ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY A FYSIKY

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ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY A FYSIKY

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On the asymptotic distribution of geodesics on surfaces of revolution.

E. R. van **Kampen** and Aurel **Wintner**, Baltimore.
(Received October 5th, 1938.)

The present note deals with the distribution questions of a geodesic on a surface of revolution in the recurrent case. It is an immediate consequence of the Kronecker approximation theorem that such a geodesic, when not periodic, is dense on a domain Θ which is either bordered by two parallel circles on the surface or is the whole surface (which then is of genus 1). It seems of some interest to go beyond this fact, by considering also the asymptotic density which bellongs to the different points of Θ. While the existence of such a density is implied by the general theory of distribution functions of almost periodic functions, it turns out that a direct consideration leads not only to an existence proof but also to the explicit representation of the density in geometrical terms.

The result admits a dynamical interpretation, since the principle of Maupertuis reduces the problem of a particle moving in a field of force of radial symmetry to the problem of geodesics on a surface of revolution whose ds^2 is determined by the force function and the energy constant. The density of the asymptotic distribution on Θ then corresponds to the density of probability belonging to the points of the circular ring in which the path described by the „rotating ellipse" is everywhere dense. Since this density is obtained in explicit form, the result might have some interest also in view of the applications of the theory of adiabatic invariants*) to the classical dynamical problem in question. In fact, the identity of a priori („geometrical") probabilities with asymptotic relative frequencies is proved, instead of being postulated.

In order to have a representation of the surface of revolution which is valid in the large without exceptions whenever the geo-

*) T. **Levi-Civita**, Drei Vorlesungen über adiabatische Invarianten, Hamburger Abhandlungen, **6** (1928), 323—366.

desic is not of the asymptotic type, it is convenient to use as Gaussian parameters the arc length along the meridian and the angle φ of rotation from a fixed meridian. Then the square of the line element is $ds^2 = d\varrho^2 + r^2 d\varphi^2$, where $r = r(\varrho)$ denotes the distance from the axis of rotation.

Local singularities of the geometry of ds^2 will be excluded by the assumption that the function r of ϱ has a continuous derivative and is positive in the ϱ-interval under consideration. Denoting by primes differentiations with respect to the arc length, those geodesics $\varrho = \varrho(s)$, $\varphi = \varphi(s)$ along which $\varrho(s)$ is not constant are identical with the solutions of the pair of equations

$$r(\varrho)^2 \varphi' = c, \qquad \varrho'^2 + c^2/r(\varrho)^2 = 1, \tag{1}$$

which express the conservation of angular momentum and energy (the vis viva is 1, since the arc length s is the time variable). Geodesics which either ~~do~~ not remain in a bounded portion of the surface or tend asymptotically to a second geodesic when $s \to +\infty$ or $s \to -\infty$ will be excluded. Writing the energy integral in the form

$$\varrho'^2 = 1 - c^2/r(\varrho)^2, \tag{2}$$

it is easy to prove that only two cases are possible:

(*i*) the expression on the right of (2), considered as a function of ϱ, has two successive simple roots, say $\varrho = a$ and $\varrho = b$, and this function of ϱ is positive for $a < \varrho < b$;

(*ii*) the function $r(\varrho)$ of ϱ is periodic, the surface is a torus, and $\varrho(s)$ is a strictly monotone function for $-\infty < s < +\infty$. Notice that a geodesic on a torus can belong to case (*i*).

In case (*i*), one sees from (2) that $r(a) = r(b) = c$, and that $r(\varrho) < c$ if $a < \varrho < b$. It also follows from (2) that the function $\varrho(s)$ of s is periodic with the period

$$\omega = 2 \int_a^b \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}} d\varrho, \tag{3}$$

and that a and b are the maximum and minimum of $\varrho(s)$. Also if $s = 0$ belongs to $\varrho = a$, then $\varrho(s)$ is an even function which is strictly increasing for $0 < s < \frac{1}{2}\omega$.

In case (*ii*), one has $r(\varrho) > c$ for every ϱ. The expression on the right of (2) is a periodic function of ϱ with the same period as $r(\varrho)$. Since the expression on the right of (2) has no zeros by assumption, it is clear from (2) that $\varrho(s)$ is, for $-\infty < s < +\infty$, strictly monotone, say increasing, and that if $\lambda > 0$ is the least positive period of $r(\varrho)$ and ω denotes the positive number

$$\omega = \int_0^\lambda \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}} d\varrho, \tag{4}$$

then there exists a function $\chi(s)$ such that

$$\varrho(s) = \lambda s/\omega + \chi(s), \text{ where } \chi(s + \omega) = \chi(s). \tag{5}$$

(That the mean motion of $\varrho(s)$ is λ/ω follows from $\varrho(s + \omega) - \varrho(s) = \lambda$.)

The above description of the cases (*i*), (*ii*) implies that if the problem of geodesics is thought of, for a fixed value of the integration constant c, as a conservative dynamical problem with a single degree of freedom, the case (*ii*) represents the case of a Lagrangian coordinate which is an angular variable (in the same sense as the coördinate of an overturning pendulum); while in the case (*i*) the coordinate ϱ is a linear coordinate which cannot be reduced to a suitable modulus.

Let the case (*i*) be considered first. Then, placing $\psi(s) = c/r(\varrho(s))^2$, the first of the integrals (1) can be written as

$$\varphi'(s) = \psi(s), \text{ where } \psi(s + \omega) = \psi(s), \text{ since } \varrho(s + \omega) = \varrho(s). \tag{6}$$

If α denotes the integral of the periodic function $\psi(s)$ over a period, and $\sigma = \sigma(s)$ the indefinite integral of the function $\psi(s) - \alpha/\omega$, then, from (6),

$$\varphi(s) = \alpha s/\omega + \sigma(s), \text{ where } \sigma(s + \omega) = \sigma(s). \tag{7}$$

Since $r(\varrho)$ is positive for $a \leqq \varrho \leqq b$, and since α/ω is the mean value of the periodic function $\psi(s) = c/r(\varrho(s))^2$, where $\min \varrho(s) = a$ and $\max \varrho(s) = b$, it is clear that the constant α is positive. Now there are two cases possible, according as α is or is not commensurable with 2π. In the first case, (7) together with $\varrho(s + \omega) = \varrho(s)$ implies that the geodesic is a closed curve on the surface; while in the second case the approximation theorem of Kronecker shows that the geodesic is everywhere dense on the portion

$$\Theta: \quad a \leqq \varrho \leqq b, \ 0 \leqq \varphi < 2\pi \tag{8}$$

of the surface. Only the latter case will be considered in what follows.

Choosing the notation such that $a = \min \varrho(s)$ is attained at $s = 0$, the even periodic function $\varrho(s)$ of period ω is strictly increasing on the interval $0 < s < \frac{1}{2}\omega$. If $p = p(s)$ is a point of the geodesic belonging to a fixed s of this interval, let $\vartheta = \vartheta(s)$, where $0 \leqq \vartheta < \frac{1}{2}\pi$, be the angle betweeen the parallel circle $\varrho =$ const. through the point $p(s)$ and the geodesic. Then $d\varrho/ds = \sin\vartheta$. Hence (2) can be written in the form

$$\sin\vartheta = \{1 - c^2/r(\varrho)^2\}^{\frac{1}{2}}; \text{ so that } \cos\vartheta = c/r(\varrho) \ (0 < s < \tfrac{1}{2}\omega), \tag{9}$$

if one chooses the orientation of the geodesic so that the integration constant c defined by (1) is non-negative. Actually, $c > 0$, since $c = 0$ would, by (1), imply that $\varphi = \varphi(s) = \text{const.}$, which contradicts the assumptions.

For fixed numbers $s_1, s_2; \varphi_1, \varphi_2$ such that

$$0 < s_1 < s_2 < \tfrac{1}{2}\,\omega; \qquad 0 \leqq \varphi_1 < \varphi_2 < 2\pi, \tag{10}$$

let Λ denote the (ϱ, φ)-region on Θ characterised by

$$\Lambda: \quad \varrho = \varrho(s),\ \varphi(s) + \varphi_1 < \varphi < \varphi(s) + \varphi_2,\ s_1 < s < s_2, \tag{11}$$

where $\varrho(s)$, $\varphi(s)$ are the Gaussian parameters along the given geodesic. For a fixed integer n, let $C_n = C_n(s_1, s_2)$ denote the arc of the geodesic determined by

$$C_n: \quad \varrho = \varrho(n\omega + s), \quad \varphi = \varphi(n\omega + s),\ s_1 < s < s_2. \tag{12}$$

It is seen from (6), (7) that this C_n is contained in the region (11) if and only if

$$\varphi_1 < \alpha n < \varphi_2 \ (\mathrm{mod}\ 2\pi). \tag{13}$$

It is also seen that if a point P of the geodesic is in the region (11) and if the function $\varrho(s)$ is increasing at P, then there exists an integer n such that P is on C_n. Hence, if s is considered as a time variable, the asymptotic relative measure of those dates s at which $\varrho(s)$ is increasing and the point $(\varrho(s), \varphi(s))$ is in Λ is

$$(2\pi\omega)^{-1}\,(\varphi_2 - \varphi_1)\,(s_2 - s_1).$$

This is clear, in view of the irrationality of $\alpha/2\pi$, from the Kronecker-Weyl approximation theorem. Now, from (2),

$$(2\pi\omega)^{-1}\,(\varphi_2 - \varphi_1)\,(s_2 - s_1) = (2\pi\omega)^{-1}\,(\varphi_2 - \varphi_1) \int_{\varrho_1}^{\varrho_2} \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}} d\varrho,$$

where $\varrho_1 = \varrho(s_1)$, $\varrho_2 = \varrho(s_2)$. On the other hand, if $|\Lambda|$ denotes the area of the subregion (11) of Θ, then

$$|\Lambda| = \frac{\varphi_2 - \varphi_1}{2\pi} \int_{\varrho_1}^{\varrho_2} 2\pi r(\varrho) d\varrho,$$

since the square of the line element on Θ is $ds^2 = d\varrho^2 + r^2 d\varphi^2$.

Hence, the asymptotic relative amount of time which the geodesic spends in Λ in such a way that $\varrho(s)$ is increasing is

$$(2\pi\omega)^{-1}\,(\varphi_2 - \varphi_1)\,(s_2 - s_1)/|\Lambda| = \frac{\displaystyle\int_{\varrho_1}^{\varrho_2} \{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}}\, d\varrho.}{\omega \displaystyle\int_{\varrho_1}^{\varrho_2} 2\pi r(\varrho) d\varrho}.$$

Letting ϱ_1 and ϱ_2 tend to a common limit ϱ, it follows that if one considers only the s-intervals at which $\varrho(s)$ is increasing, i. e. the intervals

$$n\omega < s < (n + \tfrac{1}{2})\omega,\ n = 0, \pm 1, \pm 2, \ldots, \tag{14}$$

the geodesic is asymptotically distributed on Θ in such a way as to have the function

$$(2\pi\omega)^{-1}\{1 - c^2/r(\varrho)^2\}^{-\frac{1}{2}}/r(\varrho) = (2\pi\omega)^{-1}\{r(\varrho)^2 - c^2\}^{-\frac{1}{2}} \tag{15}$$

of the position on Θ as asymptotic density of probability. It is understood that this density is referred to the area measure on Θ. Now, the density (15) depends only on the first of the Gaussian parameters ϱ, φ, i. e., the density is constant along every parallel circle of Θ. Since (15) has been calculated on the assumption that s is restricted to intervals of the form (14) and since the function $\varrho(s) = \varrho(s + \omega)$ is even, it is clear that the actual asymptotic density of the geodesic is twice as much as (15). This means in view of (3) and (9) that the asymptotic density of the geodesic on Θ is

$$(\pi\omega)^{-1}\{r(\varrho)^2 - c^2\}^{-\frac{1}{2}} = (\pi c\omega \tan\vartheta)^{-1}. \tag{16}$$

Since (4) also shows that $\tan\vartheta$ vanishes only when the geodesic reaches the bordering parallel circles $\varrho = a$ and $\varrho = b$, it follows that the density of the geodesic on Θ is a function of the portion along the meridian and becomes infinite on the boundaries of Θ. Needless to say, the integral of the density over Θ is 1, as easily verified from (16), (3), and from the fact that the surface element is $d\,|\,\Lambda\,| = r(\varrho)d\varrho d\varphi$.

The above considerations were concerned with the case (i). In the case (ii) of a torus, where $\varrho(s)$ varies monotonously from $-\infty$ to $+\infty$, the density of probability of a non-periodic geodesic is given by

$$(2\pi\omega)^{-1}\{r(\varrho)^2\}^{-\frac{1}{2}} = (2\pi c\omega \tan\vartheta)^{-1}. \tag{17}$$

It is not necessary to give a proof. In fact, the expression (15), which is in case (i) the half of the density, is in case (ii) the actual density, since in case (ii) one need not separate intervals (11) of increase of $\varrho(s)$ from the complementary intervals. Notice that while the density (16) in case (i) necessarily becomes infinite at $\varrho = a$ and $\varrho = b$, the density (17) on the torus is nowhere infinite. This agrees with the fact $\varrho'(s)$ vanishes in case (ii) when the geodesic reaches the boundary of Θ but cannot vanish in the case (ii) of the torus.

As an application, consider the path of a particle which moves in an euclidean (X, Y)-plane under the action of a central force. If R, Φ are the polar coordinates in the (x, y)-plane, $U(R)$ denotes the force function and h the energy constant, Maupertuis' principle shows that the paths of energy h can be thought of as geodesics on a surface of revolution whose squared line element is

$$ds^2 = 2(U(R) + h)\,(dR^2 + R^2 d\Phi^2). \tag{18}$$

Assuming that the path in the (x, y)-plane is bounded, neither asymptotic nor periodic, and such that the origin of the plane is not reached, the closure of the path will be a ring, $R_1 \leq R \leq R_2$, bordered by two concentric circles about the origin of the (x, y)-plane (rotating ellipse). This means that, in virtue of (18), one has to do with the case (i). In order to apply the explicit representation of the density obtained above, one merely has to replace in (18) the radius vector R by a Gaussian parameter $\varrho = \varrho(R)$ for which

$$d\varrho = \{2(U(R) + h)\}^{\frac{1}{2}}\, dR.$$

Hence, an elementary reduction shows that in the ring of the (x, y)-plane on which the path is everywhere dense the asymptotic density of the path is given by

$$2\,\{U(R) + h\}\,(\pi\omega)^{-1}\,\{2(U(R) + h)R^2 - c^2\}^{-\frac{1}{2}}, \tag{19}$$

where c is the constant of the angular momentum and the period ω of the function $R(t)$ is determined by

$$\omega = 4\int_{R_1}^{R_2} \{U(R) + h\}\,\{2(U(R) + h)R^2 - c^2\}^{-\frac{1}{2}}\, RdR.$$

It is understood that the asymptotic probability belonging to a portion of the ring in the (x, y)-plane is obtained by multiplying (19) by the euclidean area element $RdRd\Phi$, and then integrating over the portion of the ring under consideration.

* * *

O asymptotickém rozložení geodetických čar na rotační ploše.

(Obsah předešlého článku.)

Jde o studium geodetických čar na rotační ploše, které jsou hustě rozloženy v oboru Θ, při čemž Θ jest vymezen dvěma rovnoběžkami plochy anebo jest celá plocha. Rozložení takových geodetických čar v jednotlivých bodech oboru Θ vyjadřují autoři t. zv. asymptotickou hustotou, pro niž odvozují vzorce (16), (17).

Časopis pro pěstování matematiky a fysiky, roč. 72 (1947)

On regular and combinatorial imbedding.

By
Eduard Čech (Praha) and **Josef Novák** (Brno).
(Received February 11th, 1947.)

In his paper *Lattices and topological spaces* (Annals of Math. **39** (1938), 112—127) H. Wallman constructed, for an arbitrary topological space*) Q, a definite bicompact space ωQ containing Q as a dense subset. In § 3 of the present paper, we prove that ωQ may be characterised by the property that Q is both regularly and combinatorially imbedded in it. Regular imbedding is defined and analyzed in § 1, combinatorial imbedding, in § 2. In § 4, we consider the question whether two points may be separated by open subsets of ωQ.

