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On the asymptotic distribution of geodesics on surfaces of revolution.

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The present note deals with the distribution questions of a geodesic on a surface of revolution in the recurrent case. It is an immediate consequence of the Kronecker approximation theorem that such a geodesic, when not periodic, is dense on a domain Θ which is either bordered by two parallel circles on the surface or is the whole surface (which then is of genus 1). It seems of some interest to go beyond this fact, by considering also the asymptotic density which bellongs to the different points of Θ. While the existence of such a density is implied by the general theory of distribution functions of almost periodic functions, it turns out that a direct consideration leads not only to an existence proof but also to the explicit representation of the density in geometrical terms.

The result admits a dynamical interpretation, since the principle of Maupertuis reduces the problem of a particle moving in a field of force of radial symmetry to the problem of geodesics on a surface of revolution whose ds^2 is determined by the force function and the energy constant. The density of the asymptotic distribution on Θ then corresponds to the density of probability belonging to the points of the circular ring in which the path described by the „rotating ellipse“ is everywhere dense. Since this density is obtained in explicit form, the result might have some interest also in view of the applications of the theory of adiabatic invariants*) to the classical dynamical problem in question. In fact, the identity of a priori („geometrical“) probabilities with asymptotic relative frequencies is proved, instead of being postulated.

In order to have a representation of the surface of revolution which is valid in the large without exceptions whenever the geo-

*) T. **Levi-Civita**, Drei Vorlesungen über adiabatische Invarianten, Hamburger Abhandlungen, 6 (1928), 323—366.

desic is not of the asymptotic type, it is convenient to use as Gaussian parameters the arc length along the meridian and the angle φ of rotation from a fixed meridian. Then the square of the line element is $ds^2 = d\varrho^2 + r^2 d\varphi^2$, where $r = r(\varrho)$ denotes the distance from the axis of rotation.

Local singularities of the geometry of ds^2 will be excluded by the assumption that the function r of ϱ has a continuous derivative and is positive in the ϱ-interval under consideration. Denoting by primes differentiations with respect to the arc length, those geodesics $\varrho = \varrho(s)$, $\varphi = \varphi(s)$ along which $\varrho(s)$ is not constant are identical with the solutions of the pair of equations

$$r(\varrho)^2\varphi' = c, \qquad \varrho'^2 + c^2/r(\varrho)^2 = 1, \tag{1}$$

which express the conservation of angular momentum and energy (the vis viva is 1, since the arc length s is the time variable). Geodesics which either do not remain in a bounded portion of the surface or tend asymptotically to a second geodesic when $s \to +\infty$ or $s \to -\infty$ will be excluded. Writing the energy integral in the form

$$\varrho'^2 = 1 - c^2/r(\varrho)^2, \tag{2}$$

it is easy to prove that only two cases are possible:

(*i*) the expression on the right of (2), considered as a function of ϱ, has two successive simple roots, say $\varrho = a$ and $\varrho = b$, and this function of ϱ is positive for $a < \varrho < b$;

(*ii*) the function $r(\varrho)$ of ϱ is periodic, the surface is a torus, and $\varrho(s)$ is a strictly monotone function for $-\infty < s < +\infty$. Notice that a geodesic on a torus can belong to case (*i*).

In case (*i*), one sees from (2) that $r(a) = r(b) = c$, and that $r(\varrho) < c$ if $a < \varrho < b$. It also follows from (2) that the function $\varrho(s)$ of s is periodic with the period

$$\omega = 2\int_a^b \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}} d\varrho, \tag{3}$$

and that a and b are the maximum and minimum of $\varrho(s)$. Also if $s = 0$ belongs to $\varrho = a$, then $\varrho(s)$ is an even function which is strictly increasing for $0 < s < \frac{1}{2}\omega$.

In case (*ii*), one has $r(\varrho) > c$ for every ϱ. The expression on the right of (2) is a periodic function of ϱ with the same period as $r(\varrho)$. Since the expression on the right of (2) has no zeros by assumption, it is clear from (2) that $\varrho(s)$ is, for $-\infty < s < +\infty$, strictly monotone, say increasing, and that if $\lambda > 0$ is the least positive period of $r(\varrho)$ and ω denotes the positive number

$$\omega = \int_0^\lambda \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}} d\varrho, \tag{4}$$

then there exists a function $\chi(s)$ such that

$$\varrho(s) = \lambda s/\omega + \chi(s), \text{ where } \chi(s + \omega) = \chi(s). \tag{5}$$

(That the mean motion of $\varrho(s)$ is λ/ω follows from $\varrho(s + \omega) - \varrho(s) = \lambda$.)

