On the equivalence of certain types of extension of topological spaces.

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There are several types of H-closed or compact, as the case may be, extensions of a given topological space. Such extensions of a space R are: E. Čech's [1][1]) compact space βR, defined for every completely regular space R, H. Wallman's [2] compact space ωR, P. S. Alexandroff's [3] spaces αR and $\alpha' R$, the first of them defined for regular R, the second for completely regular R. In the recent paper [4] of the author a descriptive characterization is given of four types of extensions, denoted by τR, $\tau' R$, σR, $\sigma' R$, which are defined for any Hausdorff space R [2]).

It is of interest to know for what spaces R some of these eight extensions coincide. It is well known [3] that $\alpha' R = \beta R$ whenever $\alpha' R$, βR exist, i. e. for every completely regular space R. It is further known that $\omega R = \beta R$ if and only if R is normal. In the present note, necessary and sufficient conditions are given for $\beta R = \tau R$, $\beta R = \tau' R$ $\beta R = \sigma R$, $\beta R = \sigma' R$, as well as for $\omega R = \tau R$ etc. It is shown that $\beta R = \tau R$ for compact R only, $\beta R = \sigma R$ if and only if $R = R_1 + R_2$ where R_1 is compact, R_2 is discrete. The conditions for $\beta R = \tau' R$, $\beta R = \sigma' R$ show the structure of R far less clearly and could be probably replaced by simpler ones.

First of all we describe the extensions ωR, βR, τR, ...

Definitions. Let R be a topological space. A point $x \in R$ is called *semiregular* if, for every neighborhood H of x, there exists an open set G such that $x \in G \subset \operatorname{Int} \overline{G} \subset H$. A set $Q \subset R$ is said to be *regularly imbedded* (Čech and Novák [5]) in R if, for every point

[1]) The numbers in brackets refer to the list at the end of the present paper.

[2]) I take the opportunity to corect the erroneous statement of problem 1 in [4], p. 19. The problem should be stated as follows: „I do not know what conditions a space P must satisfy in order that it might be imbedded in a *H-closed Hausdorff subspace of ωP*".

$x \in R$ and every closed set $F \subset R - x$, there exists a set $A \subset Q$ such that $F \subset \overline{A} \subset R - x$. Q is said to be *combinatorially imbedded* [5] in R if $\prod_1^n \overline{F_i} = \emptyset$ whenever $F_i \subset Q$ are relatively closed and $\prod_1^n F_i = \emptyset$.

The following four theorems are known. For the first of them see [5].

Theorem 1. *Any T_1-space R may be both regularly and combinarially imbedded, in an essentially unique way, in a compact T_1-space ωR.*

Theorem 2. *Any completely regular space R may be imbedded in a compact Hausdorff space βR such that every bounded continuous real function on R may be extended to a continuous real function on βR. This imbedding is essentially unique.*

Theorem 3. *If R is normal, then $\beta R = \omega R$. If ωR is a Hausdorff space, then R is normal.*

Theorem 4. *A completely regular space R is open in βR if and only if R is locally compact.*

Definitions. Let R be a Hausdorff space, $Q \subset R$, $\overline{Q} = R$. Q is said to be *hypercombinatorially imbedded* in R if $\prod_1^n \overline{F_i} = \overline{\prod_1^n F_i}$ whenever $F_i \subset Q$ are relatively closed and $\prod_1^n F_i$ is nowhere dense in Q. Q is said to be *paracombinatorially imbedded* in R if $\prod_1^n \overline{G_i} \subset Q$ whenever $G_i \subset Q$ are relatively open and $\prod_1^n G_i = \emptyset$.

The following two lemmas and four theorems are given in [4].

Lemma 1. *Let R be a Hausdorff space, $Q \subset R$, $\overline{Q} = R$. The imbedding $Q \subset R$ is hypercombinatorial if and only if $\overline{F_1}\,\overline{F_2} = \overline{F_1 F_2}$ whenever F_1, F_2 are relatively closed subsets of Q and $F_1 F_2$ is nowhere dense in Q.*

Lemma 2. *Let R be a Hausdorff space, $Q \subset R$, $\overline{Q} = R$. The imbedding $Q \subset R$ is paracombinatorial if and only if $\overline{G_1}\,\overline{G_2} \subset Q$ whenever G_1, G_2 are relatively open subsets of Q and $G_1 G_2 = \emptyset$.*

The above lemmas assert evidently that we can put $n = 2$ in the definitions of the hypercombinatorial and paracombinatorial imbedding without changing their meaning. It is worth mentioning that an analogous lemma does not hold for the combinatorial imbedding [5].