1. Definition. A subspace Q of a space P is said to be *regulary imbedded* in P if the family $(\overline{F})$ of the closures in P of all sets F closed in Q constitutes a closed basis of P, i. e. if every set closed in P is the intersection of some subfamily of the family $(\overline{F})$. As P itself is closed in P, we have:

(1.1) *If Q is regularly imbedded in P, then Q is dense in P.*

(1.2) *If Q is regularly imbedded in P and if $Q \subset P_0 \subset P$, then Q is regularly imbedded in P_0.*

Definition. Let $Q \subset P$. The point $x \in P$ is said to be a *Q-regular point* of P if, for any set $\Phi \subset P - x$ closed in P, there exists a set F closed in Q such that $\Phi \subset \overline{F} \subset P - x$, $\overline{F}$ indicating closure in P. Clearly:

(1.3) *$Q \subset P$ is regularly imbedded in P if, and only if, (i) Q is dense in P, (ii) any point $x \in P$ is Q-regular in P.*

(1.4) *If $x \in P$ is a regular point of P, then x is Q-regular for any set Q dense in P.*

Proof. Let $\Phi \subset P - x$ be closed in P. Then $P - \Phi$ is a neighborhood of x in P. As x is a regular point of P, there exists an open

*) We consider only spaces in which the closure of any point set is closed and, for convenience, we make also the easily avoidable assumption (not made by Wallman) that each finite point set is closed.

neighborhood U of x in P such that $\overline{U} \subset P - \Phi$. The set $F = Q - U$ is closed in Q and $\overline{F} \subset P - U \subset P - x$. As $Q = QU + F$, we have $P = \overline{Q} \subset \overline{U} + \overline{F} \subset (P - \Phi) + \overline{F}$, whence $\Phi \subset \overline{F}$.

Definition. A space P is called *nearly regular* if any Q dense in P is regularly imbedded in P. From (1.3) and (1.4) we have:

(1.5) *Any regular space is nearly regular.*

Definition. A space P is called *hereditarily nearly regular* (h. n. r.) if every subspace of P is nearly regular. Since regularity is a hereditary property, (1.5) gives:

(1.6) *Any regular space is h. n. r.*

From (1.2) we see at once:

(1.7) *If every closed subspace of P is nearly regular, then P is h. n. r.*

Example 1. The space P_1 consists of the points x_{ni} $(n = 1, 2, 3, \ldots, i = 1, 2, 3, \ldots)$, $x_n (n = 1, 2, 3, \ldots)$, and z. Each point x_{ni} is an isolated point. The point x_n possesses the fundamental neighborhoods $U_{nk} (k = 1, 2, 3, \ldots)$ consisting of x_n and x_{ni} $(i \geq k)$. The point z possesses the fundamental neighborhoods V_k $(k = 1, 2, 3 \ldots)$ consisting of z and x_{ni} $(n \geq k, i \geq k)$. Clearly P_1 is a countable Hausdorff space satisfying the second countability axiom; each point except z is regular. The subspace Q_1 consisting of z and all x_{ni}'s is dense in P_1, but Q_1 is not regularly imbedded in P_1, since the set Φ consisting of all x_n's is closed in P_1, but Φ is not of the form $\Pi \overline{F}$ for any family (F) of sets closed in Q_1. Hence P_1 is not nearly regular.

Example 2. The space P_2 consists of the points x_{ni}, y_{ni} $(n = 1, 2, 3, \ldots, i = 1, 2, 3, \ldots)$, $x_n (n = 1, 2, 3, \ldots)$, and z. The points x_{ni} and y_{ni} are isolated. Each point x_n possesses the fundamental neighborhoods U_{nk} $(k = 1, 2, 3, \ldots)$ consisting of the points x_{ni} $(i \geq k)$, y_{ni} $(i \geq k)$, and x_n. The point z possesses the fundamental neighborhoods V_k $(k = 1, 2, 3, \ldots)$ consisting of the points x_{ni} $(n \geq k, i \geq k)$ and z. Again, P_2 is a countable Hausdorff space satisfying the second countability axiom and z is the only irregular point of P_2. We shall prove that P_2 is nearly regular. Let Q be any dense subset of P_2; clearly $Y \subset Q$, Y being the set of all y_{ni}'s. By (1.3) and (1.4) we have only to show that the point z is Q-regular. Let $\Phi \subset P_2 - x$ be closed in P_2. Then $F = Q\Phi + Y_0$ is closed in Q, Y_0 being the closure of Y in Q and clearly $\Phi \subset \overline{F} \subset P_2 - z$. Hence z is Q-regular. Therefore P_2 is nearly regular, but not hereditarily, which follows from example 1.

Example 3. Let M be any uncountable set. The space P_3 consists of the points $x_{n\mu}$ $(n = 1, 2, 3, \ldots, \mu \in M)$, x_n $(n = 1, 2, 3, \ldots)$, and z. The points $x_{n\mu}$ are isolated. Each point x_n possesses the fundamental neighborhoods U_{nK} consisting of the points $x_{n\mu}$

$(\mu \in M - K)$ and x_n, where K runs over the family of all finite subsets of M. The point z possesses the fundamental neighborhoods $V_k - S_k$ $(k = 1, 2, 3, \ldots)$, V_k consisting of the points $x_{n\mu}$ $(n \geqq k$, $\mu \in M)$ and S_k running over the family of all countable subsets of V_k. Clearly P_3 is a Hausdorff space and z is the only irregular point of P_3. We shall show that the space P_3 is *h. n. r.* Let Q denote any subspace of P_3 such that $z \in \overline{Q}$. By (1.3) and (1.4) we have only to show that, in the space $\overline{Q}$, the point z is Q-regular. Let $\Phi \subset \overline{Q} - z$ be closed in $\overline{Q}$, hence in P_3. Let X_n $(n = 1, 2, 3, \ldots)$ denote the set of the points $x_{n\mu}$ $(\mu \in M)$. For each n such that the set X_nQ is infinite, choose an infinite countable subset T_n of X_nQ; let $T_n = 0$ if X_nQ is finite. Then $F = Q\Phi + \left(\sum_1^\infty T_n\right)_0$, the subscript 0 indicating closure in Q, is closed in Q and it is easy to see that $\Phi \subset \overline{F} \subset P_3 - z$, which proves z to be Q-regular in $\overline{Q}$.

2. Definition. Let $n = 2, 3, 4, \ldots$. A subspace Q of a space P is said to be *n-combinatorially imbedded* in P if, for any choice $F_1, F_2, \ldots, F_n$ of relatively closed subsets of Q such that $\prod_1^n F_i = 0$ we have $\prod_1^n \overline{F}_i = 0$. Clearly m-combinatorial imbedding implies n-combinatorial imbedding for $2 \leqq n < m$. The imbedding is said to be *combinatorial* if it is n-combinatorial for each $n = 2, 3, 4, \ldots$

Definition. A subspace Q of a space P is said to be *combinatorially imbedded in P in the strong sense* if, for any choice F_1, F_2 of relatively closed subsets of Q we have $\overline{F_1F_2} = \overline{F}_1\overline{F}_2$. By an easy induction, this implies $\overline{\Pi F_i} = \Pi \overline{F}_i$ for any finite number of relatively closed $F_i \subset Q$, so that combinatorial imbedding in the strong sense implies ordinary combinatorial imbedding.

(2.1) *Let Q be 2-combinatorially imbedded in a regular space P. Then Q is combinatorially imbedded in P in the strong sense.*

Proof. Suppose, on the contrary, that there exist two relatively closed sets $F_1 \subset Q$ and $F_2 \subset Q$ such that $\overline{F_1F_2} \neq \overline{F}_1\overline{F}_2$. Then there exists a point $x \in \overline{F}_1\overline{F}_2 - \overline{F_1F_2}$. By regularity, there exists an open neighborhood U of x in P such that $\overline{U}\overline{F_1F_2} = 0$, whence $\overline{U}F_1F_2 = 0$. Clearly $x \in \overline{\Phi}_1\overline{\Phi}_2$ where the sets $\Phi_1 = F_1\overline{U}$ and $\Phi_2 = F_2\overline{U}$ are closed in Q. But this is impossible, since $\Phi_1\Phi_2 = 0$ and Q is 2-combinatorially imbedded in P.

For $n = 0, 1, 2, \ldots$ let ω_n denote the least ordinal number of power $\aleph_n$ and Z_n, the set of all ordinal numbers $\xi < \omega_n$.

Example 4. Let $P_4 = Z_1 + \omega_1$. Each $\xi \in Z_1$ possesses the fundamental system of neighborhoods $U_{\xi\eta}$ $(\eta \in Z_1,\ \eta < \xi)$, where

$U_{\xi\eta}$ consists of all ordinals ζ such that $\eta < \zeta \leqq \xi$. The point ω_1 possesses the fundamental system of neighborhoods V_ξ $(\xi \in Z_1)$, where V_ξ consists of ω_1 together with all isolated ordinals $\eta \in Z_1$, $\eta > \xi$. Clearly P_4 is a Hausdorff space and ω_1 is the only irregular point of P_4. Then Z_1 is combinatorially imbedded in P_4. Suppose, on the contrary, that there exist relatively closed sets $F_i \subset Z_1$ $(1 \leqq i \leqq n)$ such that $\prod_1^n F_i = 0 \neq \prod_1^n \overline{F}_i$. Then it is clear that no F_i is countable. But then there exist points $\xi_k \in Z_1$ $(k = 0, 1, 2, \dots)$ such that $\xi_0 < \xi_1 < \xi_2 < \dots$ and

$$\xi_{jn} \in F_1,\ \xi_{jn+1} \in F_2, \dots,\ \xi_{jn+n-1} \in F_n\ (j = 0, 1, 2, \dots)$$

which is impossible, since it implies $\lim \xi_k \in \prod_1^n F_i$. Hence Z_1 is combinatorially imbedded in P_4, but not in the strong sense. For let F_1 consist of the points

$$\xi,\ \xi + 1,\ \xi + 3,\ \xi + 5, \dots$$

and F_2, of the points

$$\xi,\ \xi + 2,\ \xi + 4,\ \xi + 6, \dots,$$

$\xi \in Z_1$ running over all non isolated ordinals. The sets $F_1 \subset Z_1$ and $F_2 \subset Z_2$ are relatively closed and we have $\omega_1 \in \overline{F}_1\overline{F}_2 - \overline{F_1F_2}$.

Lemma.*) Let $m, i_1, i_2, \dots, i_m$ be integers such that $m \geqq 1$, $0 \leqq i_1 < i_2 < \dots < i_m$. Let $S = S(i_1, i_2, \dots, i_m)$ be the cartesian product

$$(Z_{i_1} + \omega_{i_1}) \times (Z_{i_2} + \omega_{i_2}) \times \dots \times (Z_{i_m} + \omega_{i_m})$$

in its usual topology. Let A be a subset of $Z_{i_1} \times Z_{i_2} \times \dots \times Z_{i_m}$ such that $(\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_m}) \in \bar{A}$. Choose an integer r such that $1 \leqq r \leqq m$ and an ordinal $\alpha \in Z_{i_r}$. Then $\bar{A}$ contains a point $(\xi_1, \xi_2, \dots, \xi_m)$ such that $\xi_s = \omega_{i_s}$ for $1 \leqq s \leqq m$, $s \neq r$ and $\alpha < \xi_r < \omega_{i_r}$.

Proof. The lemma being trivial for $m = 1$, we may assume its valitidy for $m - 1$. Suppose first $r < m$. Since $(\omega_{i_1}, \omega_{i_2}, \dots, \omega_{i_m}) \in \bar{A}$, for a given $(\alpha_1, \alpha_2, \dots, \alpha_m) \in Z_{i_1} \times Z_{i_2} \times \dots \times Z_{i_m}$ the set A contains points $(\xi_1, \xi_2, \dots, \xi_m)$ such that $\alpha_1 < \xi_1 < \omega_{i_1}, \dots, \alpha_m < \xi_m < \omega_{i_m}$. The cardinal number of the set of all such points $(\xi_1, \xi_2, \dots, \xi_m)$ being equal to $\aleph_m$, whence greater than the cardinal number of $Z_{i_1} \times \dots \times Z_{i_{m-1}}$, the cardinal number of the set of our points will remain equal to $\aleph_m$ even if we restrict the

*) This lemma is a fairly obvious generalization of a result of A. Tychonoff (Math. Annalen 102, 1930, see Behauptung I., on p. 553 and Behauptung III., on p. 555).

first $m - 1$ coordinates to fixed, but conveniently chosen, values. Therefore, $\bar{A}$ contains points $(\xi_1, \xi_2, \ldots, \xi_m)$ such that $\alpha_s < \xi_s < \omega_{i_s}$ for $1 \leqq s \leqq m - 1$ and $\xi_m = \omega_{i_m}$. Now if B denotes the set of all $(\xi_1, \ldots, \xi_{m-1}) \in Z_{i_1} \times \ldots \times Z_{i_{m-1}}$ such that $(\xi_1, \ldots, \xi_{m-1}, \omega_{i_m}) \in \bar{A}$ we have clearly $(\omega_{i_1}, \ldots, \omega_{i_{m-1}}) \in \bar{B}$ in the space $S(i_1, \ldots, i_{m-1})$. The lemma being true for $m - 1$, $\bar{B}$ contains a point $(\xi_1, \ldots, \ldots, \xi_{m-1})$ such that $\xi_s = \omega_{i_s}$ for $1 \leqq s \leqq m - 1$, $s \neq r$ and $\alpha < < \xi_r < \omega_{i_r}$; but then $(\xi_1, \ldots, \xi_{m-1}, \omega_{i_m}) \in \bar{A}$. Secondly, let $r = m$. Choose $(\alpha_2, \ldots, \alpha_m) \in Z_{i_2} \times \ldots \times Z_{i_m}$. By transfinite induction, we may construct a transfinite sequence (p_λ) of type ω_{i_1} of points $p_\lambda = (\xi_{\lambda 1}, \ldots, \xi_{\lambda m}) \in A$ such that $\xi_{\lambda 1} < \xi_{\mu 1}, \ldots, \xi_{\lambda m} < \xi_{\mu m}$ for $\lambda < < \mu < \omega_{i_1}$ and $\xi_{\lambda_2} > \alpha_2, \ldots, \xi_{\lambda_m} > \alpha_m$ for all λ's. The point $p = = (\xi_1, \ldots, \xi_m) = \lim p_\lambda$ belongs to $\bar{A}$ and we have $\xi_1 = \omega_{i_1}$ and $\alpha_s < \xi_s < \omega_{i_s}$ for $2 \leqq s \leqq m$. Hence if B denotes the set of all $(\xi_2, \ldots, \xi_m) \in Z_{i_1} \times \ldots \times Z_{i_m}$ such that $(\omega_{i_1}, \xi_2, \ldots, \xi_m) \in \bar{A}$ we have $(\omega_{i_2}, \ldots, \omega_{i_m}) \in \bar{B}$ in the space $S(i_2, \ldots, i_m)$. The lemma being true for $m - 1$, $\bar{B}$ contains a point $(\xi_2, \ldots, \xi_m)$ such that $\xi_s = \omega_{i_s}$ for $2 \leqq s \leqq m - 1$ and $\alpha < \xi_m < \omega_{i_m}$; but then $(\omega_{i_1}, \xi_2, \ldots, \xi_m) \in \bar{A}$.

