On regular and combinatorial imbedding.

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(Received February 11th, 1947.)

In his paper *Lattices and topological spaces* (Annals of Math. **39** (1938), 112—127) H. Wallman constructed, for an arbitrary topological space*) Q, a definite bicompact space ωQ containing Q as a dense subset. In § 3 of the present paper, we prove that ωQ may be characterised by the property that Q is both regularly and combinatorially imbedded in it. Regular imbedding is defined and analyzed in § 1, combinatorial imbedding, in § 2. In § 4, we consider the question whether two points may be separated by open subsets of ωQ.

1. Definition. A subspace Q of a space P is said to be *regulary imbedded* in P if the family $(\overline{F})$ of the closures in P of all sets F closed in Q constitutes a closed basis of P, i. e. if every set closed in P is the intersection of some subfamily of the family $(\overline{F})$. As P itself is closed in P, we have:

(1.1) *If Q is regularly imbedded in P, then Q is dense in P.*

(1.2) *If Q is regularly imbedded in P and if $Q \subset P_0 \subset P$, then Q is regularly imbedded in P_0.*

Definition. Let $Q \subset P$. The point $x \in P$ is said to be a *Q-regular point* of P if, for any set $\Phi \subset P - x$ closed in P, there exists a set F closed in Q such that $\Phi \subset \overline{F} \subset P - x$, $\overline{F}$ indicating closure in P. Clearly:

(1.3) *$Q \subset P$ is regularly imbedded in P if, and only if, (i) Q is dense in P, (ii) any point $x \in P$ is Q-regular in P.*

(1.4) *If $x \in P$ is a regular point of P, then x is Q-regular for any set Q dense in P.*

Proof. Let $\Phi \subset P - x$ be closed in P. Then $P - \Phi$ is a neighborhood of x in P. As x is a regular point of P, there exists an open

*) We consider only spaces in which the closure of any point set is closed and, for convenience, we make also the easily avoidable assumption (not made by Wallman) that each finite point set is closed.

neighborhood U of x in P such that $\overline{U} \subset P - \Phi$. The set $F = Q - U$ is closed in Q and $\overline{F} \subset P - U \subset P - x$. As $Q = QU + F$, we have $P = \overline{Q} \subset \overline{U} + \overline{F} \subset (P - \Phi) + \overline{F}$, whence $\Phi \subset \overline{F}$.

Definition. A space P is called *nearly regular* if any Q dense in P is regularly imbedded in P. From (1.3) and (1.4) we have:

(1.5) *Any regular space is nearly regular.*

Definition. A space P is called *hereditarily nearly regular* (h. n. r.) if every subspace of P is nearly regular. Since regularity is a hereditary property, (1.5) gives:

(1.6) *Any regular space is h. n. r.*

From (1.2) we see at once:

(1.7) *If every closed subspace of P is nearly regular, then P is h. n. r.*

Example 1. The space P_1 consists of the points x_{ni} ($n = 1, 2, 3, \ldots$, $i = 1, 2, 3, \ldots$), x_n($n = 1, 2, 3, \ldots$), and z. Each point x_{ni} is an isolated point. The point x_n possesses the fundamental neighborhoods U_{nk}($k = 1, 2, 3, \ldots$) consisting of x_n and x_{ni} ($i \geqq k$). The point z possesses the fundamental neighborhoods V_k ($k = 1, 2, 3 \ldots$) consisting of z and x_{ni} ($n \geqq k$, $i \geqq k$). Clearly P_1 is a countable Hausdorff space satisfying the second countability axiom; each point except z is regular. The subspace Q_1 consisting of z and all x_{ni}'s is dense in P_1, but Q_1 is not regularly imbedded in P_1, since the set Φ consisting of all x_n's is closed in P_1, but Φ is not of the form $\Pi\overline{F}$ for any family (F) of sets closed in Q_1. Hence P_1 is not nearly regular.

