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Přednášky v sekcích.

I. sekce:

Matematická logika a theorie množin.

CERTAIN NOTIONS OF THE THEORY OF NUMBERS AS APPLIED TO THE PROPOSITIONAL CALCULUS.

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1.

In the propositional calculus functions containing more than two variables have been as yet very little studied.[1]) It seems advisable to elaborate general methods of studying functions containing n variables and belonging to the propositional calculus. It appears that in such studies rather unexpected assistance may be obtained from certain notions of the theory of numbers, in particular from several functions built by means of the „Entier", and among these from the notion of binary development of the natural number. The present communication refers exclusively to the two-valued propositional calculus.

The symbolism used in this paper is as follows: propositional variables are letters $pp_1p_2 \ldots qq_1q_2 \ldots rr_1r_2 \ldots ss_1s_2 \ldots$, falsum $\wedge$, verum $\vee$, negation $'$, sum $\oplus$, product $\odot$, implication $\rightarrow$, equivalence $\longleftrightarrow$, quanti-

[1]) To the exceptions known to the author belongs the paper by S. JAŚKOWSKI, *Trois contributions au calcul des propositions bivalent*, Toruń 1948. Functions containing n variables and belonging to the propositional calculus were the subject-matter of the author's communication *Functors of the Propositional Calculus*, VI Congress of Polish Mathematicians, Warsaw 1948, hereafter called „the previous communication".

fiers $\sum_{p_1 \ldots p_n}$ $\prod_{p_1 \ldots p_n}$.[2]) The definitions of difference and symmetrical difference are introduced:

$$p \ominus q =_{Df} p \odot q' \tag{1.01}$$

$$p \dot{-} q =_{Df} (p \ominus q) \oplus (q \ominus p) \tag{1.02}$$

Natural variables (i. e., variables for which only the names of natural numbers may be substituted) are letters $kk_1k_2 \ldots ll_1l_2 \ldots mm_1m_2 \ldots$ $\ldots nn_1n_2 \ldots$. By the „scheme of the propositional variable" an expression is meant which contains at least one natural variable and which becomes a propositional variable after substitution in it of a name of a natural number for every natural variable contained in it. Thus schemes of propositional variables are, e. g., the expressions $p_kq_lr_ms_n$ and the expressions $p_{k+l}\, q_{k \cdot l}\, s_k{}^l$. By the „scheme of an expression belonging to the language of the propositional calculus" an expression is meant which contains at least one natural variable and which, after substitution in it of a name of a natural number for every natural variable contained in it, becomes an expression belonging to the language of the propositional calculus. Thus schemes of expressions belonging to the language of the propositional calculus are, e. g., the expressions

$$p_k \to q_l \quad p_n \longleftrightarrow r_n \quad p_{k+l} \dot{-} r_{m \cdot 1} \,.$$

Finally, by the „schemes of a thesis belonging to the propositional calculus" an expression is meant which contains at least one natural variable and which, after substitution in it of a name of a natural number for every natural variable contained in it, becomes either a thesis belonging to the propositional calculus or an implication in which the hypothesis is a thesis belonging to the arithmetics of natural numbers, and the thesis is true in the propositional calculus. Thus schemes of theses belonging to the propositional calculus are, e. g., the two following expressions:

$$p_k \to p_k$$

$$(m + m = 2 \cdot m) \to (p_{m+m} \to p_{2 \cdot m})$$

Instead of	we write briefly
$p_1, p_2, \ldots, p_n$	$\overset{n}{\underset{k=1}{S}} p_k$
$p_1, p_2, \ldots, p_m, q_1, q_2, \ldots, q_n$	$\overset{m}{\underset{k=1}{S}} p_k \overset{n}{\underset{l=1}{S}} q_k$
$p_1 \oplus \ldots \oplus p_n$	$\sum_{k=1}^{n} p_k$

[2]) In the previous communication the author formulated the method of introducing the quantifiers into the propositional calculus.

$$p_1 \odot \ldots \odot p_n \qquad \prod_{k=1}^{n} p_k$$

$$(p_1 \odot p_2) \to q \qquad p_1 \odot p_2 \to q$$

In the study of propositional functions each containing n propositional variables an essential role is played by the following scheme of the propositional function containing $n + 2^n$ propositional variables

$$\underset{k=1}{\overset{n}{S}} p_k \in \underset{l=1}{\overset{2^n}{S}} q_l$$

recursively defined as follows[3])

$$p_1 \in q_1 q_2 =_{Df} (p_1' \to q_1) \odot (p_2 \to q_2) \tag{1.11}$$

$$\underset{k=1}{\overset{n+1}{S}} p_k \in \underset{l=1}{\overset{2^{n+1}}{S}} q_l =_{Df} p_{n+1} \in \left(\underset{k=1}{\overset{n}{S}} p_k \in \underset{l=1}{\overset{2^n}{S}} q_l\right)\left(\underset{k=1}{\overset{n}{S}} p_k \in \underset{l=2^n+1}{\overset{2^{n+1}}{S}} q_l\right) \tag{1.12}$$