Example 5.*) Let $n = 3, 4, 5, \ldots$. The space P_5 consists of all n-tuples $(\xi_1, \xi_2, \ldots, \xi_n)$ such that $\xi_i \in Z_i + \omega_i$ for $1 \leqq i \leqq n$ and $\xi_i = \omega_i$ for at least one i. We put $z = (\omega_1, \omega_2, \ldots, \omega_n)$. The point $\xi = (\xi_1, \xi_2, \ldots, \xi_n) \in P_5 - z$ possesses the fundamental system of neighborhoods $V_\xi(\eta_1, \eta_2, \ldots, \eta_n)$ $(\eta_i \in Z_i, \eta_i < \xi_i$ for $1 \leqq i \leqq n)$ consisting of all n-tuples $(\zeta_1, \zeta_2, \ldots, \zeta_n) \in P_5$ such that $\eta_i < \zeta_i \leqq \xi_i$ for $1 \leqq i \leqq n$. The point z possesses the fundamental system of neighborhoods $V(\xi_1, \xi_2, \ldots, \xi_n)$ $(\xi_i \in Z_i$ for $1 \leqq i \leqq n)$ consisting of z together with all points $(\eta_1, \eta_2, \ldots, \eta_n) \in P_5$ such that $\eta_i > \xi_i$ for $1 \leqq i \leqq n$ and $\eta_i = \omega_i$ for one and only one value of i. Clearly P_5 is a Hausdorff space and z is the only irregular point of P_5. For $1 \leqq i \leqq n$, let Φ_i consist of all points $(\xi_1, \xi_2, \ldots, \xi_n) \in P_5 - z$ such that $\xi_i = \omega_i$. Then the sets $\Phi_i \subset P_5 - z$ are relatively closed and we have $\prod_1^n \Phi_i = 0$, $\prod_1^n \bar{\Phi}_i = z$, whence $P_5 - z$ is not n-combinatorially imbedded in P_5. However, we shall show that this imbedding is $(n - 1)$-combinatorial. First, let us put $S_i = S(j_1, \ldots, \ldots, j_{n-1})$ (see the lemma above) the sequence $j_1, \ldots, j_{n-1}$ being obtained from the sequence $1, 2, \ldots, n$ by cancelling the term i. If f_i $(1 \leqq i \leqq n)$ denotes the cancelling of the i-th coordinate,

*) This example (for $n = 3$) is due to M. Katětov.

then $f_i(\Phi_i + z) = S_i$ is $1 - 1$, though not topological; however, the partial transformation $f_i(\Phi_i) = S_i - z$ is a homeomorphism. Now let the sets $F_r \subset P_5 - z$ $(1 \leqq r \leqq n - 1)$ be relatively closed and let $z \in \prod_1^{n-1} \overline{F}_r$. We have to show that $\prod_1^{n-1} F_r \neq 0$. Since $P_5 - z = \sum_1^n \Phi_i$, for each $r(1 \leqq r \leqq n - 1)$ there must exist an $i_r(1 \leqq i_r \leqq n)$ such that $z \in \overline{F_r\Phi_{i_r}}$. Now this relation valid in the space P_5 evidently implies the analogous relation $z \in \overline{f_{i_r}(F_r\Phi'_{i_r})}$ in the space S_{i_r} where $\Phi'_{i_r} = \Phi_{i_r} - \sum_{j \neq i_r} \Phi_j$. Since r assumes only $n - 1$ values, there exists an integer s such that $1 \leqq s \leqq n$ and $s \neq i_r$ for $1 \leqq r \leqq n - 1$. Using the lemma and recalling that $f_{i_r}(\Phi_i) = S_{i_r} - z$ is a homeomorphism, we see that, for any given ordinal $\alpha \in Z_s$ and for any r $(1 \leqq r \leqq n - 1)$, there exists a point $p = (\xi_1, \xi_2, \ldots, \xi_n) \in \overline{F_r\Phi_{i_r}}$ such that $\xi_i = \omega_i$ for $1 \leqq i \leqq n$, $i \neq s$ and $\alpha < \xi < \omega_s$. Of course, we have $p \in F_r\Phi_{i_r}$ since the set $F_r\Phi_{i_r} \subset P_5 - z$ is relatively closed. By induction, we may now construct an infinite sequence of points $p_k = (\xi_{k1}, \xi_{k2}, \ldots, \xi_{kn})$ such that $\xi_{ki} = \omega_i$ for $1 \leqq i \leqq n$, $i \neq s$ and all k's, $\xi_{1s} < \xi_{2s} < \xi_{3s} < \ldots < \omega_s$ and $p_k \in F_r\Phi_{i_r}$ for $1 \leqq r \leqq n - 1$, $k \equiv r \bmod (n - 1)$. There exists the limit point $p = \lim p_k$ and clearly $p \in \prod_1^{n-1} F_r$, whence $\prod_1^{n-1} F_r \neq 0$.

3. Let Q be any given topological space. We recall briefly the definition of Wallman's bicompact space $\omega Q \supset Q$. Points of $\omega Q - Q$ will be called *ideal points* and points of Q, *real points*. We have to define first the ideal points and secondly the topology of ωQ. An ideal point α is, by definition, a collection of subsets of Q (called the *coordinates of* α) having the following properties:

(*i*) the elements of the collection are non vacuous closed subsets of Q,

(*ii*) the intersection of any finite number of elements of the collection belongs itself to the collection,

(*iii*) any closed subset of Q intersecting each element of the collection belongs itself to the collection,

(*iv*) the intersection of the whole collection is vacuous.

For any open subset G of Q, let G^* consist of all real points belonging to G and of all ideal points α such that there exists some coordinate $A \subset G$ of α. If G runs over the family of all open subsets of Q then G^* runs over an open basis of ωQ, thus defininfg the topology of ωQ. For any closed subset F of Q, the closure $\overline{F}$ of F in ωQ consists of all real points belonging to F and of all ideal points α such that F is a coordinate of α.

(3.1) *The imbedding of an arbitrary topological space Q in Wallman's bicompact space ωQ is both regular and combinatorial in the strong sense.*

Proof. We begin by proving that the imbedding is regular. Q is clearly dense in ωQ. Let x be any point (real or ideal) of ωQ and let Φ be a closed subset of ωQ not containing x. By (1.3) it suffices to indicate a closed subset F of Q such that $\Phi \subset \overline{F} \subset \omega Q - x$. Since x belongs to the open subset $\omega Q - \Phi$ of ωQ, there exists an open subset G of Q such that $x \in G^* \subset \omega Q - \Phi$. Then $F = Q - G$ is a closed subset of Q. Since $x \in G^*$, we cannot have $x \in \overline{F}$. This is evident if x is real; if x is ideal, then $x \in G^*$, $x \in \overline{F}$ would mean that x has a coordinate $A \subset G$ as well as the coordinate F, which is impossible as $GF = 0$. It remains to show that $\alpha \in \overline{F}$ for any $\alpha \in \Phi$. For a real α this is a consequence of the evident relation $Q\Phi \subset Q - G = F$; if α is ideal, the inclusion $G^* \subset \omega Q - \Phi$ shows that, since $\alpha \in \Phi$, any coordinate of α meets $Q - G = F$ so that F itself is a coordinate of α whence $\alpha \in \overline{F}$.

It remains to show that the imbedding is combinatorial in the strong sense. Let F_1 and F_2 be two closed subsets of Q and let $\alpha \in \overline{F}_1\overline{F}_2$; we have to prove that $\alpha \in \overline{F_1F_2}$. This being evident for a real α, let α be ideal. Then $\alpha \in \overline{F}_1$, $\alpha \in \overline{F}_2$ means that both F_1 and F_2 are coordinates of α so that F_1F_2 is also a coordinate of α whence $\alpha \in \overline{F_1F_2}$.

(3.2) *Let the space Q be both regularly and 2-combinatorially imbedded in the bicompact space P. Then there exists a homeomorphism $f(\omega Q) \subset P$ such that $f(x) = x$ for each $x \in Q$. If the imbedding is combinatorial, we have $f(\omega Q) = P$.**)

Proof. For any $X \subset Q$, let $\overline{X}$ denote the closure of X in the space ωQ and $\tilde{X}$, the closure in the space P. For $x \in Q$, let $f(x) = x$. We next define $f(\alpha)$ for an ideal point α of ωQ. Now α is, by definition, a collection of closed subsets of Q having properties (*i*) to (*iv*). Let α^0 denote the collection of all sets $\tilde{A}$, A running over α. By properties (*i*) and (*ii*), the intersection of a finite subcollection of α^0 is never vacuous; the space P being bicompact, the intersection $\varphi(\alpha)$ of the whole collection α^0 is not vacuous either; by property (*iv*), $Q.\varphi(\alpha) = 0$. Hence $\varphi(\alpha)$ contains at least one point $\beta \in P - Q$. We have $\beta \in \tilde{A}$ for any $A \in \alpha$. Conversely, let F be a closed subset of Q such that $\beta \in \tilde{F}$. Then $\tilde{A}\tilde{F}$ contains β for any

*) We do not know whether $f(\omega Q) = P$ whenever the imbedding is 2-combinatorial.

$A \in \alpha$. The imbedding of Q in P being 2-combinatorial, it follows that $AF \neq 0$ for each $A \in \alpha$, whence $F \in \alpha$ by property (*iii*). Hence the collection α consists exactly of those closed subsets A of Q for which the relation $\beta \in \tilde{A}$ holds true. Now by regularity of the imbedding of Q in P, the one point closed subset (β) of P is the intersection of all such $\tilde{A}$'s. It follows that the set $\varphi(\alpha)$ consists of just the one point β and we may put $f(\alpha) = \beta$. The transformation $f(\omega Q) \subset P$ so defined is clearly $1 - 1$ and $f(x) = x$ for each $x \in Q$. Let us put $f(\omega Q) = P_0$ so that $Q \subset P_0 \subset P$.

For any closed subset F of Q we must have $f(\bar{F}) = P_0.\tilde{F}$. Suppose first that $\beta \in P_0.\tilde{F}$; we have to prove that $\beta \in f(\bar{F})$. If $\beta \in Q$, then $\beta \in F \subset f(\bar{F})$; hence suppose $\beta \in P_0 - Q$. By definition of P_0, there exists an ideal point α of ωQ such that $\beta = f(x)$; α consists of all closed subsets A of Q such that $\beta \in \tilde{A}$; since $\beta \in \tilde{F}$, we have $F \in \alpha$, whence $\alpha \in \bar{F}$ and $\beta = f(\alpha) \subset f(\bar{F})$. Conversely, let $\beta \in f(\bar{F})$ so that $\beta \in P_0$; we have to prove that $\beta \in \tilde{F}$. There exists an $\alpha \in \bar{F}$ such that $\beta = f(\alpha)$. If α is real, we have $\beta = \alpha \in F \subset f(\bar{F})$. If α is ideal, then $\alpha \in \bar{F}$ means $F \in \alpha$, whence $\beta = f(\alpha) \in \tilde{F}$.

Let C_0 be a closed subset of P_0. There exists a closed C of P such that $C_0 = P_0 . C$. The imbedding of Q in P being regular, there exists a family φ of closed subsets F of Q such that $C = \Pi\tilde{F}$, whence $C_0 = \Pi P_0 . \tilde{F}$, F running over φ. But $P_0 . \tilde{F} = f(\bar{F})$ and the transformation f being $1 - 1$, we have $C_0 = \Pi f(\bar{F}) = f(\Pi\bar{F})$. Hence each closed subset C_0 of P_0 has the form $C_0 = f(\Phi)$, Φ being closed in ωQ. Conversely, let Φ be closed in ωQ. By (3.1), the imbedding of Q in ωQ is regular. Hence there exists a family φ of closed subsets F of Q such that $\Phi = \Pi\bar{F}$. The transformation f being $1 - 1$, we have $C_0 = f(\Phi) = \Pi f(\bar{F}) = \Pi P_0.\tilde{F} = P_0.\Pi\tilde{F}$. The set C_0 is the intersection of P_0 and a closed subset of P; therefore, C_0 is closed in P_0. Consequently, the closed subsets of P_0 are precisely the sets $f(\Phi)$ with Φ closed in ωQ, which proves that the transformation f is topological.

Now suppose that the imbedding of Q in P is combinatorial and choose $\beta \in P$. We have to prove that $\beta \in f(\omega Q)$. This being evident for $\beta \in Q$, suppose $\beta \in P - Q$. The imbedding of Q in P being regular, there exists a family φ of closed subsets F of Q such that $\beta = \Pi\tilde{F}$ for $F \in \Phi$. Since $\beta \in P - Q$, we must have $\Pi F = 0$. Now for any finite subfamily $F_1, F_2, \ldots, F_n$ of Φ, we have $\beta \in \prod_1^n \tilde{F}_i$, whence $\prod_1^n F_i \neq 0$, the imbedding of Q in P being combinatorial.

As the space ωQ is bicompact, there must exist a point $\alpha \in \Pi \overline{F}$ for $F \in \Phi$. Since $f(\overline{F}) = P_0\tilde{F} \subset \tilde{F}$, we have $f(\alpha) \subset \Pi \tilde{F} = \beta$, whence $\beta = f(\alpha) \in f(\omega Q)$.

4. Two points a and b of a space P will be said to be *H-separated* if there exist two open sets G_1 and G_2 such that $a \in G_1$, $b \in G_2$, $G_1G_2 = 0$. A Hausdorff space is then a space such that any two distinct points are H-separated. As was shown by Wallman (l. c.), the space ωQ is a Hausdorff space if, and only if, the space Q is normal. We consider here the question of H-separability in ωQ of two real points, a real and an ideal point, and two ideal points. Clearly two H-separated points of a space P are H-separated in every subspace of P containing them.

For a Hausdorff space Q, two real points are always H-separated in ωQ. This is a consequence of the following trivial theorem.

(4.1) *If two points a and b are H-separated in a dense subspace Q of a space P, a and b are H-separated in P.*

Proof. There exist two open subsets H_1 and H_2 of Q such that $a \in H_1$, $b \in H_2$, $H_1H_2 = 0$. The sets $F_1 = Q - H_1$ and $F_2 = Q - H_2$ are closed in Q and $a \in Q - F_1$, $b \in Q - F_2$, $F_1 + F_2 = Q$. Therefore $a \in P - \overline{F}_1$, $b \in P - \overline{F}_2$, $\overline{F}_1 + \overline{F}_2 = P$. The sets $G_1 = P - \overline{F}_1$ and $G_2 = P - \overline{F}_2$ are open in P and $a \in G_1$, $b \in G_2$, $G_1G_2 = 0$.

(4.2) *A point $a \in Q$ is regular in ωQ if, and only if, it is regular in Q.*

Proof. If a is regular in ωQ then, of course, a is regular in $Q \subset \omega Q$ as well. Let a be regular in Q. If U is any neighborhood of a in ωQ, there exists a neighborhood G of a in Q such that $G^* \subset U$. Since a in regular in Q, there exists a neighborhood H of a in Q the closure of which in Q is contained in G. It is easy to see that H^* is a neighborhood of a in ωQ the closure of which is contained in G^*, whence in U.

(4.3) *If a is an irregular point of the bicompact space P, there exists a point $b \in P - a$ such that a and b are not H-separated.*

Proof. There exists a neighborhood U of a such that $\overline{V} - U \neq 0$ for every neighborhood V of a. If V_i $(1 \leq i \leq n)$ are neighborhoods of a, then $\prod_1^n V_i$ is also a neighborhood of a, whence

$$\prod_1^n (\overline{V}_i - U) \supset \overline{\prod_1^n V_i} - U \neq 0.$$

The space being bicompact, there exists a point b such that $b \in \overline{V} - U$ for every neighborhood V of a. It is easy to see that a and b are not H-separated.

If the space Q is regular, we see from (4.2) that a real and an ideal point are always H-separated in ωQ. If Q is an irregular Hausdorff space, we see from (4.1) and (4.3) that a real and an ideal point are not always H-separated. If the regular space Q is not normal, then two ideal points cannot be always H-separated, since otherwise ωQ would be a Hausdorff space, which it is not.