Example 2. The space P_2 consists of the points x_{ni}, y_{ni} ($n = 1, 2, 3, \ldots$, $i = 1, 2, 3, \ldots$), x_n($n = 1, 2, 3, \ldots$), and z. The points x_{ni} and y_{ni} are isolated. Each point x_n possesses the fundamental neighborhoods U_{nk} ($k = 1, 2, 3, \ldots$) consisting of the points x_{ni} ($i \geqq k$), y_{ni} ($i \geqq k$), and x_n. The point z possesses the fundamental neighborhoods V_k ($k = 1, 2, 3, \ldots$) consisting of the points x_{ni} ($n \geqq k$, $i \geqq k$) and z. Again, P_2 is a countable Hausdorff space satisfying the second countability axiom and z is the only irregular point of P_2. We shall prove that P_2 is nearly regular. Let Q be any dense subset of P_2; clearly $Y \subset Q$, Y being the set of all y_{ni}'s. By (1.3) and (1.4) we have only to show that the point z is Q-regular. Let $\Phi \subset P_2 - x$ be closed in P_2. Then $F = Q\Phi + Y_0$ is closed in Q, Y_0 being the closure of Y in Q and clearly $\Phi \subset \overline{F} \subset P_2 - z$. Hence z is Q-regular. Therefore P_2 is nearly regular, but not hereditarily, which follows from example 1.

Example 3. Let M be any uncountable set. The space P_3 consists of the points $x_{n\mu}$ ($n = 1, 2, 3, \ldots$, $\mu \in M$), x_n ($n = 1, 2, 3, \ldots$), and z. The points $x_{n\mu}$ are isolated. Each point x_n possesses the fundamental neighborhoods U_{nK} consisting of the points $x_{n\mu}$

($\mu \in M - K$) and x_n, where K runs over the family of all finite subsets of M. The point z possesses the fundamental neighborhoods $V_k - S_k$ ($k = 1, 2, 3, \ldots$), V_k consisting of the points $x_{n\mu}$ ($n \geq k$, $\mu \in M$) and S_k running over the family of all countable subsets of V_k. Clearly P_3 is a Hausdorff space and z is the only irregular point of P_3. We shall show that the space P_3 is *h. n. r.* Let Q denote any subspace of P_3 such that $z \in \overline{Q}$. By (1.3) and (1.4) we have only to show that, in the space $\overline{Q}$, the point z is Q-regular. Let $\Phi \subset \overline{Q} - z$ be closed in $\overline{Q}$, hence in P_3. Let X_n ($n = 1, 2, 3, \ldots$) denote the set of the points $x_{n\mu}$ ($\mu \in M$). For each n such that the set X_nQ is infinite, choose an infinite countable subset T_n of X_nQ; let $T_n = 0$ if X_nQ is finite. Then $F = Q\Phi + \left(\sum_1^\infty T_n\right)_0$, the subscript 0 indicating closure in Q, is closed in Q and it is easy to see that $\Phi \subset \overline{F} \subset P_3 - z$, which proves z to be Q-regular in $\overline{Q}$.

2. Definition. Let $n = 2, 3, 4, \ldots$. A subspace Q of a space P is said to be *n-combinatorially imbedded* in P if, for any choice $F_1, F_2, \ldots, F_n$ of relatively closed subsets of Q such that $\prod_1^n F_i = 0$ we have $\prod_1^n \overline{F}_i = 0$. Clearly m-combinatorial imbedding implies n-combinatorial imbedding for $2 \leq n < m$. The imbedding is said to be *combinatorial* if it is n-combinatorial for each $n = 2, 3, 4, \ldots$

Definition. A subspace Q of a space P is said to be *combinatorially imbedded in P in the strong sense* if, for any choice F_1, F_2 of relatively closed subsets of Q we have $\overline{F_1F_2} = \overline{F}_1\overline{F}_2$. By an easy induction, this implies $\overline{\Pi F_i} = \Pi\overline{F}_i$ for any finite number of relatively closed $F_i \subset Q$, so that combinatorial imbedding in the strong sense implies ordinary combinatorial imbedding.

