\alpha_{w+1}, c) \leqq 2$, $\mu(\alpha_z, c) = 1$.

d_{12}) Let L be infinite. Construct in it a subgraph (S, σ) of the first kind which contains a and b_v, and order its vertices in a sequence (15) in which α_w has distance at most 1 from a certain point c' of this subgraph, belonging to a certain component from $(L - S, \varrho \cap D(L - S))$ (thus $\mu(c', \alpha_w) \leqq 1$).

d_2) Let $b_v \in L' \neq L$.

d_{21}) If L is finite, order it in a sequence (15) such that again $\alpha_{v+1} = a$ and $\mu(\alpha_w, c) \leqq$ $\leqq 2$.

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[3]) We can choose a point analogous to a' such that $a' \neq a$ because we have $\aleph_0$ possibilities for the choice of this vertex (according to the definition of a graph on the system of sets K^h).

d_{211}) Let L' be infinite. Then construct in it a subgraph (S, σ) of the first kind which contains b_ν and let c' be its vertex assigned to some component from $(L' - S, \varrho \cap \cap D(L' - S))$. Order S in a sequence

$$(17) \qquad \alpha_{w+1}, \dots, \alpha_z ,$$

where $\mu(\alpha_{w+1}, c) \leqq 2$, $\mu(\alpha_z, c') \leqq 1$ and for which (7) holds.

d_{212}) Let L' be finite. Order it in a sequence (16) with the required properties.

d_{22}) Let L be infinite. Then construct in L a subgraph of the first kind and order its vertices in a sequence (15) with the required properties.

We order the component L' as in the case d_{21}.

In all the cases d) we obtain a sequence

$$(18) \qquad a_1, \dots, a_n, \alpha_1, \dots, \alpha_v, \alpha_{v+1}, \dots, \alpha_w$$

or

$$(19) \qquad a_1, \dots, a_n, \alpha_1, \dots, \alpha_v, \alpha_{v+1}, \dots, \alpha_w, \alpha_{w+1}, \dots, \alpha_z .$$

As in all cases $\mu(\alpha_v, \alpha_{v+1}) \leqq k$ and $\mu(\alpha_w, \alpha_{w+1}) \leqq 4$, the sequences (18) or (19) satisfy (7). If we denote the set of all members in the sequence (18) or (19) by G_{n+1}, then $(G_{n+1}, \varrho \cap D(G_{n+1}))$ is a graph of the first kind. Also, if some c'' in a graph (G_n, ϱ_n) belongs to infinite by many components, again there exist in $(G - G_{n+1}, \varrho \cap (G - - G_{n+1}))$ infinitely many components with distance 1 from c''. In (18) or (19) the elements are from at most two components assigned to c'' in $(G - G_n, \varrho \cap D(G - G_n))$. The infinite components L from $(G - G_n, \varrho \cap D(G - G_n))$ whose elements appear in (18) resp. (19), are now replaced by the components of the graph of the first kind obtained in the construction of the sequences (18) and (19). If to such a component L there belongs a vertex c'' which also belongs in (G_n, ϱ_n) to a component L' whose elements do not occur in our sequence, then again we assign c'' to the component L' in the graph (G_{n+1}, ϱ_{n+1}) $(\varrho_{n+1} = \varrho \cap D(G_{n+1}))$. The assignement of vertices from G_{n+1} to the new components let be taken over from the single subgraphs of the first kind obtained in the construction of the sequences (18) and (19). Then the induction assumption concerning the last element of a sequence (i.e. α_w or α_z) is also satisfied, when for c_{j_n} one takes c or c' (obtained in d_{12}), d_{211}) and similarly for d_{22})).

4.7. Theorem 4.6 does not hold for $k = 1, 2, 3$. An example for $k = 1$ may be found on fig. 1, where K represents a complete enumerable graph, an example for $k = 3$ on fig. 2.