Example 6. Let Q be an irregular Hausdorff space containing a finite subset K such that the subspace $Q - K$ is normal; e. g. $Q = P_1$, $K = z$ (see example 1 above). Then two different ideal points α and β are always H-separated in ωQ. For there exists a coordinate F_1 of α and a coordinate F_2 of β such that $F_1F_2 = 0$. Then $F_1 - K$ is a coordinate of α, $F_2 - K$ is a coordinate of β, and $F_1 - K$ and $F_2 - K$ are disjoint closed subsets of the normal space $Q - K$. Hence there exist two open subsets G_1 and G_2 of $Q - K$ such that $F_1 - K \subset G_1$, $F_2 - K \subset G_2$, $G_1G_2 \neq 0$. Since $Q - K$ is open in Q, G_1 and G_2 are so also. Hence $G_1{}^*$ is a neighborhood of α in ωQ, $G_2{}^*$ is a neighborhood of β in ωQ, and $G_1{}^*G_2{}^* = 0$.

* * *

O regulárním a kombinatorickém vnoření.

(Obsah předešlého článku.)

V pojednání *Lattices and topological spaces* (Annals of Math. **39** (1938), 112—127) přiřadil H. Wallman libovolnému topologickému prostoru Q určitý bikompaktní prostor ωQ. V tomto článku dokazujeme, že bikompaktní prostor ωQ je charakterisován tím, že Q je do něho vnořen regulárně a kombinatoricky. Při tom pravíme, že Q je vnořen regulárně do prostoru P, jestliže každá množina uzavřená v P je průnikem uzávěrů množin uzavřených v Q a pravíme, že Q je vnořen kombinatoricky do prostoru P, jestliže konečně mnoho disjunktních relativně uzavřených částí Q má vždy disjunktní uzávěry v P. Udáváme také několik příkladů objasňujících pojmy regulárního a kombinatorického vnoření.

On H-closed extensions of topological spaces.

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Like many other notions in General Topology, the H-closed spaces are due to Alexandroff and Urysohn [1][1]). In their paper they stated two problems concerning H-closed spaces: (i) is a space every closed subspace of which is H-closed necessarily compact? (ii) may any Hausdorff space P be imbedded as a dense subset in a H-closed space R? A step towards the solution of (ii) was made by Tychonoff [2] showing that P may be imbedded in a H-closed space R (without being, in general, necessarily dense in R). In his important paper [3] M. H. Stone solved both (i) and (ii) and showed moreover that there exists a H-closed space $R \supset P$ which is a strict extension (loc. cit., definition 14) of P and has the same character (i. e. the minimal power of an open base) as P. Another notion introduced by Stone in connection with his algebraic considerations is the semiregularity which is less restrictive than regularity but simplifies the theory of H-closed extensions considerably. In his paper M. H. Stone uses an elaborate algebraic theory. A part of his results was proved in a similar but more direct way by Fomin [4].

In his paper [2] A. Tychonoff showed that any completely regular space may be imbedded in a compact space (as a matter of fact, he showed more, namely that there exists an universal compact space of character $\aleph$ for any infinite cardinal $\aleph$). A further important result is due to E. Čech [5] who proved that any completely regular space P possesses a compact extension[2]) βP such that $\overline{P} = \beta P$ and that every bounded continuous real function on P may be extended to a continuous real function on βP; the space βP is uniquely determined by these properties. Later on, H. Wallman [6] proved that every topological space P may be imbedded

[1]) The numbers in brackets refer to the list at the end of this paper.
[2]) If P is a subspace of a space $R \supset P$, then we say that P is *imbedded* in R or that R is an *extension* of P.

as a dense set in a compact space ωP. The space ωP is a Hausdorff space if and only if P is normal; in this case $\omega P = \beta P$. The space ωP possesses the same homology theory as P.

The question arises whether a H-closed extension of a similar kind exists for Hausdorff spaces. This problem is solved in the author's paper [7]. Every Hausdorff space P possesses a H-closed extension τP such that $(i) \overline{P} = \tau P$; (ii) every mapping f of P into a Hausdorff space S such that $\overline{f(P)} = S$ may be extended to a mapping of a subspace $P' \subset \tau P$ onto S. The space τP is uniquely determined by the properties (i) and (ii).

The extensions βP, ωP, τP have been so far characterized either by their „construction" (e. g. the „ideal" points of ωP correspond to certain collections — the so called maximal basic sets — of closed subsets of P) or by certain properties of mappings of P, viz. by possibility of their continuous extending. This cannot be considered as a wholly satisfactory descriptive characterization of the extensions βP, ωP, τP. Such a characterization for ωP was not given till recently by Čech and Novák [8]. The space ωP is characterized by P being imbedded in ωP both combinatorially, which means that $\prod_1^n \overline{F}_i = 0$ whenever F_i are relatively closed in P and $\prod_1^n F_i = 0$, and regularly, which means that every closed set $\Phi \subset \omega P$ may be represented as intersection of a family of sets $\overline{F}$, $F \subset P$.

In the present paper I intend to give an analogous descriptive characterization of the space τP and three other types of H-closed extension which are obtained by imposing different conditions concerning relative semiregularity.

In § 1 of the present paper semiregularity of a point relatively to a set is defined and examined. It is shown that M. H. Stone's strict extension and E. Čech's regular imbedding are equivalent notions and may be both expressed in terms of relative semiregularity. Certain modifications (we call them SR-modifications) are considered, transforming a given Hausdorff space into a space satisfying appropriate relative semiregularity conditions. A modification of this kind occurs implicitly already in the author's paper [7].

In § 2 hypercombinatorial and paracombinatorial imbedding are examined which are closely related to Čech's combinatorial imbedding. Whereas however there is a difference between n-combinatorial ($n = 2, 3, \ldots$), combinatorial and combinatorial in the strong sense imbedding (Čech and Novák [8]), the analogous notions coincide for hypercombinatorial and paracombinatorial imbedding as shown in (2,1) and (2,4).

In § 3 a descriptive characterization of the extensions τP, $\tau' P$, σP, $\sigma' P$ is given. The space τP occurs already in [7]. The spaces σP and $\sigma' P$ may be obtained as Fomin's [4] spaces $\sigma(P)$ by taking for the basic collection $\{G\}$ the family of all open sets for σP and the family of all regularly open sets for $\sigma' P$.

There remain several unsolved problems.

1. I do not know what conditions a space P must satisfy in order that it might be imbedded combinatorially in a H-closed space. If such an extension exists it need not be unique. Example: Let P_0 be the space of all pairs of ordinals (ξ, η), $\xi \leqq \omega_0$, $\eta \leqq \omega_1$, $(\xi, \eta) \neq (\omega_0, \omega_1)$ with the usual topology. The set $\omega P_0 - P_0$ contains exactly two points and by cancelling any of them a H-closed space is obtained. The two spaces are not topologically equivalent but P_0 is combinatorially imbedded in either of them. — It seems probable that an extension of this kind is possible and unique if and only if P is normal (in that case it coincides, of course, with ωP).

2. It is perhaps of some interest to examine the conditions under which several of the spaces τP, $\tau' P$, σP, $\sigma' P$, ωP, βP coincide. It is known only that $\omega P = \beta P$ if and only if P is normal. The conditions for the other equivalences should be far more restrictive.

3. If completely regular spaces P_1 and P_2 satisfy the first countability axiom, then $\beta P_1 = \beta P_2$ implies $P_1 = P_2$ ($=$ denotes topological equivalence here). It could be of some interest to find sufficiently broad conditions under which a similar implication holds for τP and the other H-closed extensions.

§ 1.

All spaces considered are Hausdorff spaces even if it is not explicitly stated. The signs $\Rightarrow$ and $\Leftrightarrow$ denote logical implication and equivalence.

Definitions. Let P be a space, $M \subset P$. A set $G \subset P$ is said to be *regularly open* (Kuratowski [9]) if $G = \operatorname{Int} \overline{G}$[3]) and is said to be *regularly open relatively to M* if $G = \operatorname{Int} (G + M\overline{G})$.

A point $x \in P$ is called *semiregular relatively to M* if whenever G is open and $x \in G$ there exists an open set H such that $x \in H \subset \operatorname{Int} (H + M\overline{H}) \subset G$.

If every point $x \in P$ is semiregular relatively to M, then P is said to be *semiregular relatively to M*. If a point $x \in P$ is semiregular relatively to P, then it is called simply *semiregular*. If

[3]) Int A is the interior of the set A, i. e. the set $P - \overline{P - A}$.

every point $x \in P$ is semiregular, then the space P is said to be *semiregular*. This definition is evidently equivalent with M. H. Stone's [3] definition of semiregularity.

(1.1) *If* G_i $(i = 1, \ldots n)$ *are regularly open relatively to* M, *then* $\prod_1^n G_i$ *is so as well.*

Proof. Denoting $\prod_1^n G_i$ by H we have $H + M\overline{H} \subset \prod_1^n (G_i + M\overline{G}_i)$, whence $\operatorname{Int}(H + M\overline{H}) \subset \operatorname{Int}\prod_1^n (G_i + M\overline{G}_i) = \prod_1^n \operatorname{Int}(G_i + M\overline{G}_i) = \prod_1^n G_i = H$.

Clearly:

(1.2) *For any open* $G \subset P$ *the set* $\operatorname{Int}(G + M\overline{G})$ *is regularly open relatively to* M.

(1.3) *A point* $x \in P$ *is semiregular relatively to* M *if and only if it possesses fundamentel neighborhoods which are regularly open relatively to* M.

Proof. If x is semiregular relatively to M, then for any open G, $x \in G$, there exists an open H such that $x \in H \subset H_1 = \operatorname{Int}(\overline{H} + M\overline{H}) \subset G$. By (1.2) H_1 is regularly open relatively to M. The other half of the lemma is obvious.

Definitions. Let P be a space, $Q \subset P$. Q is said to be *regularly imbedded* [8] in P if for any closed set $F \subset P$ and every $x \in P - F$ there exists $A \subset Q$ such that $F \subset \overline{A} \subset P - x$. P is said to be a *strict extension* [3] of Q if $\overline{Q} = P$ and for any open $G \subset P$ and every $x \in G$ there exists an open neighborhood H of x such that $\operatorname{Int}(H + A) \subset G$ whenever A is nowhere dense and $AQ = 0$.

(1.4) *Let* S *be dense in* P. S *is regularly imbedded in* P *if and only if* P *is semiregular relatively to* $P - S$.

Proof. I. Let P be semiregular relatively to $M = P - S$ and let $F \subset P$ be closed, $x \in P - F$. Then there exists an open set G such that $x \in G$, $F.\operatorname{Int}(G + M\overline{G}) = 0$. Setting $A = P - (G + M\overline{G}) = (S - G) + (M - \overline{G})$ we have $F \subset \overline{A} \subset P - x$. Since $\overline{M - \overline{G}} \subset \overline{P - \overline{G}} = \overline{S - \overline{G}}$ we have $\overline{A} = \overline{S - G}$. Hence the imbedding $S \subset P$ is regular. II. Let S be regularly imbedded in P and let $H \subset P$ be open, $x \in H$. There exists a set $A \subset S$ such that $P - H = F \subset \overline{A} \subset P - x$. Setting $G = P - \overline{A}$, $M = P - S$ we have $H \supset P - \overline{A} = \operatorname{Int}(P - S\overline{A}) = \operatorname{Int}(G + M) = \operatorname{Int}[G + M\overline{G} + (M - \overline{G})] = \operatorname{Int}(G + M\overline{G})$ which proves the theorem.

(1.5) *Let* $Q \subset P$, $\overline{Q} = P$. *P is a strict extension of Q if and only if P is semiregular relatively to* $P - Q$.

Proof. The implication: strict extension $\Rightarrow$ relative semiregularity follows at once from the above definition of strict extension by setting $A = (\overline{H} - H)(P - Q)$. Let P be semiregular relatively to $P - Q$ and let G be open, $x \in G$. There exists an open set H such that $x \in H \subset \text{Int}\,[H + (\overline{H} - Q)] \subset G$. Now let $A \subset P$ be nowhere dense, $AQ = 0$. Then $\text{Int}\,(H + A) \subset \text{Int}\,(H + A\overline{H}) \subset \text{Int}\,[H + (\overline{H} - Q)] \subset G$. Hence P is a strict extension of Q.

(1.4) and (1.5) imply:

(1.6) *Let* $S \subset P$, $\overline{S} = P$. *P is a strict extension of S if and only if S is regularly imbedded in P.*

Definition. Let P be a space, $Q \subset P$, $M \subset P$. Let $\mathfrak{G}$ denote the family of all open sets $G \subset P$ such that every $x \in QG$ possesses a neighborhood $H \subset G$ which is regularly open relatively to M. The space P' which is obtained by choosing $\mathfrak{G}$ as an open base will be called the *SR-modification of the space P on the set Q relatively to M.*

Since for open sets G and H $GH = 0 \Rightarrow \text{Int}\,\overline{G}.\,\text{Int}\,\overline{H} = 0$ and by (1.1) $G_1 \in \mathfrak{G}$, $G_2 \in \mathfrak{G} \Rightarrow G_1G_2 \in \mathfrak{G}$, any SR-modification of a Hausdorff space is a Hausdorff space again. If P' is a SR-modification of P, then the identical transformation $P \rightarrow P'$ is a mapping.

We may consider any set $A \subset P$ either as a subset of P or as a subset of P'. If A is closed, open, ... if considered as a subset of P (or P') we shall say, for convenience, that A is closed, open, ... *in* P (or *in* P').

(1.7) *Let P be a space, $Q \subset P$, $M \subset P$ and denote by P' the SR-modification of P on Q relatively to M. Then* (i) *if G is open in P, then G has the same closure both in P and in P'*; (ii) *if G is regularly open in P relatively to M, then it is open in P'*; (iii) *if G is open in P' and regularly open in P relatively to a set M_1, then G is regularly open relatively to M_1 in the space P' as well.*

Proof. For any $A \subset P$ denote by $\bar{A}$, A^*, $\text{Int}\,A$, $\text{Int}^*\,A$ the closure and the interior of A in P and in P' respectively. Then clearly $\overline{G} \subset G^*$; if $x \in P - \overline{G}$, we have $HG = 0$, where $H = P - \overline{G}$, hence $\overline{H}G$ which implies, by the definition of SR-modification, $x \in P - G^*$. Hence $G^* = \overline{G}$.

If $G = \text{Int}\,(G + \overline{G}M)$, then by (1.2) and by the definition of SR-modification G is open in P'. If $G = \text{Int}\,(G + \overline{G}M_1)$ and G is open in P', then $G \subset \text{Int}^*\,(G + G^*M_1) = \text{Int}^*\,(G + \overline{G}M_1) \subset \subset \text{Int}\,(G + \overline{G}M_1) = G$, hence G is regularly open in P' relatively to M_1.

(1.7) and (1.3) imply

(1.8) *Let P' be the SR-modification of P on Q relatively to M. Then every point $x \in Q$ is semiregular in P' relatively to M.*

(1.5), (i) implies

(1.9) *Let P' be a SR-modification of P. A set $G \subset P$ is regularly open in P' if and only if it is regularly open in P. If a point $x \in P$ is semiregular in P then it is so in P' as well.*

(1.10) *If a subspace $Q \subset P$ is semiregular, then its topology remains unchanged under an arbitrary SR-modification of P.*

Proof. Let $H \subset Q$ be a relative neighborhood of a point $x \in Q$ in Q. Since Q is semiregular there exists a set $G \subset Q$ such that $x \in G \subset H$ and G is regularly open in Q. Let G_0 be open in P, $G = QG_0$ and denote Int $\overline{G_0}$ by G_1. We have $G = Q - \overline{Q - Q\overline{G}} = Q.(P - \overline{Q(P - \overline{G})}) = Q.(P - \overline{P - \overline{G_0}}) = QG_1$ which proves the theorem since (1.2) and (1.5) imply that G_1 is open in P', P' denoting an arbitrary SR-modification of P.

(1.11) *Let $Q \subset P$, $M \subset P$. The topology of both $P_1 = P - \overline{Q}$ and $P_2 = P - M$ remains unchanged by the SR-modification of P on Q relatively to M.*

Proof. If G is open in P, then by (1.2) and (1.5) $H = \text{Int}\,(G + \overline{G}M)$ is open in P', P' denoting the SR-modification of P on Q relatively to M, and $HP_2 = GP_2$ which proves the theorem for P_2. For P_1, it follows immediately from the definition of SR-modification.