(2.1) *Let Q be 2-combinatorially imbedded in a regular space P. Then Q is combinatorially imbedded in P in the strong sense.*

Proof. Suppose, on the contrary, that there exist two relatively closed sets $F_1 \subset Q$ and $F_2 \subset Q$ such that $\overline{F_1F_2} \neq \overline{F}_1\overline{F}_2$. Then there exists a point $x \in \overline{F}_1\overline{F}_2 - \overline{F_1F_2}$. By regularity, there exists an open neighborhood U of x in P such that $\overline{U}\overline{F_1F_2} = 0$, whence $\overline{U}F_1F_2 = 0$. Clearly $x \in \overline{\Phi}_1\overline{\Phi}_2$ where the sets $\Phi_1 = F_1\overline{U}$ and $\Phi_2 = F_2\overline{U}$ are closed in Q. But this is impossible, since $\Phi_1\Phi_2 = 0$ and Q is 2-combinatorially imbedded in P.

For $n = 0, 1, 2, \ldots$ let ω_n denote the least ordinal number of power $\aleph_n$ and Z_n, the set of all ordinal numbers $\xi < \omega_n$.

Example 4. Let $P_4 = Z_1 + \omega_1$. Each $\xi \in Z_1$ possesses the fundamental system of neighborhoods $U_{\xi\eta}$ ($\eta \in Z_1$, $\eta < \xi$), where

$U_{\xi\eta}$ consists of all ordinals ζ such that $\eta < \zeta \leqq \xi$. The point ω_1 possesses the fundamental system of neighborhoods V_ξ $(\xi \in Z_1)$, where V_ξ consists of ω_1 together with all isolated ordinals $\eta \in Z_1$, $\eta > \xi$. Clearly P_4 is a Hausdorff space and ω_1 is the only irregular point of P_4. Then Z_1 is combinatorially imbedded in P_4. Suppose, on the contrary, that there exist relatively closed sets $F_i \subset Z_1$ $(1 \leqq i \leqq n)$ such that $\prod_1^n F_i = 0 \neq \prod_1^n \overline{F}_i$. Then it is clear that no F_i is countable. But then there exist points $\xi_k \in Z_1$ $(k = 0, 1, 2, \ldots)$ such that $\xi_0 < \xi_1 < \xi_2 < \ldots$ and

$$\xi_{jn} \in F_1,\ \xi_{jn+1} \in F_2, \ldots, \xi_{jn+n-1} \in F_n \ (j = 0, 1, 2, \ldots)$$

which is impossible, since it implies $\lim \xi_k \in \prod_1^n F_i$. Hence Z_1 is combinatorially imbedded in P_4, but not in the strong sense. For let F_1 consist of the points

$$\xi,\ \xi + 1,\ \xi + 3,\ \xi + 5, \ldots$$

and F_2, of the points

$$\xi,\ \xi + 2,\ \xi + 4,\ \xi + 6, \ldots,$$

$\xi \in Z_1$ running over all non isolated ordinals. The sets $F_1 \subset Z_1$ and $F_2 \subset Z_2$ are relatively closed and we have $\omega_1 \in \overline{F}_1\overline{F}_2 - \overline{F_1F_2}$.

Lemma.*) Let $m, i_1, i_2, \ldots, i_m$ be integers such that $m \geqq 1$, $0 \leqq i_1 < i_2 < \ldots < i_m$. Let $S = S(i_1, i_2, \ldots, i_m)$ be the cartesian product

$$(Z_{i_1} + \omega_{i_1}) \times (Z_{i_2} + \omega_{i_2}) \times \ldots \times (Z_{i_m} + \omega_{i_m})$$

in its usual topology. Let A be a subset of $Z_{i_1} \times Z_{i_2} \times \ldots \times Z_{i_m}$ such that $(\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_m}) \in \bar{A}$. Choose an integer r such that $1 \leqq r \leqq m$ and an ordinal $\alpha \in Z_{i_r}$. Then $\bar{A}$ contains a point $(\xi_1, \xi_2, \ldots, \xi_m)$ such that $\xi_s = \omega_{i_s}$ for $1 \leqq s \leqq m$, $s \neq r$ and $\alpha < \xi_r < \omega_{i_r}$.