Suppose for instance that it is possible to order the points of the graph on a figure 2 in a sequence $\alpha_1, \alpha_2, \dots, \alpha_n, \dots$ such that $\mu(\alpha_i, \alpha_{i+1}) \leqq 3$. Let $b = \alpha_{i_1}$, $c = \alpha_{i_2}$. Let $N > i_1, i_2$. Let $n_1, n_2, \dots$ be the sequence of all those $n > N$ for which $\alpha_n = a_i$, $\alpha_{n+1} = d_k$ (for certain i, k). Let n_j be an arbitrary member of this sequence and $\alpha_{n_j+1} = d_k$ for a certain k. Let $f_k = \alpha_i$. We shall show that

$$(20) \qquad i < n_j + 1 \quad \text{or} \quad i = n_j + 2 .$$

Assume that (20) does not hold. Then evidently $\alpha_{i-1} = e_k$ and then necessarily $\alpha_{i+1} = c$, which is a contradiction with $i_2 < i + 1$ and $c = d_{i_2}$.

Further we shall show that if $\alpha_n = f_i$ for $n > n_j$, $n_{j+1} > n$, then $d_i = \alpha_{n'}$ with $n' < n_{j+1}$. If $n' < n$ there is nothing to prove, if $n' > n$ then $\alpha_{n-1} = e_i$ and $\alpha_{n+1} = d_i$, and thus $n' = n + 1 < n_{j+1}$.

Therefore there exists a j' such that $n > n_{j'}$, $f_i = \alpha_n$, $d_i = \alpha_{n'} \Rightarrow n' < n$. Let $j > j'$ and $\alpha_{n_j+1} = d_{i_1}$. According to the assertion proved above, $\alpha_{n_j+2} = f_{i_1}$, $\alpha_{n_j+3} = e_{i_1}$. Thus $\alpha_{n_j+4} = d_{i_2}$ for some i_2 and again $\alpha_{n_j+5} = f_{i_2}$, $\alpha_{n_j+6} = e_{i_2}$. By induction it follows that α_n with $n > n_j$ is always one of the elements of the form d_i, c_i, f_i; but this is impossible since the are infinitely many a_i.

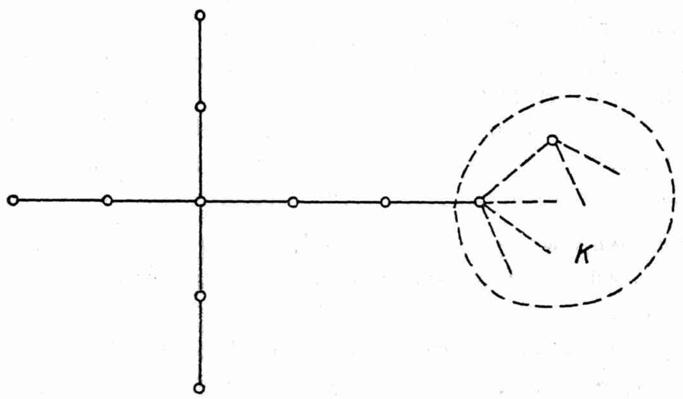
It is easy to prove that our graph satisfies the suppositions of theorem 4.6 for $k = 3$.

4.8. A sequence $p = \{d_1, d_2, \ldots\}$ will be called an *A-sequence* when one of following cases occurs

1. $d_i = 1$ for all i.
2. There is an index i_1 such that $d_{i_1} = 2$, and $d_i = 1$ for $i \neq i_1$.
3. There is an index i_1 such that $d_{i_1} = d_{i_1+1} = 2$, and $d_i = 1$ for $i \neq i_1$, $i \neq i_1 + 1$.

We shall say that an enumerable connected graph (G, ϱ) is of type A if there exist $B \subset G$ and $b \in G$ such that:

1. card $B \geqq 2$, card $(G - B) \geqq 2$.
2. b non $\in B$.
3. $\varrho = D(G) - \{(b, b') : b' \in B\}$ or $B = \{b_1, b_2\}$, $\varrho = D(G) - D(\{b, b_1, b_2\})$.



Obr. 1.

Then

a) Let $\pi = \{\alpha_1, \ldots, \alpha_n, \ldots\} \in \pi(G)$. Then $p(\pi)$ is an A-sequence.

b) Let p be an A-sequence. Then there exists a $\pi \in \pi(G)$ such that $p(\pi) = p$.