§ 2.

Definitions. Let Q be a dense subspace of a space P. Then Q is said to be

(i) *combinatorially imbedded* [8] in P if whenever $F_i \subset Q$ are relatively closed in Q_n

$$\prod_1^n F_i = 0 \Rightarrow \prod_1^n \overline{F}_i = 0 \quad (n = 2, 3, \ldots);$$

(ii) *combinatorially imbedded in P in the strong sense* [8] if whenever F_1, F_2 are relatively closed in Q we have

$$\overline{F_1 F_2} = \overline{F}_1 \overline{F}_2;$$

(iii) *hypercombinatorially imbedded in P* if whenever $F_i \subset Q$ are relatively closed in Q we have

$$\prod_1^n F_i \text{ nowhere dense in } Q \Rightarrow \prod_1^n \overline{F}_i = \overline{\prod_1^n F_i} \quad (n = 2, 3, \ldots);$$

(iv) *paracombinatorially imbedded in* P if for any relatively open sets $G_i \subset Q$ we have

$$\prod_1^n G_i = 0 \Rightarrow \prod_1^n \bar{G}_i \subset Q. \quad (n = 2, 3, \ldots).$$

(2.1) *Let* Q *be dense in* P. Q *is hypercombinatorially imbedded in* P *if and only if one of the following equivalent conditions holds:*

(i) *whenever* F_1, F_2 *are relatively closed subsets of* Q *and* F_1F_2 *is nowhere dense in* Q *we have* $\bar{F}_1\bar{F}_2 = \overline{F_1F_2}$;

(ii) *if* $F_i \subset Q$ *are relatively closed in* Q, *then*

$$\prod_1^n \bar{F}_i - Q = \overline{\prod_1^n G_i} - Q,$$

where G_i *denotes the relative interior of* F_i *in* Q $(n = 1, 2, 3, \ldots)$.

Proof. If (i) holds and $x \in \bar{F}_1\bar{F}_2 - Q$, then setting $A_i = F_i - G_1G_2$ we have $x \in (\bar{G}_1\bar{G}_2 - Q) + (\bar{A}_1\bar{A}_2 - Q)$ which implies $x \in \overline{G_1G_2} - Q$ since $\overline{A_1A_2} - Q = 0$, A_i being nowhere dense. Hence (ii) holds for $n = 2$. Now let (ii) be true for $n = 2, 3, \ldots m$ and let F_i $(i = 1, 2, \ldots m + 1)$ be relatively closed in Q. Then $\prod_1^m \bar{F}_i - Q = \overline{\prod_1^m G_i} - Q$ and setting $\Phi = Q\overline{\prod_1^m G_i}$, $\Gamma =$ relative interior of Φ in Q we have $\prod_1^m \bar{F}_i - Q = \bar{\Phi} = Q$, $\prod_1^{m+1} \bar{F}_i - Q = \bar{\Phi}\overline{F_{n+1}} - Q = \overline{\Gamma G_{m+1}} - Q$. Since the sets G_i are regularly open in Q, so is $\prod_1^m G_i$ by (1.1), therefore $\Gamma = \prod_1^m G_i$, hence $\prod_1^{m+1} \bar{F}_i - Q = \overline{\Gamma G_{m+1}} - Q = \overline{\prod_1^{m+1} G_i} - Q$. This yields by induction the implication (i) $\Rightarrow$ (ii) which proves the theorem since evidently (ii) $\Rightarrow$ hypercombinatorial imbedding $\Rightarrow$ (i).

The following obvious lemma is useful sometimes.

(2.2) *Whenever* $G_i \subset R$ *are open in* R *we have*

$$\prod_1^n \operatorname{Int} \bar{G}_i = \operatorname{Int} \prod_1^n \bar{G}_i = \operatorname{Int} \overline{\prod_1^n G_i}.$$

Proof. We have only to prove these equalities for $n = 2$. Evidently $\operatorname{Int} \bar{G}_1\bar{G}_2 = \operatorname{Int} \bar{G}_1 \operatorname{Int} \bar{G}_2$. Denoting this set by H we have $H \supset \operatorname{Int} \overline{G_1G_2}$, $\overline{G_1G_2} = \overline{\bar{G}_1G_2} \supset \overline{HG_2} = \overline{H\bar{G}_2} \supset \bar{H}$; hence $\operatorname{Int} \overline{G_1G_2} \supset H$, $H = \overline{\bar{G}_1\bar{G}_2}$.

(2.3) ***Hypercombinatorial imbedding is both paracombinatorial and combinatorial in the strong sense.***

Proof. Let Q be hypercombinatorially imbedded in P. Let G_i $(i = 1, \dots n)$ be relatively open in Q, $\prod_1^n G_i = 0$. Denoting $Q\overline{G_i}$ by A_i we have $\prod_1^n \overline{G_i} = \prod_1^n \overline{A_i} = \prod_1^n A_i \subset Q$ since by (2.2) $\prod_1^n A_i$ is nowhere dense in Q. Hence the imbedding $Q \subset P$ is paracombinatorial. Now let F_1, F_2 be relatively closed in Q and denote by H_i the relative interior of F_i. Then by (2.1) $\overline{F_1}\overline{F_2} - Q = \overline{H_1}\overline{H_2} - Q$, hence $\overline{F_1}\overline{F_2} = \overline{F_1 F_2}$ which proves the theorem.

(2.4) *Let Q be dense in P. Q is paracombinatorially imbedded in P if and only if one of the following conditions holds:*

(i) *whenever G_1, G_2 are relatively open in Q and $G_1G_2 = 0$ we have $\overline{G_1}\overline{G_2} \subset Q$;*

(ii) *for any choice of relatively open $G_i \subset Q$ we have*

$$\prod_1^n \overline{G_i} - \overline{\prod_1^n G_i} \subset Q \quad (n = 2, 3, \dots).$$

Proof. Let the implication (*) $\prod_1^n G_i = 0 \Rightarrow \prod_1^n \overline{G_i} \subset Q$ hold for $n = 2, \dots m$. Then we have, for arbitrary relatively open $G_i \subset Q$, $\prod_1^m (G_i - A) = 0$, where $A = \overline{\prod_1^m G_i}$, whence $\prod_1^m \overline{G_i} - \overline{\prod_1^m G_i} = \prod_1^m \overline{G_i} - A \subset \prod_1^m \overline{G_i - A} \subset Q$. If $H_i \subset Q$ are relatively open, $\prod_1^{m+1} H_i = 0$, then $\prod_1^{m+1} \overline{H_i} \subset (\prod_1^m \overline{H_i} - \overline{\prod_1^m H_i}) + \overline{\prod_1^m H_i}.\overline{H_{m+1}} \subset Q$, hence (*) holds for $n = m + 1$. This yields, by induction, (i) $\Rightarrow$ (ii) which proves the theorem, since clearly (ii) $\Rightarrow$ paracombinatorial imbedding $\Rightarrow$ (i).

(2.5) ***A paracombinatorial imbedding $Q \subset P$ is hypercombinatorial if and only if every relatively closed set $F \subset Q$ which is nowhere dense in Q is closed in P.***

Proof. We have only to prove that the condition is sufficient. Let F_1, F_2 be relatively closed in Q, F_1F_2 nowhere dense in Q. Denoting the relative interior of F_i by G_i we have $G_1G_2 = 0$, $\overline{F_i} - Q = (\overline{F_i - G_i} - Q) + (\overline{G_i} - Q) = \overline{G_i} - Q$, hence $\overline{F_1}\overline{F_2} - Q = \overline{G_1}\overline{G_2} - Q = 0$ which implies by (2.1) that the imbedding is hypercombinatorial.

The following theorems (2.6), (2.7), (2.8) are well known; (2.6) is due to Alexandroff and Urysohn [1], (2.7) and (2.8) are given in [7].

Definition. A Hausdorff space P is called *H-closed* if P is closed in any Hausdorff space in which it is imbedded.

(2.6) *A Hausdorff space P is H-closed if and only if every open covering $\{G\}$ contains a finite subcollection $\{G_i\}$ such that* $\sum_1^n \overline{G}_i = P$.

Proof. Let P be H-closed and let $\{G\}$ be an open covering. Let $R = P + \alpha$ and let the point α possess fundamental neighborhoods $P - \sum_1^n \overline{G}_i + \alpha$. Then R is a Hausdorff space, P is imbedded in R, therefore closed. Hence there exist G_i such that $P - \sum_1^n \overline{G}_i = 0$.

If P is not H-closed, there exists a Hausdorff space $R \supset P$ such that P is not closed in R. Let $\alpha \in \overline{P} - P$. The family $\{G\}$ of all $G = P - \overline{H}$ where H is a neighborhood of α in R is an open covering of P. For arbitrary $G_i \in \{G\}$, $G_i = P - \overline{H}_i$ $(i = 1, \dots n)$, we have $\overline{G}_i \subset R - H_i$, $\sum_1^n \overline{G}_i \subset R - \prod_1^n H_i$, $P - \sum_1^n \overline{G}_i \supset P . \prod_1^n H_i \neq 0$, since $\prod_1^n H_i$ is a neighborhood of α.

(2.7) *If P is H-closed and $G \subset P$ is open, then $Q = \overline{G}$ is H-closed.*

Proof. Let $\{H\}$ be an open covering of the space P_1. Then the collection consisting of the set $P - Q$ and of all $P - \overline{Q - H}$ is an open covering of P, hence there exist H_i $(i = 1, \dots n)$ such that $\sum_1^n \overline{\Gamma_i} + \overline{P - Q} = P$, where $\Gamma_i = P - \overline{Q - H_i}$, therefore $\sum_1^n \overline{\Gamma_i} \supset$ $\supset P - \overline{P - Q} \supset G$, $\sum_1^n \overline{\Gamma_i} \supset \overline{G} = Q$. Since $\Gamma_i Q = H_i$ we obtain $\sum_1^n \overline{H}_i \supset \sum_1^n \overline{\Gamma_i G} = \overline{\overline{G \sum_1^n \Gamma_i}} = \overline{G} = Q$ which proves the theorem.

(2.8) *If a H-closed space P is continuously mapped on a Hausdorff space R, then R is H-closed.*

Proof. Denote by f a mapping of P onto R. Let $\{G\}$ be an open covering of R. Then $\{f^{-1}(G)\}$ is an open covering of P, hence there exist G_i such that $\sum_1^n \overline{f^{-1}(G_i)} = P$, whence $\sum_1^n \overline{G}_i = R$. Therefore by (2.6) R is closed.

(2.9) *If a collection* $\{G\}$ *of open subsets of a H-closed space* R *has the finite intersection property* (i. e. $\prod_1^n G_i \neq 0$ for any choice of $G_i \in \{G\}$), *then the intersection of all* $\overline{G}$ *is non-empty.*

Proof. If $\Pi\overline{G}$ were empty, then the collection $\{R - \overline{G}\}$ would be an open covering of R, hence by (2.6) there would exist G_i such that $\sum_1^n \overline{R - \overline{G_i}} = R$, whence $\prod_1^n G_i = 0$ which is not possible.

(2.10) *Let* P *be H-closed and let* f *be a* $1 - 1$ *mapping of* P *onto a Hausdorff space* R. *Then* $G \subset P$ *is regularly open if and only if* $f(G) \subset R$ *is regularly open.*

Proof. Denote $f(G)$ by H. The set $f(\overline{G})$ is closed by (2.7) and (2.8); hence $f(\overline{G}) = \overline{H}$, $R - \overline{H} = f(P - \overline{G})$ and since $\overline{f(P - \overline{G})}$ is closed by (2.7) and (2.8) we have $\overline{R - \overline{H}} = f(\overline{P - \overline{G}})$, therefore $\operatorname{Int} \overline{H} = f\,(\operatorname{Int} \overline{G})$ which proves the lemma.

(2.11) *Let* Q *be paracombinatorially imbedded in a H-closed space* P. *Let* f *be a* $1 - 1$ *mapping of* P *onto* R. *Then the imbedding* $f(Q) \subset R$ *is paracombinatorial.*

Proof. Let H_1, H_2 be relatively open in $S = f(Q)$, $H_1 H_2 = 0$ and denote $f^{-1}(H_i)$ by G_i. If $f(a) = b \in \overline{H}_1 - S$, then $a \in \overline{G}_1 - Q$ (cf. the proof of 2.10), hence $a \in P - \overline{G}_2$, $b \in R - f(\overline{G}_2)$. Since $f(\overline{G}_2)$ is closed we have $b \in R - \overline{H}_2$, therefore $\overline{H}_1 \overline{H}_2 \subset S$ which proves the lemma.

Example 1. Q denotes the plane; A, B, C denote the set of all $(x, y) \in Q$ such that $y > 0$, $y = 0$, $y < 0$ respectively. $P_1 = \omega Q$ is Wallman's [6] compact space. The imbedding $Q \subset P_1$ is combinatorial in the strong sense [8], but is not paracombinatorial since A and C are relatively open in Q, $AC = 0$, but $\overline{A}\overline{C} - Q \supset \overline{B} - B \neq 0$.

(2.12) *If a normal space* Q *is paracombinatorially imbedded in* P, *then the imbedding is combinatorial.*

Proof. If $F_i \subset Q$ are relatively closed in Q and $\prod_1^n F_i = 0$, then there exist[4]) relatively open sets $G_i \subset Q$ such that $G_i \supset F_i$, $\prod_1^n G_i = 0$, hence $\prod_1^n \overline{G}_i \subset Q$, $\prod_1^n \overline{F}_i \subset Q$, $\prod_1^n \overline{F}_i = \prod_1^n F_i = 0$.

Example 2. Denote by I the discrete space of natural numbers. Choose a point $a \in \beta I - I$, βI denoting Čech's [5] compact

[4]) This is a well known property of normal spaces.

space. P_2 is the set βI with the topology defined in the following way: the points $n \in I$ are isolated; the fundamental neighborhoods of a are the same as in βI; any point $x \in \beta I - I - a$ possesses fundamental neighborhoods $GI + x$, G being a neighborhood of x in βI. Denote $P_2 - a$ by Q. Whenever G_i are open in Q and $a \in \overline{G}_1\overline{G}_2$ we have $a \in \overline{G_1 I}$, $a \in \overline{G_2 I}$, hence $G_1 G_2 I \neq 0$. Therefore the imbedding $Q \subset P_2$ is paracombinatorial.

Now choose for every infinite $A \subset I$ a point $x(A) \in \overline{A} - A$, $x(A) \neq a$, and denote by F the set of all $x(A)$. Then F is closed in Q, $a \in \overline{F}$ (since a clearly belongs to the closure of $\overline{F}$ in βI) and the power of F does not exceed the power c of the family of all subsets of I. Now $\Phi = Q - I - F$ is closed in Q and for any infinite $A \subset I$ we have $\overline{A}\Phi \neq 0$ since (Pospíšil [10]) $\overline{A}$ has the power 2^c. Hence $a \in \overline{\Phi}$, $a \in \overline{F}\overline{\Phi}$ which implies that the imbedding $Q \subset P_2$ is not combinatorial, not even 2-combinatorial [8].

Example 3. Choose again a point $a \in \beta I - I$. The space P_3 consists of the points x_{mn}, x_m, z (m, $n = 1, 2, \ldots$). The points x_{mn} are isolated; every point x_m possesses fundamental neighborhoods U_{mG} consisting of x_m and all x_{mn}, $n \in G$, G running over all neighborhoods of a in βI. The point z possesses fundamental neighborhoods $U_{\{G_k\}}$ consisting of z and of the points x_m and x_{mn} such that $n \in G_m$, $m \in G_0$, where $\{G_k\}$ runs over all sequences of neighborhoods of a in βI. It is easy to show that P_3 is regular, hence, being countable, normal.

Denote $P_3 - z$ by Q. If $F \subset Q$ is relatively closed, $z \in \overline{F}$, then denoting by F^* the set of all $x_n \in F$ we have easily $z \in \overline{F^*}$.