Proof. The lemma being trivial for $m = 1$, we may assume its valididy for $m - 1$. Suppose first $r < m$. Since $(\omega_{i_1}, \omega_{i_2}, \ldots, \omega_{i_m}) \in \bar{A}$, for a given $(\alpha_1, \alpha_2, \ldots, \alpha_m) \in Z_{i_1} \times Z_{i_2} \times \ldots \times Z_{i_m}$ the set A contains points $(\xi_1, \xi_2, \ldots, \xi_m)$ such that $\alpha_1 < \xi_1 < \omega_{i_1}, \ldots, \alpha_m < \xi_m < \omega_{i_m}$. The cardinal number of the set of all such points $(\xi_1, \xi_2, \ldots, \xi_m)$ being equal to $\aleph_m$, whence greater than the cardinal number of $Z_{i_1} \times \ldots \times Z_{i_{m-1}}$, the cardinal number of the set of our points will remain equal to $\aleph_m$ even if we restrict the

*) This lemma is a fairly obvious generalization of a result of A. Tychonoff (Math. Annalen 102, 1930, see Behauptung I., on p. 553 and Behauptung III., on p. 555).

first $m - 1$ coordinates to fixed, but conveniently chosen, values. Therefore, $\bar{A}$ contains points $(\xi_1, \xi_2, \ldots, \xi_m)$ such that $\alpha_s < \xi_s < \omega_{i_s}$ for $1 \leqq s \leqq m - 1$ and $\xi_m = \omega_{i_m}$. Now if B denotes the set of all $(\xi_1, \ldots, \xi_{m-1}) \in Z_{i_1} \times \ldots \times Z_{i_{m-1}}$ such that $(\xi_1, \ldots, \xi_{m-1}, \omega_{i_m}) \in \bar{A}$ we have clearly $(\omega_{i_1}, \ldots, \omega_{i_{m-1}}) \in \bar{B}$ in the space $S(i_1, \ldots, i_{m-1})$. The lemma being true for $m - 1$, $\bar{B}$ contains a point $(\xi_1, \ldots, \ldots, \xi_{m-1})$ such that $\xi_s = \omega_{i_s}$ for $1 \leqq s \leqq m - 1$, $s \neq r$ and $\alpha < < \xi_r < \omega_{i_r}$; but then $(\xi_1, \ldots, \xi_{m-1}, \omega_{i_m}) \in \bar{A}$. Secondly, let $r = m$. Choose $(\alpha_2, \ldots, \alpha_m) \in Z_{i_2} \times \ldots \times Z_{i_m}$. By transfinite induction, we may construct a transfinite sequence (p_λ) of type ω_{i_1} of points $p_\lambda = (\xi_{\lambda 1}, \ldots, \xi_{\lambda m}) \in A$ such that $\xi_{\lambda 1} < \xi_{\mu 1}, \ldots, \xi_{\lambda m} < \xi_{\mu m}$ for $\lambda < < \mu < \omega_{i_1}$ and $\xi_{\lambda_2} > \alpha_2, \ldots, \xi_{\lambda_m} > \alpha_m$ for all λ's. The point $p = = (\xi_1, \ldots, \xi_m) = \lim p_\lambda$ belongs to $\bar{A}$ and we have $\xi_1 = \omega_{i_1}$ and $\alpha_s < \xi_s < \omega_{i_s}$ for $2 \leqq s \leqq m$. Hence if B denotes the set of all $(\xi_2, \ldots, \xi_m) \in Z_{i_1} \times \ldots \times Z_{i_m}$ such that $(\omega_{i_1}, \xi_2, \ldots, \xi_m) \in \bar{A}$ we have $(\omega_{i_2}, \ldots, \omega_{i_m}) \in \bar{B}$ in the space $S(i_2, \ldots, i_m)$. The lemma being true for $m - 1$, $\bar{B}$ contains a point $(\xi_2, \ldots, \xi_m)$ such that $\xi_s = \omega_{i_s}$ for $2 \leqq s \leqq m - 1$ and $\alpha < \xi_m < \omega_{i_m}$; but then $(\omega_{i_1}, \xi_2, \ldots, \xi_m) \in \bar{A}$.