Theorem 5. *Any Hausdorff space R may be hypercombinatorially imbedded in a H-closed*[3]) *space τR such that R is open in τR and the subspace $\tau R - R$ is discrete. The imbedding is essentially unique.*

Theorem 6. *Any Hausdorff space R may be paracombinatorially imbedded in a H-closed space $\tau' R$ such that R is open in $\tau' R$ and every point $x \in \tau' R - R$ is semiregular. This imbedding is essentially unique.*

Theorem 7. *Any Hausdorff space R may be imbedded both hypercombinatorially and regularly in a H-closed space σR. This imbedding is essentially unique.*

Theorem 8. *Any Hausdorff space R may be imbedded both paracombinatorially and regularly in a H-closed space $\sigma' R$ such that every point $x \in \sigma' R - R$ is semiregular. This imbedding is essentially unique.*

Now we proceed to establish the conditions for the equivalence $\beta R = \tau R, \ldots$

Lemma 3. *If every nowhere dense closed subset of a regular space R is compact, then R is normal.*

Proof. Let F_1, F_2 be disjoint closed subsets of R. Denote Int F_1 by G, $F_1 - G$ by K. For each point $x \in K$ choose an open set $H(x)$ such that $x \in H(x)$, $\overline{H(x)} F_2 = \emptyset$. Since K is compact there exist x_i such that $\sum_1^n H(x_i) \supset K$. Setting $H = G + \sum_1^n H(x_i)$ we have $H \supset F_1$, $\overline{H} F_2 = \emptyset$. Hence R is normal.

Definition. A subset M of a topological space R is called *regularly nowhere dense* if $M = \overline{G_1}\, \overline{G_2}$ where G_1, G_2 are open, $G_1 G_2 = \emptyset$.

Lemma 4. *If every regularly nowhere dense closed subset of a regular space R is compact, then, for every pair G, H of open sets such that $\overline{G} \subset H$, there exists a continuous real function f on R such that $f(x) = 0$ for $x \in G$, $f(x) = 1$* for $x \in R - H$.

Proof. Denote Int $\overline{G}$ by G_0, $\overline{G} - G_0$ by K. For each point $x \in K$ choose an open set $U(x)$ such that $x \in U(x) \subset \overline{U(x)} \subset H$. Since K is closed and regularly nowhere dense, therefore compact, there exist $x_i \in K$ such that $\sum_1^n U(x_i) \supset K$. Seting $U = G_0 + \sum_1^n U(x_i)$ we have $\overline{G} \subset U \subset \overline{U} \subset H$. The rest of the proof is now completely analogous to that of the well known Urysohn's lemma.

Theorem 9. *Let R be a completely regular space. The imbedding $R \subset \beta R$ is hypercombinatorial (paracombinatorial) if and only if*

[3]) A Hausdorff space R is called H-closed if it is closed in any Hausdorff space in which it is imbedded.

every nowhere dense (regularly nowhere dense) closed subset of R is compact.

Proof. I. Let the imbedding $R \subset \beta R$ be hypercombinatorial. If $F \subset R$ is nowhere dense and closed (in R), then $F = \overline{F}$ and since βR is compact, so is F.

II. Let the imbedding $R \subset \beta R$ be paracombinatorial. If $F \subset R$ is closed and regularly nowhere dense (in R), then $F = R\overline{G_1}\,\overline{G_2}$, where G_1, G_2 are disjoint open subsets of R. Therefore $\overline{F} \subset \overline{G_1}\,\overline{G_2} \subset R$, whence $\overline{F} = F$. Thus F is compact.

III. Suppose that every nowhere dense closed set $F \subset R$ is compact. Let F_1, F_2 be closed subsets of R and let $F = F_1 F_2$ be nowhere dense. Choose a point $x \in \overline{F_1}\,\overline{F_2}$. If we had $x \in \beta R - \overline{F}$, there would exist an open (in βR) set H such that $H \supset \overline{F}$, $x \in \beta R - \overline{H}$, hence $x \in \overline{F_1 - H}\,\overline{F_2 - H}$. This contradicts the fact that, R being normal by lemma 3, there exists by theorem 2 a continuous real function f on βR such that $f(x) = 0$ for $x \in F_1 - H$, $f(x) = 1$ for $x \in F_2 - H$. Therefore $\overline{F_1}\,\overline{F_2} = \overline{F} = F = F_1 F_2$. Hence by lemma 1 the imbedding $R \subset \beta R$ is hypercombinatorial.

IV. Suppose that every regularly nowhere dense closed set $F \subset R$ is compact. Let G_1, G_2 be disjoint open subsets of R. Denote $R\,\overline{G_1}\,\overline{G_2}$ by F; F is compact, hence $\overline{F} = F$. Suppose that $\overline{G_1}\,\overline{G_2} \neq F$; choose a point $x \in \overline{G_1}\,\overline{G_2} - \overline{F}$. Then there exists an open set H such that $H \supset \overline{F}$, $x \in \beta R - \overline{H}$, $x \in \overline{G_1 - \overline{H}}$, $x \in \overline{G_2 - \overline{H}}$. This is a contradiction since by lemma 4 and theorem 2 there exists a continuous real function f on βR such that $f(x) = 0$ for $x \in G_1 - \overline{H}$, $f(x) = 1$ for $x \in R - G_2 - \overline{H}$. Hence $\overline{G_1}\,\overline{G_2} = F \subset R$ which by lemma 2 proves that the imbedding $R \subset \beta R$ is paracombinatorial.