Since the imbedding of the set A of all x_n in $A + z$ is clearly combinatorial, this proves that the imbedding $Q \subset P_3$ is combinatorial.

Now let G_1, G_2 be open in Q, $z \in \overline{G}_1\overline{G}_2$. Then $Q\overline{G}_1\overline{G}_2 \neq 0$ (since the imbedding is combinatorial), hence either there exists a $x_{mn} \in \overline{G}_1\overline{G}_2$ which implies $x_{mn} \in G_1 G_2$, $G_1 G_2 \neq 0$, or there exists a $x_m \in \overline{G}_1\overline{G}_2$ which implies again $G_1 G_2 \neq 0$ since, for a given m, the imbedding of the set B_m of all x_{mn} in $B_m + x_m$ is clearly paracombinatorial. Therefore the imbedding $Q \subset P_3$ is paracombinatorial.

The set A is relatively closed and nowhere dense in Q, $z \in \overline{A} - A$. Hence Q is not hypercombinatorially imbedded in P_3.

§ 3.

(3.1) *Any Hausdorff space P may be hypercombinationally imbedded in a H-closed space $R = \tau P$ such that P is open in τP and the subspace $\tau P - P$ is discrete. This imbedding is essentially unique,* *i. e. if a space $R_1 \supset P$ possesses the same properties, then there exists a topological transformation of R onto R_1 which is identity on P.*

If P is paracombinatorially imbedded in a Hausdorff space S, then there exists a $1 - 1$ mapping f of a set $T \subset \tau P$, $T \supset P$, onto S such that $f(x) = x$ for $x \in P$; if S is H-closed, then $T = \tau P$.

Proof. I. Suppose that P is not H-closed since otherwise the theorem is trivial. A collection $\mathfrak{A}$ of open sets $A \subset P$ will be called an α-collection if (i) $A \in \mathfrak{A} \Rightarrow A \neq 0$; (ii) $A_1 \in \mathfrak{A}$, $A_2 \in \mathfrak{A} \Rightarrow A_1A_2 \in \mathfrak{A}$; (iii) the intersection of all $\overline{A}$, $A \in \mathfrak{A}$, is void. A maximal α-collection will be called a β-collection. By Zorn's theorem every α-collection is contained in a β-collection.

The space $\tau P = R$ consists of the points $\tau_{\mathfrak{B}}$ each of them corresponding to a β-collection $\mathfrak{B}$ and of all $x \in P$. Fundamental neighborhoods of the points $x \in P$ in R are their neighborhoods in P. Every $\tau_{\mathfrak{B}}$ possesses fundamental neighborhoods $B + \tau_{\mathfrak{B}}$, where $B \in \mathfrak{B}$. Clearly R is a Hausdorff space, P is open in R and the subspace $R - P$ is discrete.

Now let $\mathfrak{G}$ be an open covering of R and denote by $\mathfrak{A}$ the collection of the sets $P - \sum_1^n \overline{G}_i$, $G_i \in \mathfrak{G}$. $\mathfrak{A}$ is no α-collection since otherwise there would exist a β-collection $\mathfrak{B} \supset \mathfrak{A}$ and we would have the implications $G \in \mathfrak{G} \Rightarrow P - \overline{G} \in \mathfrak{A} \Rightarrow \tau_{\mathfrak{B}} \in \overline{P - \overline{G}} = \overline{R - \overline{G}} \subset R - G$ which is impossible. Evidently $\mathfrak{A}$ possesses the properties (ii) and (iii) of an α-collection; hence $\mathfrak{A}$ does not possess the property (i), i. e. $0 \in \mathfrak{A}$ which proves by (2.6) that R is H-closed.

Let F_1 and F_2 be relatively closed subsets of P and let $\tau_{\mathfrak{B}} \in \overline{F}_1\overline{F}_2$. Then $P - F_i$ non $\in \mathfrak{B}$ $(i = 1, 2)$ and since $\mathfrak{B}$ is a maximal α-collection there exists a set $B_i \in \mathfrak{B}$ such that $B_i(P - F_i) = 0$. Hence $B_i \subset F_i$, $B_1B_2 \in \mathfrak{B}$, $0 \neq B_1B_2 \subset F_1F_2$, therefore F_1F_2 is not nowhere dense. This proves by (2.1) that the imbedding $P \subset R$ is hypercombinatorial.

II. Now let P be paracombinatorially imbedded in a space S. For every $y \in S - P$ denote by $\mathfrak{B}(y)$ the collection of all open $A \subset P$ such that $y \in \overline{A}$. The intersection of all $P\overline{A}$, $A \in \mathfrak{B}(y)$, is void. If $A_1 \in \mathfrak{B}(y)$, $A_2 \in \mathfrak{B}(y)$, then $y \in \overline{A}_1\overline{A}_2$, hence by (2.4) $y \in \overline{A_1A_2}$, $A_1A_2 \in \mathfrak{B}(y)$. Therefore $\mathfrak{B}(y)$ is an α-collection. If $B \subset P$ is relatively open and $A \in \mathfrak{B}(y) \Rightarrow BA \neq 0$, then clearly $y \in \overline{B}$, whence $B \in \mathfrak{B}(y)$. Hence $\mathfrak{B}(y)$ is a β-collection.

For any $y \in S - P$ we set $\tau_y = \tau_{\mathfrak{B}(y)}$ and denote by T the set consisting of the points τ_y and of all $x \in P$. Clearly $y \neq y' \Rightarrow \tau_y \neq \tau_{y'}$. We set $f(\alpha_y) = y$ and $f(x) = x$ for $x \in P$; thus f is a $1 - 1$ transformation, $f(T) = S$. If G is an open neighborhood of a point $y \in S - P$, then $y \in \overline{GP}$, $GP \in \mathfrak{B}(y)$, $GP + \tau_y$ is a neighborhood of $\alpha_y = f^{-1}(y)$ in R. If G is an open neighborhood (in S) of a point $x \in P$, then $f^{-1}(G) \supset GP$ and GP is a neighborhood of x in R. Hence f is a continuous mapping.

If S is H-closed, let $x = \tau_{\mathfrak{B}} \in R - P$. Denote by C the intersection of closures (in S) of all $B \in \mathfrak{B}$. Then by (2.7) and (2.9) $C \neq 0$ since S is H-closed. If $y \in C$, then $B \in \mathfrak{B} \Rightarrow y \in \overline{B}$, whence $\mathfrak{B} \subset \mathfrak{B}(y)$, therefore $\mathfrak{B} = \mathfrak{B}(y)$ since $\mathfrak{B}$ is a β-collection. This implies $x = \alpha_y$, whence $T = R$.

III. Let P be hypercombinatorially, hence by (2.3) paracombinatorially, imbedded in a H-closed space R_1 such that P is open in R_1 and $R_1 - P$ is discrete. There exists, by II., a $1 - 1$ mapping f of R onto R_1 such that $f(x) = x$ for $x \in P$. Let $z = \tau_{\mathfrak{B}} \in$ $\in R - P$; then $B \in \mathfrak{B} \Rightarrow y = f(z) \in \overline{f(B)} = \overline{B}$. The imbedding $P \subset R_1$ being hypercombinatorial, $B \in \mathfrak{B} \Rightarrow y$ non $\in \overline{P - B}$; hence $R_1 -$ $- (P - B)$ is a neighborhood of y for any $B \in \mathfrak{B}$. Since $R_1 - P$ is discrete, $y + P$ is a neighborhood of y, hence so is $(y + P)\,[R_1 -$ $-(P - B)] = y + B = f(z + B)$. Therefore f is a topological transformation. This completes the proof.

Remark. It is immediately seen from the first part of this proof that τP is identical with the H-closed extension described in [7], 2.1.

(3.2) *Any Hausdorff space P may be paracombinatorially imbedded in a H-closed space $\tau' P$ such that P is open in $\tau' P$ and every point $x \in \tau' P - P$ is semiregular. The imbedding $P \subset \tau' P$ is essentially unique, and the SR-modification of τP on the set $\tau P - P$ may be taken as $\tau' P$.*

Proof. Denote by $\tau' P = R$ the SR-modification of τP on the set $\tau P - P$. Then by (1.11) P is imbedded in R. By (2.11) the imbedding is paracombinatorial and by (1.8) every $x \in R - P$ is semiregular. R is H-closed by (2.8).

If a space $R_1 \subset P$ possesses the above properties, then by (3.1) there exists a $1 - 1$ mapping f of R onto R_1 such that $f(x) = x$ for $x \in P$. Both in R_1 and in R the family consisting of all open sets contained in P and of all regularly open sets is an open base, since P is open and every point of its complement is semiregular. This implies by (2.10) that f is a topological transformation, i. e. the imbedding $P \subset R$ is essentially unique.

(3.3) *Any Hausdorff space P may be imbedded both regularly and hypercombinatorially in a H-closed space σP. This imbedding is essentially unique, and the SR-modification of τP relatively to $\tau P - P$ may be taken as σP.*

Proof. Denote by $\sigma P = R$ the SR-modification of τP relatively to $\tau P - P$. Then by (1.11) P is imbedded in R; the imbedding is regular by (1.8) and (1.4) and paracombinatorial by (2.11); R is H-closed by (2.8). If $F \subset P$ is closed, then $G = \tau P - F$ is open in τP and, for any $x \in G$, the set $H = GP + x$ is open in τP and $H + (\overline{H} - P) \subset G$, $\overline{H}$ denoting the closure of H in τP, hence G is open in R. Therefore F is closed in R. This implies by (2.5) that the imbedding $P \subset R$ is hypercombinatorial.

Now let P be both regularly and hypercombinatorially imbedded in a H-closed space R_1. By (3.1) there exists a $1 - 1$ mapping f of τP onto R_1 such that $f(x) = x$ for $x \in P$. Since the imbedding $P \subset R$ is regular, the family $\mathfrak{G}$ consisting of all $R - \overline{P - G}$, $G \subset P$ relatively open, is an open base of R. Since the imbedding $P \subset R$ is hypercombinatorial, $R - \overline{P - G} = G + (\overline{G} - P)$. Similarly, the family $\mathfrak{G}_1$ consisting of all sets $R_1 - \tilde{F} = G + (\tilde{G} - P)$, where $F = P - G$, G is open in P, and $\tilde{G}$, $\tilde{F}$ denote the closures of G, F in R_1, is an open base of R_1. Now it is easily seen that $f(\overline{G}) = \tilde{G}$ for any relatively open $G \subset P$ (since the closure of G in τP is H-closed it must be equal both to $\overline{G}$ and $f^{-1}(\tilde{G})$). Therefore we have the equivalence $H \in \mathfrak{G} \Leftrightarrow f(H) \in \mathfrak{G}_1$. Hence f is a topological transformation.

(3.4) *Any Hausdorff space P may be both regularly and paracombinatorially imbedded in a H-closed space $\sigma'P$ such that every point $x \in \sigma'P - P$ is semiregular. The imbedding is essentially unique and the SR-modification of $\tau'P$ relatively to $\tau'P - P$ may be taken as $\sigma'P$.*

Proof. Denote by $\sigma'P = R$ the SR-modification of $\tau'P$ relatively to $\tau'P - P$. Then by (1.11) P is imbedded in R and the imbedding is paracombinatorial by (2.11) and regular by (1.8) and (1.4). The points $x \in R - P$ are semiregular by (1.8). R is H-closed by (2.8).

Now let P be both regularly and paracombinatorially imbedded in R_1 and let every point $x \in R_1 - P$ be semiregular. By (3.1) there exists a $1 - 1$ mapping f of R onto R_1. For every relatively closed set $F \subset P$ the set $\overline{F} - F$ ($\overline{F}$ denoting the closure in R) consists of all points $y \in R - P$ such that $G\overline{F} \neq 0$ for any regularly open $G \subset R$ containing y. Since the same holds for R_1 we have by (2.10) $f(\overline{F}) = \tilde{F}$, where F is relatively closed in P and $\tilde{F}$ denotes closure in R_1. Since P is regularly imbedded both in R and R_1 we have $f(\overline{A}) = \tilde{A}$ for any $A \subset R$. Hence f is a topological transformation.

(3.5) *Any semiregular Hausdorff space P may be paracombinato-*

rially imbedded in a semiregular H-closed space R. The imbedding is essentially unique and we may set $R = \sigma' P$.

Proof. Let R be the SR-modification of τP on τP relatively to τP. Then by (1.10) P is imbedded in R and the imbedding is paracombinatorial by (2.11). R is H-closed by (2.8) and semiregular by (1.8). Hence by (1.4) P is regularly imbedded in R. This implies by (3.4) the topological equivalence $R = \sigma' P$ and the essential uniqueness of R.

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O H-uzavřených obalech topologických prostorů.

(Obsah předešlého článku.)

Nechť P je AHF-prostor. Říkáme, že bod $x \in P$ je *poloregulární*, když ke každému okolí H bodu x existuje otevřená množina G taková, že $x \in G \subset \operatorname{Int} \overline{G} \subset H$.

Nechť $Q \subset P$. Říkáme, že množina Q je *regulárně vnořena* do P, když každá uzavřená množina $F \subset P$ je průnikem některých množin tvaru $\overline{A}$, $A \subset Q$.

Říkáme, že množina Q je *hyperkombinatoricky vnořena* do P, když $\overline{Q} = P$ a pro libovolné uzavřené v Q množiny $F_i \subset Q$ platí: je-li $\prod_1^n F_i$ řídká v Q, pak $\prod_1^n \overline{F_i} = \prod_1^n F_i$.

Říkáme, že množina Q je *parakombinatoricky vnořena* do P, když $\overline{Q} = P$ a pro libovolné otevřené v Q množiny $G_i \subset Q$ platí: $\prod_1^n G_i = 0 \Rightarrow \prod_1^n \overline{G}_i \subset Q$.

Nazýváme AHF-prostor P *H-uzavřeným*, je-li P množina uzavřená v libovolném AHF-prostoru R, do něhož je prostor P vnořen.

Hlavním výsledkem práce jsou tyto věty:

Každý AHF-prostor P lze hyperkombinatoricky vnořit (a to v podstatě jediným způsobem) do H-uzavřeného prostoru τP takového, že množina P je otevřená v τP a všechny body prostoru $\tau P - P$ jsou isolované (v $\tau P - P$).

Každý AHF-prostor P lze parakombinatoricky vnořit (a to v podstatě jediným způsobem) do H-uzavřeného prostoru $\tau' P$ takového, že množina P je otevřená v $\tau' P$ a každý bod $x \in \tau' P - P$ je poloregulární.

Každý AHF-prostor P lze hyperkombinatoricky a regulárně vnořit (a to v podstatě jediným způsobem) do H-uzavřeného prostoru σP.

Každý AHF-prostor P lze parakombinatoricky a regulárně vnořit (a to v podstatě jediným způsobem) do H-uzavřeného prostoru $\sigma' P$ takového, že každý bod $x \in \sigma' P - P$ je poloregulární.

Transformace geodetických zeměpisných souřadnic na mezinárodní elipsoid.

Ing. Dr Václav Elznic, Praha.

(Došlo 15. 3. 1947.)

Otázkou transformace zeměpisných souřadnic geodetických s jedné plochy elipsoidické na druhou jsem se zabýval v úvodu svých tabulek „*Transeuro*" (Tabulky pro řešení hlavní geodetické úlohy na mezinárodním elipsoidu v zeměpisných šířkách 35^0 — 70^0; vyšlo v časopise „Zprávy o technické službě", čís. 20., roč. 1944*)), kde jsem také uvedl důvod této úvahy.