Example 5.*) Let $n = 3, 4, 5, \ldots$. The space P_5 consists of all n-tuples $(\xi_1, \xi_2, \ldots, \xi_n)$ such that $\xi_i \in Z_i + \omega_i$ for $1 \leqq i \leqq n$ and $\xi_i = \omega_i$ for at least one i. We put $z = (\omega_1, \omega_2, \ldots, \omega_n)$. The point $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in P_5 - z$ possesses the fundamental system of neighborhoods $V_\xi(\eta_1, \eta_2, \ldots, \eta_n)$ $(\eta_i \in Z_i, \eta_i < \xi_i$ for $1 \leqq i \leqq n)$ consisting of all n-tuples $(\zeta_1, \zeta_2, \ldots, \zeta_n) \in P_5$ such that $\eta_i < \zeta_i \leqq \xi_i$ for $1 \leqq i \leqq n$. The point z possesses the fundamental system of neighborhoods $V(\xi_1, \xi_2, \ldots, \xi_n)$ $(\xi_i \in Z_i$ for $1 \leqq i \leqq n)$ consisting of z together with all points $(\eta_1, \eta_2, \ldots, \eta_n) \in P_5$ such that $\eta_i > \xi_i$ for $1 \leqq i \leqq n$ and $\eta_i = \omega_i$ for one and only one value of i. Clearly P_5 is a Hausdorff space and z is the only irregular point of P_5. For $1 \leqq i \leqq n$, let Φ_i consist of all points $(\xi_1, \xi_2, \ldots, \xi_n) \in P_5 - z$ such that $\xi_i = \omega_i$. Then the sets $\Phi_i \subset P_5 - z$ are relatively closed and we have $\prod_1^n \Phi_i = 0$, $\prod_1^n \bar{\Phi}_i = z$, whence $P_5 - z$ is not n-combinatorially imbedded in P_5. However, we shall show that this imbedding is $(n - 1)$-combinatorial. First, let us put $S_i = S(j_1, \ldots, \ldots, j_{n-1})$ (see the lemma above) the sequence $j_1, \ldots, j_{n-1}$ being obtained from the sequence $1, 2, \ldots, n$ by cancelling the term i. If f_i $(1 \leqq i \leqq n)$ denotes the cancelling of the i-th coordinate,

*) This example (for $n = 3$) is due to M. Katětov.

then $f_i(\Phi_i + z) = S_i$ is $1 - 1$, though not topological; however, the partial transformation $f_i(\Phi_i) = S_i - z$ is a homeomorphism. Now let the sets $F_r \subset P_5 - z$ $(1 \leqq r \leqq n - 1)$ be relatively closed and let $z \in \prod\limits_1^{n-1} \overline{F}_r$. We have to show that $\prod\limits_1^{n-1} F_r \neq 0$. Since $P_5 - z = \sum\limits_1^{n} \Phi_i$, for each $r (1 \leqq r \leqq n - 1)$ there must exist an $i_r (1 \leqq i_r \leqq n)$ such that $z \in \overline{F_r \Phi_{i_r}}$. Now this relation valid in the space P_5 evidently implies the analogous relation $z \in \overline{f_{i_r}(F_r \Phi'_{i_r})}$ in the space S_{i_r} where $\Phi'_{i_r} = \Phi_{i_r} - \sum\limits_{j \neq i_r} \Phi_j$. Since r assumes only $n - 1$ values, there exists an integer s such that $1 \leqq s \leqq n$ and $s \neq i_r$ for $1 \leqq r \leqq n - 1$. Using the lemma and recalling that $f_{i_r}(\Phi_i) = S_{i_r} - z$ is a homeomorphism, we see that, for any given ordinal $\alpha \in Z_s$ and for any r $(1 \leqq r \leqq n - 1)$, there exists a point $p = (\xi_1, \xi_2, \dots, \xi_n) \in \overline{F_r \Phi_{i_r}}$ such that $\xi_i = \omega_i$ for $1 \leqq i \leqq n$, $i \neq s$ and $\alpha < \xi < \omega_s$. Of course, we have $p \in F_r \Phi_{i_r}$ since the set $F_r \Phi_{i_r} \subset P_5 - z$ is relatively closed. By induction, we may now construct an infinite sequence of points $p_k = (\xi_{k1}, \xi_{k2}, \dots, \xi_{kn})$ such that $\xi_{ki} = \omega_i$ for $1 \leqq i \leqq n$, $i \neq s$ and all k's, $\xi_{1s} < \xi_{2s} < \xi_{3s} < \dots < \omega_s$ and $p_k \in F_r \Phi_{i_r}$ for $1 \leqq r \leqq n - 1$, $k \equiv r \bmod (n - 1)$. There exists the limit point $p = \lim p_k$ and clearly $p \in \prod\limits_1^{n-1} F_r$, whence $\prod\limits_1^{n-1} F_r \neq 0$.