The above description of the cases (i), (ii) implies that if the problem of geodesics is thought of, for a fixed value of the integration constant c, as a conservative dynamical problem with a single degree of freedom, the case (ii) represents the case of a Lagrangian coordinate which is an angular variable (in the same sense as the coördinate of an overturning pendulum); while in the case (i) the coordinate ϱ is a linear coordinate which cannot be reduced to a suitable modulus.

Let the case (i) be considered first. Then, placing $\psi(s) = c/r(\varrho(s))^2$, the first of the integrals (1) can be written as

$$\varphi'(s) = \psi(s), \text{ where } \psi(s + \omega) = \psi(s), \text{ since } \varrho(s + \omega) = \varrho(s). \tag{6}$$

If α denotes the integral of the periodic function $\psi(s)$ over a period, and $\sigma = \sigma(s)$ the indefinite integral of the function $\psi(s) - \alpha/\omega$, then, from (6),

$$\varphi(s) = \alpha s/\omega + \sigma(s), \text{ where } \sigma(s + \omega) = \sigma(s). \tag{7}$$

Since $r(\varrho)$ is positive for $a \leqq \varrho \leqq b$, and since α/ω is the mean value of the periodic function $\psi(s) = c/r(\varrho(s))^2$, where $\min \varrho(s) = a$ and $\max \varrho(s) = b$, it is clear that the constant α is positive. Now there are two cases possible, according as α is or is not commensurable with 2π. In the first case, (7) together with $\varrho(s + \omega) = \varrho(s)$ implies that the geodesic is a closed curve on the surface; while in the second case the approximation theorem of Kronecker shows that the geodesic is everywhere dense on the portion

$$\Theta: \quad a \leqq \varrho \leqq b, \ 0 \leqq \varphi < 2\pi \tag{8}$$

of the surface. Only the latter case will be considered in what follows.

Choosing the notation such that $a = \min \varrho(s)$ is attained at $s = 0$, the even periodic function $\varrho(s)$ of period ω is strictly increasing on the interval $0 < s < \frac{1}{2}\omega$. If $p = p(s)$ is a point of the geodesic belonging to a fixed s of this interval, let $\vartheta = \vartheta(s)$, where $0 \leqq \vartheta < \frac{1}{2}\pi$, be the angle betweeen the parallel circle $\varrho = \text{const.}$ through the point $p(s)$ and the geodesic. Then $d\varrho/ds = \sin\vartheta$. Hence (2) can be written in the form

$$\sin\vartheta = \{1 - c^2/r(\varrho)^2\}^{\frac{1}{2}}; \text{ so that } \cos\vartheta = c/r(\varrho) \ (0 < s < \tfrac{1}{2}\omega), \tag{9}$$

if one chooses the orientation of the geodesic so that the integration constant c defined by (1) is non-negative. Actually, $c > 0$, since $c = 0$ would, by (1), imply that $\varphi = \varphi(s) = \text{const.}$, which contradicts the assumptions.

For fixed numbers $s_1, s_2; \varphi_1, \varphi_2$ such that

$$0 < s_1 < s_2 < \tfrac{1}{2}\,\omega; \qquad 0 \leqq \varphi_1 < \varphi_2 < 2\pi, \tag{10}$$

let Λ denote the (ϱ, φ)-region on Θ characterised by

$$\Lambda: \quad \varrho = \varrho(s),\ \varphi(s) + \varphi_1 < \varphi < \varphi(s) + \varphi_2,\ s_1 < s < s_2, \tag{11}$$

where $\varrho(s)$, $\varphi(s)$ are the Gaussian parameters along the given geodesic. For a fixed integer n, let $C_n = C_n(s_1, s_2)$ denote the arc of the geodesic determined by

$$C_n: \quad \varrho = \varrho(n\omega + s), \quad \varphi = \varphi(n\omega + s),\ s_1 < s < s_2. \tag{12}$$