Při svém zasedání r. 1924 v Madridě rozhodla „Union géodésique et géophysique international", že napříště se mají všechny geodetické práce vykonávati na jednotné referenční ploše mezinárodního elipsoidu, jehož rozměry vypočetl z astronomicko-geodetické sítě USA r. 1910 sir John Hayford[1]) takto:

$$\text{velká poloosa} \ldots\ldots\ldots a = 6378\,388 \text{ m} \pm 35 \text{ m},$$
$$\text{malá poloosa} \ldots\ldots\ldots b = 6356\,912 \text{ m},$$
$$\text{zploštění} \ldots\ldots\ldots \frac{a-b}{a} = i = 1 : 297 \pm 0{,}8.$$

I když připustíme oprávněnost námitek (zejména ruských geodetů, hlavně prof. F. N. Krasovského) proti těmto rozměrům a proti označení „mezinárodního" elipsoidu (Hayford použil pouze sítí USA a vůbec nedbal rozsáhlých sítí evropských s význačnými oblouky poledníkovými a rovnoběžkovými, na př. meridianového oblouku západoevropského, oblouku Struveova, rovnoběžkového oblouku podél 52^0, ale nepoužil ani jiných dobrých měření, jako v Přední Indii a pod.), přece musíme připustiti, že nelze vůbec očekávati větších odchylek od skutečného všeobecného elipsoidu, vypočteného ze všech dosud známých měření, ze kterých dnes nejvýznačnější místo zaujímá rozsáhlá řetězcová síť SSSR. Důkazem toho jsou Krasovského studie v tomto směru (uveřejněné

*) Nyní „Zprávy veřejné služby technické".

[1]) John Hayford: The Figure of the Earth, Washington 1910. Hayfordem uváděnou střední chybu $\pm$ 18 m opravil na $\pm$ 35 Helmert.

v roce 1935 v časopise Geodezist), ke kterým použil a upravil výsledky všech do té doby známých měření, a vypočetl, že dávají

sítě SSSR.........	$a = 6378\,182 \pm 96$ m, $i = 1 : 298{,}97 \pm 2{,}0$ 6378 097 m pro $i = 1 : 297$
sítě USA.........	$a = 6378\,383 \pm 52$ m, $i = 1 : 297{,}70 \pm 1{,}6$ 6378 371 m pro $i = 1 : 297$
sítě Evropy a SSSR	$a = 6378\,247 \pm 58$ m, $i = 1 : 300{,}59 \pm 1{,}4$ 6378 129 m pro $i = 1 : 297$
sítě Evropy a USA	$a = 6378\,373 \pm 35$ m, $i = 1 : 298{,}24 \pm 1{,}1$ 6378 356 m pro $i = 1 : 297$
sítě Evropy, SSSR a USA	$a = 6378\,338 \pm 32$ m, $i = 1 : 299{,}97 \pm 0{,}8$ 6378 268 m pro $i = 1 : 297$.

Tím vlastně potvrdil, že rozměry Hayfordovy dobře vyhovují i oblastem euro-asijské pevniny. Nelze ovšem vyvrátiti námitku, že mezinárodní elipsoid byl odvozen pouze ze sítě jediného zeměDílu, a rozhodnutí Mezinárodní geodetické a geofysikální Unie se týkalo výpočtu z geodetického materiálu 14 let starého. V roce 1910 měl Hayford k disposici pouze 20 000 km triangulačních řetězců, zatím co do r. 1922 přibylo dalších 7 500 km, a od r. 1922—30 přibývalo ročně 1 400 km, od r. 1930 dokonce ročně 4 000 km řetězců.

Hayford nepoužil ani jiných prací v té době hotových. Dnes ovšem stav daleko pokročil. Tak Japonsko, které začalo triangulovat v r. 1888 má na své ostrovní říši úplnou síť I. řádu, která dává oblouk 16° dlouhý. Kanada začala s budováním triangulací r. 1906 a r. 1936 měla již 10 000 km triangulačních řetězců. Podobně Mexiko, Brasilie a Argentina mají značnou část země pokrytou novými triangulacemi. V Africe se buduje meridianový oblouk z Kaira k Mysu Dobré Naděje podél 30. poledníku a jeho veliká část je hotova. V Evropě se budoval podle Boškovičova návrhu oblouk z Kréty až k Murmaňsku a studovala se možnost spojení s obloukem africkým; tím by vznikl gigantický meridianový oblouk amplitudy 109°.

Dále je celá střední Evropa pokryta soustavnou plošnou triangulací a již v r. 1900 bylo zde k disposici 7 000 trojúhelníků stupňového měření. Od té doby přibylo úžasné množství sítí. Jen v SSSR bylo v r. 1935 na 30 000 km triangulačních řetězců I. řádu vysoké přesnosti, a ročně jich přibývá asi 5 000 km. Síť v té době obsahovala 73 základen a každá z nich měla oba koncové body určeny astronomickými zeměpisnými souřadnicemi a azimutem; kromě toho bylo v té době určeno dalších 176 Laplaceových bodů. V roce 1936 ukončil SSSR dosud nedosaženou délku rovnoběžkového oblouku mezi 52. a 54. rovnoběžkou a mezi 27. a 137. poledníkem, tedy o amplitudě 110°. Připojíme-li k němu evropský oblouk

52. rovnoběžky, činí amplituda oblouku z Irska až po Chabarovsk plných 145°. Kromě toho Krasovský namítá proti Hayfordovu výpočtu, že v Evropě je nepoměrně příznivější poloha sítí, a zejména sítí v SSSR, kde v nepřehledných rovinách činí střední hodnota tížnicové odchylky kolem 2″ a jen výjimečně dosahuje 10″. Naproti tomu Hayfordovy výpočty nejsou ušetřeny tížnicových odchylek přesahujících 100″, a několikadesítkové odchylky jsou pravidlem.

Přesto potvrzují výpočty Krasovského dobrou práci Hayfordovu, a hlavně praktickou cenu a správnost isostatické redukce, kterou tu Hayford poprvé užil ve velkém rozsahu podle vlastní methody. Ovšem ve střední Evropě užívaný elipsoid Besselův s rozměry

$$a = 6377\,397{,}15 \text{ m}$$
$$b = 6356\,078{,}96 \text{ m}$$
$$i = 1 : 299{,}15$$

je skutečným rozměrům Země velmi vzdálen, a evropské pevnině by spíše vyhovoval elipsoid Clarkův s Besselovým zploštěním. I tuto alternativu Krasovský připouští, neboť se mu zdá Besselovo zploštění lepší než Hayfordovo, a pouhá změna rozměrů velké poloosy by transformace velmi zjednodušila. Rovnice (1) na str. 37 by podržely pouze prvý člen, takže by

$$d\varphi = A\, s \cos \alpha_2,$$
$$d\lambda = B\, s \sin \alpha_2,$$
$$d\alpha = C\, s \sin \alpha_2,$$

neboť součinitele A, B, C by bylo lze tabelovati ve vhodném intervalu.

Nehodlám posuzovati oprávněnost námitek, ale jistě každý geodet a matematik musí připustiti skutečnost již s ohledem na rozhodnutí Mezinárodní geodetické a geofysikální Unie, že Besselův i jiné elipsoidy dosud užívané bude nutno opustiti a veškeré výpočty resp. výsledky převésti na elipsoid mezinárodní, ať již Hayfordův či jiný, pokročí-li vývoj této otázky dále. Hayfordova elipsoidu používá zatím jen Belgie, Bulharsko, Dánsko, Finsko, Italie, Portugalsko a Rumunsko.

Z řady různých rozměrů Země jsou pro mapy evropských států užívány dodnes tyto referenční elipsoidické plochy:

Airy (1830)	$a = 6377\,542{,}178$ m $b = 6356\,235{,}765$ m $i = 1 : 299{,}325$	Velká Britannie
Bessel (1841)	$a = 6377\,397{,}155$ m $b = 6356\,078{,}963$ m $i = 1 : 299{,}153$	ČSR, SSSR, Polsko, Jugoslovie, Řecko, Italie, Albansko, Holandsko, Norsko, Portugalsko, Rumunsko, Švýcarsko, Maďarsko, Německo.

Hayford (1910)	a = 6378 388,000 m b = 6356 911,946 m i = 1 : 297,000	Belgie od r. 1924, Bulharsko 1919, Dánsko 1934, Finsko 1924, Italie 1930, Portugalsko 1927, Rumunsko od r. 1924.
Clarke I. (1880)	a = 6378 249,2 m b = 6356 515,0 m i = 1 : 293,466	Francie, některé staré sibiřské triangulace SSSR.
Clarke II. (1880)	a = 6378 253,00 m b = 6356 518,33 m i = 1 : 293,46	Turecko, Rumunsko (1916).
Dánský elipsoid	a = 6377 104,43 m b = 6355 847,42 m i = 1 : 300	Dánsko do r. 1934.
Delambre (1806)	a = 6376 985 m b = 6356 323 m i = 1 : 308,647	Belgie do r. 1880.
Holandský elipsoid	a = 6376 950,4 m b = 6356 356,1 m i = 1 : 309,65	Holandsko do r. 1885
Plessis	a = 6376 523,3 m b = 6355 862,8 m i = 1 : 308,64	Francie (staré triangulace 1818 až 1855)
Schmidt	a = 6376 804,38 m b = 6355 690,52 m i = 1 : 302,02	Švýcarsko (staré triangulace z r. 1840)
Struve (1860)	a = 6378 298,3 m b = 6356 657,1 m i = 1 : 294,73	Španělsko
Svanberg	a = 6376 797 m b = 6355 838 m i = 1 : 304,25	Švédsko (od r. 1920 Besselův)

Tato pestrá směs rozměrů čeká na svoji mezinárodní unifikaci a musí se jí jednou dočkat, což vyžaduje nejen vědecký názor na geodetické základy kartografických prací, ale i praktický požadavek jednotné referenční plochy pro mezinárodní geodetické práce, spojení triangulací různých států a tím vytvoření mezinárodní trigonometrické sítě.

Nejednotnost geodetických prací je zvýšena ještě různými zobrazovacími způsoby na stejné referenční ploše. Zdá se, že budoucnost náleží válcové zobrazovací metodě Gauß-Krügerově, které dnes používá SSSR, Jugoslavie, Německo, Polsko, Italie, Norsko, Švédsko a Portugalsko (od r. 1924). Ale v Evropě nalezneme ještě celou řadu dalších způsobů zobrazovacích: Tak Velká Britannie má „British Modefied System", Polsko ještě Roussilhovu stereografickou projekci, Řecko azimutální projekci Hattovu, konformní kuželové zobrazení a Lambertovo modif. zobrazení, na jihu Italie modif. zobrazení Lambertovo, Albanie má italské zobrazení Bonneovo, Holandsko

Bonneovo zobrazení a konformní dvojitou projekci stereografickou, Portugalsko do r. 1927 zobrazení Bonneovo, Rumunsko od r. 1924 stereografickou projekci, ale část rovněž v zobrazení Bonneově, dunajskou oblast v modif. zobrazení Lambertově, Švýcarsko má zobrazení Bonneovo, Maďarsko stereografickou projekci, Belgie Lambertovo zobrazení, Dánsko zobrazení Buchwaldovo, od r. 1934 konformní zobrazení kuželové, Francie zobrazení Lambertovo, staré triangulace v zobrazení Bonneově, Turecko Bonneovo, teprve nově Gauß-Krügerovo a Španělsko zobrazení Lambertovo. Konečně jak známo, má ČSR obecné konformní zobrazení kuželové.

Lze si jen přáti, aby v zájmu jednotících prací byla evropská mezinárodní astronomicko-geodetická síť jako celek vyrovnána v době nejkratší (k tomu cíli zřídila Mezinárodní geodetická a geofysikální Unie komisi č. 12 ,,pro souborné vyrovnání evropských sítí,,), a aby současně zmizela nejednotnost zobrazovacích ploch i prací kartografických. Je to v zájmu hospodářském i vědeckém, aby nová Evropa vešla v život s novým astronomicko-geodetickým a kartografickým základem.

Z celého úvodu je jasné, že unifikace referenční plochy elipsoidické je a bude otázkou nedaleké budoucnosti. Proto úloha v tomto pojednání řešená má nejen teoretický, ale i praktický význam. Že se zabývám převodem geodetických zeměpisných souřadnic na společnou novou plochu elipsoidickou vyplývá právě z popsaného kaleidoskopu referenčních ploch zobrazovacích nejrůznějšího druhu, se zcela samostatnými počátky bez vzájemné souvislosti. Do jisté míry a s jistými výhradami jsou jedině zeměpisné souřadnice společným systémem a vůbec lze míti za to, že nejspolehlivější převod souřadnic různého původu je transformace přes zeměpisné souřadnice, když zobrazovací soustavy jsou přesně matematicky definovány.

Změnou parametrů elipsoidické plochy se změní i geodetické zeměpisné souřadnice a to podle diferenciálních vzorců, které jsou uvedeny v knize: Jordan-Eggert: ,,Handbuch der Vermesssungskunde", sv. III/2, str. 430—434 v tomto tvaru:

$$
\begin{aligned}
d\varphi_2 &= \frac{\varrho}{aM_2}\, s \cos\alpha_2\, da + \\
&\quad + \left(2(\varphi_2 - \varphi_1) - 3(\varphi_2 - \varphi_1)\sin^2\varphi + \frac{l^2}{2\varrho}\, sin^3\varphi \cos\varphi\right) di \\
d\lambda_2 &= \frac{\varrho}{aN_2}\, s \sin\alpha_2 \sec\varphi_2\, da - l \sin^2\varphi_1 \frac{\cos\varphi_1}{\cos\varphi_2}\, di \qquad (1) \\
d\alpha_2 &= \frac{\varrho}{aN_2}\, s \sin\alpha_2 \operatorname{tg}\varphi_2\, da - \\
&\quad - \left(\frac{l \sin^2\varphi_1 \sin\varphi_2 \cos\varphi_1}{\cos\varphi_2} - \frac{l}{\varrho}(\varphi_2 - \varphi_1)\cos^3\varphi_1\right) di
\end{aligned}
$$

V uvedených rovnicích je řada známých označení, které v dalším pro udržení souvislosti ponecháme: M poloměr poledníkové křivosti, N poloměr příčné křivosti, φ zeměpisná šířka; připojené indexy vyznačují příslušnost bodů *1*, *2* geodetické čáry délky s, α_2 zpětný azimut, radiant $\varrho = 206\,264{,}80625$, $l = \lambda_2 - \lambda_1$, t. j. rozdíl zeměpisných délek, a velká poloosa průřezové elipsy, da je změna rozměru velké poloosy, di změna jejího zploštění. V dalším se ještě setkáme se Schreiberovými symboly $(1) = \varrho/M$, $(2) = \varrho/N$.

Vzorce nejsou právě vhodně upraveny pro číselný výpočet, a pro kontrolu geodetického přenášení zeměpisných souřadnic jsem je upravil v jednodušší a přehlednější tvar:

$$
\begin{aligned}
d\varphi_2 &= (1)_2\, s \cos\alpha_2 \,.\, \frac{da}{a} + (2 - 3 \sin^2 \varphi_m)\, \Delta\varphi \,.\, di \\
d\lambda_2 &= (2)_2\, s \sin\alpha_2 \sec\varphi_2 \,.\, \frac{da}{a} - \sin^2\varphi_1 \frac{\cos\varphi_1}{\cos\varphi_2}\, \Delta\lambda \,.\, di \qquad (2) \\
d\alpha_2 &= (2)_2\, s \sin\alpha_2 \operatorname{tg}\varphi_2 \,.\, \frac{da}{a} - \sin^3\varphi_m \frac{\cos\varphi_1}{\cos\varphi_2}\, \Delta\lambda \,.\, di
\end{aligned}
$$

Z nových označení tu přichází střední zeměpisná šířka $\varphi_m = \frac{1}{2}(\varphi_1 + \varphi_2)$, a rozdíl zeměpisných šířek koncových bodů geodet. čáry je označen $\Delta\varphi = \varphi_2 - \varphi_1$, podobně rozdíl délek je $\Delta\lambda = \lambda_2 - \lambda_1 = l$.

Členy, které jsem vypustil ze složitých rovnic původních jsou vesměs řádu nižšího než 0,0001″, takže upravené diferenciální rovnice udržují přesnost požadovanou při přenosu geodetických zeměpisných souřadnic v síti bodů I. řádu, t. j. $\pm$ 0,0001″ v souřadnicích a $\pm$ 0,001″ v azimutě. Rovnice jsou tím jednodušší, že pro určitou dvojici elipsoidů jsou da/a a di konstantami, jak plyne z příkladů.