3. Let Q be any given topological space. We recall briefly the definition of Wallman's bicompact space $\omega Q \supset Q$. Points of $\omega Q - Q$ will be called *ideal points* and points of Q, *real points*. We have to define first the ideal points and secondly the topology of ωQ. An ideal point α is, by definition, a collection of subsets of Q (called the *coordinates of* α) having the following properties:

(*i*) the elements of the collection are non vacuous closed subsets of Q,

(*ii*) the intersection of any finite number of elements of the collection belongs itself to the collection,

(*iii*) any closed subset of Q intersecting each element of the collection belongs itself to the collection,

(*iv*) the intersection of the whole collection is vacuous.

For any open subset G of Q, let G^* consist of all real points belonging to G and of all ideal points α such that there exists some coordinate $A \subset G$ of α. If G runs over the family of all open subsets of Q then G^* runs over an open basis of ωQ, thus defininfg the topology of ωQ. For any closed subset F of Q, the closure $\overline{F}$ of F in ωQ consists of all real points belonging to F and of all ideal points α such that F is a coordinate of α.

(3.1) *The imbedding of an arbitrary topological space Q in Wallman's bicompact space ωQ is both regular and combinatorial in the strong sense.*

Proof. We begin by proving that the imbedding is regular. Q is clearly dense in ωQ. Let x be any point (real or ideal) of ωQ and let Φ be a closed subset of ωQ not containing x. By (1.3) it suffices to indicate a closed subset F of Q such that $\Phi \subset \overline{F} \subset \omega Q - x$. Since x belongs to the open subset $\omega Q - \Phi$ of ωQ, there exists an open subset G of Q such that $x \in G^* \subset \omega Q - \Phi$. Then $F = Q - G$ is a closed subset of Q. Since $x \in G^*$, we cannot have $x \in \overline{F}$. This is evident if x is real; if x is ideal, then $x \in G^*$, $x \in \overline{F}$ would mean that x has a coordinate $A \subset G$ as well as the coordinate F, which is impossible as $GF = 0$. It remains to show that $\alpha \in \overline{F}$ for any $\alpha \in \Phi$. For a real α this is a consequence of the evident relation $Q\Phi \subset Q - G = F$; if α is ideal, the inclusion $G^* \subset \omega Q - \Phi$ shows that, since $\alpha \in \Phi$, any coordinate of α meets $Q - G = F$ so that F itself is a coordinate of α whence $\alpha \in \overline{F}$.

It remains to show that the imbedding is combinatorial in the strong sense. Let F_1 and F_2 be two closed subsets of Q and let $\alpha \in \overline{F}_1\overline{F}_2$; we have to prove that $\alpha \in \overline{F_1F_2}$. This being evident for a real α, let α be ideal. Then $\alpha \in \overline{F}_1$, $\alpha \in \overline{F}_2$ means that both F_1 and F_2 are coordinates of α so that F_1F_2 is also a coordinate of α whence $\alpha \in \overline{F_1F_2}$.