From the theorems 4, 6, 7, 8, 9 we obtain the following

Theorem 10. *Let R be a completely regular space. Then* (i) $\beta R = \tau' R$ *if and only if R is locally compact and every regularly nowhere dense closed set $F \subset R$ is compact;*

(ii) $\beta R = \sigma R$ *if and only if every nowhere dense closed set $F \subset R$ is compact;*

(iii) $\beta R = \sigma' R$ *if and only if every regularly nowhere dense closed set $F \subset R$ is compact.*

In the theorem 11 we succeed to replace the condition for $\beta R = \sigma R$ by a more illuminating one. As to $\beta R = \tau' R$ it is clear that if $R = R_1 + R_2$ where R_1 is compact, R_2 is closed dicrete, then the conditions for $\beta R = \tau' R$ are satisfied. I do not know whether they may be satisfied by a space R which does not admit of a decomposition of the above kind.

Lemma 5. *In order that every nowhere dense closed subset of a Hausdorff space R should be compact it is necessary and sufficient that the set of all non-isolated points of R be compact.*

Proof. The sufficiency being evident, we have only to prove the necessity of the condition. Denote by S the set of all non-isolated points of R. Let F_ξ be, for every ordinal $\xi < \alpha$, a non-empty closed subset of S; let $F_\xi \supset F_\eta$ for $\xi < \eta < \alpha$. We have to prove $\prod_\xi F_\xi \neq \emptyset$. If, for some ξ, $F_\eta (F_\xi - \operatorname{Int} F_\xi) \neq \emptyset$ for every η, $\xi < \eta < \alpha$, then we obtain $\prod_\eta F_\eta \neq \emptyset$ since $F_\xi - \operatorname{Int} F_\xi$ is nowhere dense and closed, therefore compact. Hence we may suppose that there exists, for every $\xi < \alpha$, a ξ' such that $\xi < \xi' < \alpha$, $F_{\xi'} \subset \operatorname{Int} F_\xi$. Further we may suppose, for convenience, replacing if necessary $\{F_\xi\}$ by an appropriate subcollection, that $F_{\xi+1} \subset \operatorname{Int} F_\xi$, $F_{\xi+1} \neq F_\xi$ for every $\xi < \alpha$. For each $\xi < \alpha$, choose a point $a_\xi \in \operatorname{Int} F_\xi - F_{\xi+1}$ and denote by A the set of all a_ξ. Evidently a_ξ non $\in G_\eta = \operatorname{Int} F_\eta - F_{\eta+1}$ whenever $\eta < \alpha$, $\eta \neq \xi$. Hence every point $x \in A$ is an isolated point of the set A, but is not an isolated point of the whole space R since $A \subset S$. Hence A is nowhere dense and so is $B = \overline{A}$ as well. Therefore B is compact and from $F_\xi B \neq \emptyset$ we obtain $\prod_\xi F_\xi \neq \emptyset$.

Theorem 10 and the above lemma imply

Theorem 11. *Let R be a completely regular space. $\beta R = \sigma R$ if and only if the set of all non-isolated points of R is compact.*

Lemma 6. *If the set of non-isolated points of a locally compact Hausdorff space R is compact, then $R = R_1 + R_2$ where R_1, R_2 are disjoint closed sets, R_1 is compact, R_2 is discrete.*

Proof. Denote by S the set of all non-isolated points of R. For every point $x \in S$ choose an open set $G(x)$ such that $x \in G(x)$ and $\overline{G(x)}$ is compact. Since S is compact, there exist x_i such that $H = \sum_1^n G(x_i) \supset S$. The set $R - H$ is both closed and open since it contains isolated points only. Hence $H = \overline{H} = \sum_1^n \overline{G(x_i)}$ is compact. Setting $R_1 = H$, $R_2 = R - H$ we obtain the required decomposition.

Theorem 12. *Let R be a completely regular space. $\beta R = \tau R$ if and only if R is compact.*

Proof. If R is compact, $\tau R = R = \beta R$. If $\beta R = \tau R$, then by theorem 5 and 9 and lemma 5 the set all non-isolated points of R is compact. Hence by theorem 5 and 4 and lemma 6 we obtain $R = R_1 + R_2$ where R_1, R_2 are disjoint closed sets, R_1 is compact, R_2 is discrete. This yields $\beta R_2 = \tau R_2$ which is possible only for