In conformity with this scheme we obtain, e. g.,

$$p_1 p_2 \in q_1 q_2 q_3 q_4 =_{Df} p_2 \in (p_1 \in q_1 q_2)(p_1 \in q_3 q_4). \tag{1.13}$$

Three operations on natural numbers, all of them built by means of the „Entier" (E), will be useful:

$$m * n =_{Df} E\frac{n-1}{2^{m-1}} - 2 \cdot E\frac{n-1}{2^m} \tag{1.21}$$

$$m \,\mathrm{X}\, n =_{Df} m - 2^n \cdot E\frac{m-1}{2^n} \tag{1.22}$$

$$m \,\mathrm{Y}\, n =_{Df} 1 + E\frac{m-1}{2^n} \tag{1.23}$$

The expression $m * n$ is read „the m-th digit (figure) of the binary development of the number $n - 1$".

The expression p^n is defined so that

$$p^0 \longleftrightarrow p' \qquad p^1 \longleftrightarrow p \tag{1.31}$$

For values other than 0 or 1 of the exponent n the expression p^n is indefinite. Since, as can be easily noticed,

$$(m * n = 0) \oplus (m * n = 1) \tag{1.41}$$

then the expression p^{m*n} possesses definite value for all correct substitutions for the propositional variable p and the natural variables m, n.

2.

The following schemes of theses belonging to the propositional calculus have been proved:

[3]) Compare the previous communication, propositions 3. 11. — 3. 12.

$$\left(\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}q_l\right) \longleftrightarrow \sum_{l=1}^{2^n}\left(\prod_{k=1}^{n} p_k^{k*l}\right)\odot q_l \tag{2.11}$$

$$\left(\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}q_l\right) \longleftrightarrow \prod_{l=1}^{2^n}\left(\prod_{k=1}^{n} p_k^{k*l}\right) \to q_l \tag{2.12}$$

For instance,

$$(p_1p_2 \in q_1q_2q_3q_4) \longleftrightarrow \tag{2.13}$$

$$\longleftrightarrow (p_1' \odot p_2' \odot q_1) \oplus (p_1 \odot p_2' \odot q_2) \oplus (p_1' \odot p_2 \odot q_3) \oplus (p_1 \odot p_2 \odot q_4)$$

$$(p_1p_2 \in q_1q_2q_3q_4) \longleftrightarrow \tag{2.14}$$

$$\longleftrightarrow (p_1' \odot p_2' \to q_1) \odot (p_1 \odot p_2' \to q_2) \odot (p_1' \odot p_2 \to q_3) \odot (p_1 \odot p_2 \to q_4).$$

Each of the schemes 2.11, 2.12 may well replace the recursive scheme 1.11—1.12.

$$\sum_{p_1 \ldots p_n}\left(\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}q_l\right) \longleftrightarrow \sum_{l=1}^{2^n} q_l \tag{2.21}$$

For instance,

$$\sum_{p_1p_2}(p_1p_2 \in q_1q_2q_3q_4) \longleftrightarrow q_1 \oplus q_2 \oplus q_3 \oplus q_4 \tag{2.22}$$

$$\prod_{p_1 \ldots p_n}\left(\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}q_l\right) \longleftrightarrow \prod_{l=1}^{2^n} q_l. \tag{2.23}$$

For instance,

$$\prod_{p_1p_2}(p_1p_2 \in q_1q_2q_3q_4) \longleftrightarrow q_1 \odot q_2 \odot q_3 \odot q_4 \tag{2.24}$$

$$\left(\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}q_l\right)' \longleftrightarrow \left(\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}q'_l\right). \tag{2.31}$$

For instance,

$$(p_1p_2 \in q_1q_2q_3q_4)' \longleftrightarrow (p_1p_2 \in q_1'q_2'q_3'q_4'). \tag{2.32}$$

If the following expression

$$\left[\left(\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}q_l\right) \circ \left(\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}r_l\right)\right] \longleftrightarrow \left[\underset{k=1}{\overset{n}{S}}p_k \in \underset{l=1}{\overset{2^n}{S}}(q_l \circ r_l)\right] \tag{2.41}$$

is considered, then, after substitution in it for the „circle" of one of the following symbols:

$$\oplus \quad \odot \quad \to \quad \longleftrightarrow \quad \ominus \quad \dot{-}$$

a scheme of a thesis belonging to the propositional calculus is obtained,[4]) for insta ce

$$[(p_1 \in q_1q_2) \oplus (p_1 \in r_1r_2)] \longleftrightarrow [p_1 \in (q_1 \oplus r_1)(q_2 \oplus r_2)] \tag{2.42}$$

$$[(p_1p_2 \in q_1q_2q_3q_4) \dot{-} (p_1p_2 \in r_1r_2r_3r_4)] \longleftrightarrow$$

$$\longleftrightarrow [p_1p_2 \in (q_1 \dot{-} r_1)(q_2 \dot{-} r_2)(q_3 \dot{-} r_3)(q_4 \dot{-} r_4)]. \tag{2.43}$$

[4]) Compare the previous communication, scheme 3. 31.