(3.2) *Let the space Q be both regularly and 2-combinatorially imbedded in the bicompact space P. Then there exists a homeomorphism $f(\omega Q) \subset P$ such that $f(x) = x$ for each $x \in Q$. If the imbedding is combinatorial, we have $f(\omega Q) = P$.*)*

Proof. For any $X \subset Q$, let $\overline{X}$ denote the closure of X in the space ωQ and $\tilde{X}$, the closure in the space P. For $x \in Q$, let $f(x) = x$. We next define $f(\alpha)$ for an ideal point α of ωQ. Now α is, by definition, a collection of closed subsets of Q having properties (*i*) to (*iv*). Let α^0 denote the collection of all sets $\tilde{A}$, A running over α. By properties (*i*) and (*ii*), the intersection of a finite subcollection of α^0 is never vacuous; the space P being bicompact, the intersection $\varphi(\alpha)$ of the whole collection α^0 is not vacuous either; by property (*iv*), $Q.\varphi(\alpha) = 0$. Hence $\varphi(\alpha)$ contains at least one point $\beta \in P - Q$. We have $\beta \in \tilde{A}$ for any $A \in \alpha$. Conversely, let F be a closed subset of Q such that $\beta \in \tilde{F}$. Then $\tilde{A}\tilde{F}$ contains β for any

*) We do not know whether $f(\omega Q) = P$ whenever the imbedding is 2-combinatorial.

$A \in \alpha$. The imbedding of Q in P being 2-combinatorial, it follows that $AF \neq 0$ for each $A \in \alpha$, whence $F \in \alpha$ by property (*iii*). Hence the collection α consists exactly of those closed subsets A of Q for which the relation $\beta \in \tilde{A}$ holds true. Now by regularity of the imbedding of Q in P, the one point closed subset (β) of P is the intersection of all such $\tilde{A}$'s. It follows that the set $\varphi(\alpha)$ consists of just the one point β and we may put $f(\alpha) = \beta$. The transformation $f(\omega Q) \subset P$ so defined is clearly $1 - 1$ and $f(x) = x$ for each $x \in Q$. Let us put $f(\omega Q) = P_0$ so that $Q \subset P_0 \subset P$.

For any closed subset F of Q we must have $f(\overline{F}) = P_0 . \tilde{F}$. Suppose first that $\beta \in P_0 . \tilde{F}$; we have to prove that $\beta \in f(\overline{F})$. If $\beta \in Q$, then $\beta \in F \subset f(\overline{F})$; hence suppose $\beta \in P_0 - Q$. By definition of P_0, there exists an ideal point α of ωQ such that $\beta = f(x)$; α consists of all closed subsets A of Q such that $\beta \in \tilde{A}$; since $\beta \in \tilde{F}$, we have $F \in \alpha$, whence $\alpha \in \overline{F}$ and $\beta = f(\alpha) \subset f(\overline{F})$. Conversely, let $\beta \in f(\overline{F})$ so that $\beta \in P_0$; we have to prove that $\beta \in \tilde{F}$. There exists an $\alpha \in \overline{F}$ such that $\beta = f(\alpha)$. If α is real, we have $\beta = \alpha \in F \subset f(\overline{F})$. If α is ideal, then $\alpha \in \overline{F}$ means $F \in \alpha$, whence $\beta = f(\alpha) \in \tilde{F}$.

Let C_0 be a closed subset of P_0. There exists a closed C of P such that $C_0 = P_0 . C$. The imbedding of Q in P being regular, there exists a family φ of closed subsets F of Q such that $C = \Pi \tilde{F}$, whence $C_0 = \Pi P_0 . \tilde{F}$, F running over φ. But $P_0 . \tilde{F} = f(\overline{F})$ and the transformation f being $1 - 1$, we have $C_0 = \Pi f(\overline{F}) = f(\Pi \overline{F})$. Hence each closed subset C_0 of P_0 has the form $C_0 = f(\Phi)$, Φ being closed in ωQ. Conversely, let Φ be closed in ωQ. By (3.1), the imbedding of Q in ωQ is regular. Hence there exists a family φ of closed subsets F of Q such that $\Phi = \Pi \overline{F}$. The transformation f being $1 - 1$, we have $C_0 = f(\Phi) = \Pi f(\overline{F}) = \Pi P_0 . \tilde{F} = P_0 . \Pi \tilde{F}$. The set C_0 is the intersection of P_0 and a closed subset of P; therefore, C_0 is closed in P_0. Consequently, the closed subsets of P_0 are precisely the sets $f(\Phi)$ with Φ closed in ωQ, which proves that the transformation f is topological.