It is seen from (6), (7) that this C_n is contained in the region (11) if and only if

$$\varphi_1 < \alpha n < \varphi_2 \ (\text{mod } 2\pi). \tag{13}$$

It is also seen that if a point P of the geodesic is in the region (11) and if the function $\varrho(s)$ is increasing at P, then there exists an integer n such that P is on C_n. Hence, if s is considered as a time variable, the asymptotic relative measure of those dates s at which $\varrho(s)$ is increasing and the point $(\varrho(s), \varphi(s))$ is in Λ is

$$(2\pi\omega)^{-1}\,(\varphi_2 - \varphi_1)\,(s_2 - s_1).$$

This is clear, in view of the irrationality of $\alpha/2\pi$, from the Kronecker-Weyl approximation theorem. Now, from (2),

$$(2\pi\omega)^{-1}\,(\varphi_2 - \varphi_1)\,(s_2 - s_1) = (2\pi\omega)^{-1}\,(\varphi_2 - \varphi_1) \int_{\varrho_1}^{\varrho_2} \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}}\, d\varrho,$$

where $\varrho_1 = \varrho(s_1)$, $\varrho_2 = \varrho(s_2)$. On the other hand, if $|\Lambda|$ denotes the area of the subregion (11) of Θ, then

$$|\Lambda| = \frac{\varphi_2 - \varphi_1}{2\pi} \int_{\varrho_1}^{\varrho_2} 2\pi r(\varrho)\,d\varrho,$$

since the square of the line element on Θ is $ds^2 = d\varrho^2 + r^2 d\varphi^2$.

Hence, the asymptotic relative amount of time which the geodesic spends in Λ in such a way that $\varrho(s)$ is increasing is

$$(2\pi\omega)^{-1}\,(\varphi_2 - \varphi_1)\,(s_2 - s_1)/|\Lambda| = \frac{\displaystyle\int_{\varrho_1}^{\varrho_2} \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}}\, d\varrho}{\displaystyle\omega \int_{\varrho_1}^{\varrho_2} 2\pi r(\varrho)\,d\varrho}.$$

Letting ϱ_1 and ϱ_2 tend to a common limit ϱ, it follows that if one considers only the s-intervals at which $\varrho(s)$ is increasing, i. e. the intervals

$$n\omega < s < (n + \tfrac{1}{2})\omega,\ n = 0, \pm 1, \pm 2, \ldots, \tag{14}$$

the geodesic is asymptotically distributed on Θ in such a way as to have the function

$$(2\pi\omega)^{-1} \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}}/r(\varrho) = (2\pi\omega)^{-1} \{r(\varrho)^2 - c^2\}^{-\frac{1}{2}} \tag{15}$$

of the position on Θ as asymptotic density of probability. It is understood that this density is referred to the area measure on Θ. Now, the density (15) depends only on the first of the Gaussian parameters ϱ, φ, i. e., the density is constant along every parallel circle of Θ. Since (15) has been calculated on the assumption that s is restricted to intervals of the form (14) and since the function $\varrho(s) = \varrho(s + \omega)$ is even, it is clear that the actual asymptotic density of the geodesic is twice as much as (15). This means in view of (3) and (9) that the asymptotic density of the geodesic on Θ is

$$(\pi\omega)^{-1} \{r(\varrho)^2 - c^2\}^{-\frac{1}{2}} = (\pi c\omega \tan \vartheta)^{-1}. \tag{16}$$

Since (4) also shows that $\tan\vartheta$ vanishes only when the geodesic reaches the bordering parallel circles $\varrho = a$ and $\varrho = b$, it follows that the density of the geodesic on Θ is a function of the portion along the meridian and becomes infinite on the boundaries of Θ. Needless to say, the integral of the density over Θ is 1, as easily verified from (16), (3), and from the fact that the surface element is $d \mid \Lambda \mid = r(\varrho) d\varrho d\varphi$.

The above considerations were concerned with the case (i). In the case (ii) of a torus, where $\varrho(s)$ varies monotonously from $-\infty$ to $+\infty$, the density of probability of a non-periodic geodesic is given by

$$(2\pi\omega)^{-1} \{r(\varrho)^2\}^{-\frac{1}{2}} = (2\pi c\omega \tan \vartheta)^{-1}. \tag{17}$$

It is not necessary to give a proof. In fact, the expression (15), which is in case (i) the half of the density, is in case (ii) the actual density, since in case (ii) one need not separate intervals (11) of increase of $\varrho(s)$ from the complementary intervals. Notice that while the density (16) in case (i) necessarily becomes infinite at $\varrho = a$ and $\varrho = b$, the density (17) on the torus is nowhere infinite. This agrees with the fact $\varrho'(s)$ vanishes in case (ii) when the geodesic reaches the boundary of Θ but cannot vanish in the case (ii) of the torus.

As an application, consider the path of a particle which moves in an euclidean (X, Y)-plane under the action of a central force. If R, Φ are the polar coordinates in the (x, y)-plane, $U(R)$ denotes