Now suppose that the imbedding of Q in P is combinatorial and choose $\beta \in P$. We have to prove that $\beta \in f(\omega Q)$. This being evident for $\beta \in Q$, suppose $\beta \in P - Q$. The imbedding of Q in P being regular, there exists a family φ of closed subsets F of Q such that $\beta = \Pi \tilde{F}$ for $F \in \Phi$. Since $\beta \in P - Q$, we must have $\Pi F = 0$. Now for any finite subfamily $F_1, F_2, \ldots, F_n$ of Φ, we have $\beta \in \prod_1^n \tilde{F}_i$, whence $\prod_1^n F_i \neq 0$, the imbedding of Q in P being combinatorial.

As the space ωQ is bicompact, there must exist a point $\alpha \in \Pi \bar{F}$ for $F \in \Phi$. Since $f(\bar{F}) = P_0\bar{F} \subset \tilde{F}$, we have $f(\alpha) \subset \Pi\tilde{F} = \beta$, whence $\beta = f(\alpha) \in f(\omega Q)$.

4. Two points a and b of a space P will be said to be *H-separated* if there exist two open sets G_1 and G_2 such that $a \in G_1$, $b \in G_2$, $G_1G_2 = 0$. A Hausdorff space is then a space such that any two distinct points are H-separated. As was shown by Wallman (l. c.), the space ωQ is a Hausdorff space if, and only if, the space Q is normal. We consider here the question of H-separability in ωQ of two real points, a real and an ideal point, and two ideal points. Clearly two H-separated points of a space P are H-separated in every subspace of P containing them.

For a Hausdorff space Q, two real points are always H-separated in ωQ. This is a consequence of the following trivial theorem.

(4.1) *If two points a and b are H-separated in a dense subspace Q of a space P, a and b are H-separated in P.*

Proof. There exist two open subsets H_1 and H_2 of Q such that $a \in H_1$, $b \in H_2$, $H_1H_2 = 0$. The sets $F_1 = Q - H_1$ and $F_2 = Q - H_2$ are closed in Q and $a \in Q - F_1$, $b \in Q - F_2$, $F_1 + F_2 = Q$. Therefore $a \in P - \bar{F}_1$, $b \in P - \bar{F}_2$, $\bar{F}_1 + \bar{F}_2 = P$. The sets $G_1 = P - \bar{F}_1$ and $G_2 = P - \bar{F}_2$ are open in P and $a \in G_1$, $b \in G_2$, $G_1G_2 = 0$.

(4.2) *A point $a \in Q$ is regular in ωQ if, and only if, it is regular in Q.*

Proof. If a is regular in ωQ then, of course, a is regular in $Q \subset \omega Q$ as well. Let a be regular in Q. If U is any neighborhood of a in ωQ, there exists a neighborhood G of a in Q such that $G^* \subset U$. Since a in regular in Q, there exists a neighborhood H of a in Q the closure of which in Q is contained in G. It is easy to see that H^* is a neighborhood of a in ωQ the closure of which is contained in G^*, whence in U.

(4.3) *If a is an irregular point of the bicompact space P, there exists a point $b \in P - a$ such that a and b are not H-separated.*

Proof. There exists a neighborhood U of a such that $\bar{V} - U \neq 0$ for every neighborhood V of a. If V_i $(1 \leq i \leq n)$ are neighborhoods of a, then $\prod_1^n V_i$ is also a neighborhood of a, whence

$$\prod_1^n (\bar{V}_i - U) \supset \overline{\prod_1^n V_i} - U \neq 0.$$

The space being bicompact, there exists a point b such that $b \in \bar{V} - U$ for every neighborhood V of a. It is easy to see that a and b are not H-separated.