O. Schreiber ve svých tabulkách pro výpočet geodetických zeměpisných souřadnic na Besselově elipsoidu uvádí příklad, kde je dán počáteční bod souřadnicemi $\varphi_1 = 57^0$ a $\lambda_1 = 31^0$; geodetická čára délky $s = 120$ km má azimut $\alpha_1 = 135^0$. Stejnou úlohu řeší Ölander v podobných tabulkách pro elipsoid Hayfordův. Vlivem různých rozměrů elipsoidů jsou i výsledky výpočtu odlišné, jak vidíme z tohoto srovnání:

Hayford:	$\varphi_2 = 56^013'49{,}4628''$	Hayford:	$\lambda_2 = 32^022'05{,}2005''$
Bessel:	$\varphi_2 = 56^013'49{,}0218''$	Bessel:	$\lambda_2 = 32^022'06{,}0327''$
rozdíl:	$d\varphi_2 = +\,0{,}4410''$	rozdíl:	$d\lambda_2 = -\,0{,}8322''$

Hayford:	$\alpha_2 = 316^008'32{,}663''$
Bessel:	$\alpha_2 = 316^008'33{,}355''$
rozdíl:	$d\alpha_2 = -\,0{,}692''$.

Kontrolu správnosti výpočtu lze provésti za pomocí našich vzorců (2) pro $d\varphi_2$, $d\lambda_2$, $d\alpha_2$:

$$\text{Hayford: } a = 6378\,388{,}000 \text{ m}$$
$$\text{Bessel: } a = 6377\,397{,}154 \text{ m}$$

$$\frac{\mathrm{d}a}{a} = \frac{6378\,388{,}00 - 6377\,397{,}15}{6377\,397{,}15} = +\ 0{,}00015\,53684$$

$$\begin{array}{ll} \text{Bessel:} & i = 0{,}0033\,4277 \\ \text{Hayford:} & i = 0{,}0033\,6700 \\ \hline & \mathrm{d}i = -\ 0{,}0000\,2423 \end{array}$$

$$\begin{array}{ll} (1)_2 = 0{,}03233\,545 & \varphi_1 = 57^0 \\ (2)_2 = 0{,}03226\,836 & \varphi_2 = 56^0 13'49{,}0218'' \\ \Delta\lambda = 1^0 22'06{,}0327'' & \Delta\varphi = -\ 46'10{,}9782'' \\ = 4926{,}0327'' & = -\ 2770{,}9782'' \end{array}$$

$$\varphi_m = 56^0 36'55''.$$

Vyčíslením vzorců dostaneme:

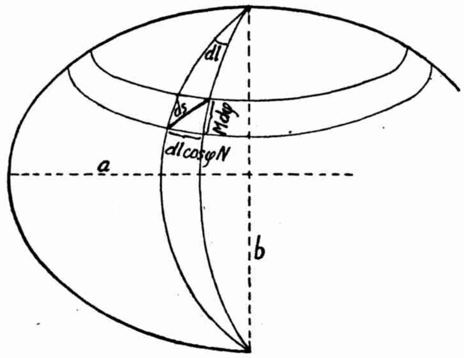
$$\begin{array}{lll} \mathrm{d}\varphi_2 = +\ 0{,}4409'' & \text{má býti} & +\ 0{,}4410'' \\ \mathrm{d}\lambda_2 = -\ 0{,}8322'' & & -\ 0{,}8322'' \\ \mathrm{d}\alpha_2 = -\ 0{,}692'' & & -\ 0{,}692''. \end{array}$$

Pro transformaci souřadnic bodů celostátních triangulací s jedné elipsoidické plochy na druhou je třeba zavésti vhodnější, rychlejší, hlavně ale systematický a jednotný způsob a postup, vyhovující přesností i bodům značně vzdáleným od centrálního bodu státní triangulace. Jak z dalšího vyplyne, může býti „centrální„ bod zcela libovolný, dokonce i fiktivní, protože nemusí býti souřadnicovým počátkem původní nebo snad nové soustavy. Jako nejvhodnější transformační způsob můžeme označiti takový, jehož transformační rovnice by obsahovaly výhradně členy s argumentem elementů centrálního bodu a nikoliv transformovaných bodů, čímž složité výrazy nabudou charakteru konstantních součinitelů.

A ještě jednu poznámku musíme připojiti k otázce „transformace“ zeměpisných souřadnic na mezinárodní elipsoid. Na rozdíl od transformací souřadnic různých zobrazovacích rovin, nepředpokládáme tu žádné totožné body obou soustav pro vytvoření transformačních rovnic (na př. Helmertova způsobu podobnosti, Tissotova způsobu affinní transformace, Gaußovy konformní transformace, Merkelovy projektivní transformace, a pod.), nýbrž tu jde o skutečné „převedení“ geodetických zeměpisných souřadnic s jedné plochy na druhou. Stupňová síť zeměpisných souřadnic na obou elipsoidických plochách se sice nemění, ale v důsledku různých rozměrů obou těles jsou její rozměry různé.

Převod můžeme uskutečniti tím způsobem, že zvolíme vhodný centrální bod (na př. průsečík stupně poledníku se stupněm rovnoběžky, které jsou na obou elipsoidech společnými křivkami), a spojíme jej se všemi body triangulace. Pro každou spojnici vypočteme délku a azimut na prvém (původním) elipsoidě, a s těmito prvky provedeme přenos souřadnic na elipsoidu druhém (novém). Navržený způsob je velmi způsobilý pro praktický výpočet, umíme-li vhodně odstraniti prvou část úlohy, t. j. výpočet délky strany a azimutu, která při velkém počtu bodů a značných délkách stran by činila nemalé počtářské potíže (okrajové body triangulace u nás by měly od centrálního bodu na př. $\varphi_0 = 49^0$, $\lambda_0 = 17^0$ vzdálenosti až 400 km).

Abychom tuto nesnáz odstranili, vyjdeme z vět o geodetické čáře, kterou v naší úloze skutečně představuje každá spojnice trigonometrického bodu s bodem centrálním. Pro element ds křivky (nerozhoduje jaké) na elipsoidě v místě s poloměrem poledníkové křivosti M a příčné křivosti N platí podle náčrtu:



$$ds^2 = M^2 d\varphi^2 + N^2 \cos^2 \varphi \, dl^2$$

čili

$$ds = \sqrt{M^2\left(\frac{d\varphi}{dl}\right)^2 + N^2 \cos^2 \varphi} \, dl = U \, dl.$$

Integrací této funkce obdržíme výraz pro délku oblouku:

$$s = \int_{l_1}^{l_2} U . dl.$$

Řešení se stane určitým, když stanovíme charakter křivky; jak známo, je takovou křivkou pro účely geodetické nejkratší spojnice dvou bodů na elipsoidické ploše. Místo funkce $\varphi = f(l)$ položíme řadu funkcí s proměnlivým parametrem ε, které představují řadu křivek, které všechny procházejí oběma body:

$$\varphi_1 = f(l_1, \varepsilon), \qquad \varphi_2 = f(l_2, \varepsilon), \qquad \text{atd.};$$

z nich je minimální ta, pro kterou $\varepsilon = 0$, t. j. pro kterou $\partial s / \partial \varepsilon$ pro

$\varepsilon = 0$ je nula. Diferencování integrálu dává

$$\frac{\partial s}{\partial \varepsilon} = \int_{l_1}^{l_2} \frac{\partial U}{\partial \varepsilon} \, dl$$

a jelikož U je funkcí φ a $d\varphi/dl$, dostaneme výraz geodetické čáry

$$\frac{\partial s}{\partial \varepsilon} = \int_{l_1}^{l_2} \left(\frac{\partial U}{\partial \varphi} \cdot \frac{\partial \varphi}{\partial \varepsilon} + \frac{\partial U}{\partial \varphi} \cdot \frac{\partial^2 \varphi}{\partial l \partial \varepsilon} \right) dl$$

pro kterou známe diferenciální vzorce (viz Jordan: Handbuch der Vermessungskunde, III/2, str. 69):

$$\begin{aligned} ds \cos \alpha &= M . d\varphi \\ ds \sin \alpha &= N \cos \varphi . dl \\ d\alpha &= dl . \sin \varphi \end{aligned} \tag{3}$$

ve kterých jsou:

$$M = \frac{c}{V^3}, \quad N = \frac{c}{V}, \quad V = \sqrt{1 + e'^2 \cos^2 \varphi}$$

$$e' \cos \varphi = \eta, \quad \text{čili} \quad V^2 = 1 + \eta^2, \quad e'^2 = \frac{a^2 - b^2}{b^2}, \quad c = \frac{a^2}{b},$$

dále $t = \operatorname{tg} \varphi$, $u = s \cos \alpha$, $v = s \sin \alpha$, (pro geodetickou čáru délky s a azimutu α).

Na základě diferenciálních rovnic (3) odvodil Jordan (l. c. III/2, § 18) rovnice, které vyjadřují vztah mezi zeměpisnými souřadnicemi koncových bodů geodetické čáry a její délkou i azimutem. Jordanovy rovnice převeďme na sférickou plochu poloměru N a obdržíme:

$$\begin{aligned} \frac{\varphi_2 - \varphi_1}{V^2} &= \frac{\varrho}{N} u - \frac{3\varrho}{2N^2} \eta^2 t u^2 - \frac{\varrho t}{2N^2} v^2 - \frac{\varrho}{6N^3} (1 + 3t^2 + \eta^2 - \\ &\quad - 9\eta^2 t^2) uv^2 - \frac{\varrho \eta^2}{2N^3} (1 - t^2) u^3 + \frac{\varrho t}{24N^4} (1 + 3t^2) v^4 - \\ &\quad - \frac{\varrho t}{6N^4} (2 + 3t^2) u^2 v^2; \\ l \cos \varphi &= \frac{\varrho}{N} v + \frac{\varrho t}{N^2} uv - \frac{\varrho t^2}{3N^3} v^3 + \frac{\varrho}{3N^3} (1 + 3t^2 + \eta^2) u^2 v - \\ &\quad - \frac{\varrho t}{3N^4} (1 + 3t^2) uv^3 + \frac{\varrho t}{3N^4} (2 + 3t^2) u^3 v; \\ \Delta\alpha &= \frac{\varrho t}{N} v + \frac{\varrho}{2N^2} (1 + 2t^2 + \eta^2) uv - \frac{\varrho t}{6N^3} (1 + 2t^2 + \eta^2) v^3 + \\ &\quad + \frac{\varrho t}{6N^3} (5 + 6t^2) u^2 v - \frac{\varrho}{24N^4} (1 + 20t^2 + 24t^4) uv^3 + \\ &\quad + \frac{\varrho}{24N^4} (5 + 28t^2 + 24t^4) u^3 v. \end{aligned} \tag{4}$$

V geodetické literatuře nemáme dosud výrazy pro obrácené řady, vyplývající z řešení 2. hlavní geodetické úlohy, kterých dosud nebylo třeba. Nazveme-li $\varphi_2 - \varphi_1 = \Delta\varphi$, $\lambda_2 - \lambda_1 = \Delta\lambda$, $\alpha_2 - \alpha_1 = \Delta\alpha$, obdržíme inversí geodetických řad (4) tyto rovnice:

$$\frac{u}{N} = \frac{1}{\varrho V^2} \,.\, \Delta\varphi + \frac{3\eta^2 t}{2\varrho^2 V^2} \,.\, \Delta\varphi^2 + \frac{1}{2\varrho^2}\cos^2\varphi \,.\, \Delta\lambda^2 +$$

$$+ \frac{\eta^2}{2\varrho^3}(1 - t^2) \,.\, \Delta\varphi^3 + \frac{\cos^2\varphi}{6\varrho^3}(1 - 3t^2 + 3\eta^2 t^2) \,.\, \Delta\varphi\Delta\lambda^2 -$$

$$- \frac{t\cos^2\varphi}{3\varrho^4} \,.\, \Delta\varphi^2\Delta\lambda^2 + \frac{t\,.\cos^4\varphi}{24\varrho^4}(1 - t^2) \,.\, \Delta\lambda^4; \qquad (5)$$

$$\frac{v}{N\cos\varphi} = \frac{\Delta\lambda}{\varrho} - \frac{t}{\varrho^2 V^2} \,.\, \Delta\varphi\Delta\lambda - \frac{\sin^2\varphi}{6\varrho^3} \,.\, \Delta\lambda^3 - \frac{1}{6\varrho^3}(2 - 2\eta^2 +$$

$$+ 9\eta^2 t^2) \,.\, \Delta\varphi^2\Delta\lambda - \frac{t\cos^2\varphi}{6\varrho^4}(1 - t^2) \,.\, \Delta\varphi\Delta\lambda^3;$$

$$\Delta\alpha = \sin\varphi \,.\, \Delta\lambda + \tfrac{1}{2}\cos\varphi \,.\, \Delta\varphi\Delta\lambda + \tfrac{1}{12}\cos^2\sin\varphi \,.\, \Delta\lambda^3.$$

V rovnicích (4) i (5) se všechny koeficienty vztahují k šířce φ_1. Jelikož podle dřívějšího $V^2 = 1 + \eta^2$, a pro geodetické čáry do 500 až 600 km můžeme s ohledem na danou úlohu podržeti členy nejvýš 3. řádu, lze rovnice (4) vyjádřiti analyticky ve zkrácené formě takto:

$$\Delta\varphi = \frac{1+\eta^2}{N}u - \frac{t(1+\eta^2)}{2N^2}v^2 - \frac{3t\eta^2}{2N^2}u^2 - \frac{(1+3t^2)+(1-9t^2\eta^2)}{6N^3}uv^2$$

$$(6)$$

$$\Delta\lambda = \frac{1}{N\cos\varphi}v + \frac{t}{N^2\cos\varphi}uv - \frac{t^2}{3N^3\cos\varphi}v^3 + \frac{(1+3t^2)+\eta^2}{3N^3\cos\varphi}u^2 v.$$

Teprve v této úpravě jsou rovnice vhodné pro vyjádření změn, které se projeví v $\Delta\varphi$ a $\Delta\lambda$ změnou parametrů elipsoidu. Původně známé změny v rozměrech elipsoidu da a di se tu neuplatňují, protože nyní je elipsoid zemský definován hodnotou příčné křivosti N a funkcí numerické výstřednosti $\eta^2 = f(e'^2)$.

Považujme parciální diferenciální podíly rovnic (6) podle N a η^2 za skutečné odchylky; diferencováním dostáváme:

$$d\varphi = \left(-\frac{1+\eta^2}{N^2}u + \frac{t(1+\eta^2)}{N^3}v^2 + \frac{3t\eta^2}{N^3}u^2 + \frac{1+3t^2}{2N^4}uv^2\right)dN$$

$$+ \left(\frac{1}{N}u - \frac{t}{2N^2}v^2 - \frac{3t}{2N^2}u^2 - \frac{1-9t^2}{6N^3}uv^2\right)d(\eta^2); \qquad (7)$$

$$d\lambda = \left(-\frac{1}{N^2\cos\varphi}\,v - \frac{2t}{N^3\cos\varphi}\,uv + \frac{t^2}{N^4\cos\varphi}\,v^3 - \right.$$
$$\left. - \frac{1+3t^2+\eta^2}{N^4\cos\varphi}\,u^2v\right) dN + \left(\frac{1}{3N^3\cos\varphi}\,u^2v\right) d(\eta^2).$$

Dosadíme-li konečně $u = s\cos\varphi$ a $v = s\sin\varphi$ z rovnic (5) do (7) dostáváme po úpravě a vyloučení číselně nepatrných členů tyto rovnice:

$$d\varphi = \left[-\frac{dN}{N} + (1-\eta^2)\,d(\eta^2)\right]\Delta\varphi + \left[\frac{dN}{N}\,\frac{2}{3}\,\frac{t\eta^2}{\varrho} - \right.$$
$$\left. - \frac{3}{2}\,\frac{t(1-3\eta^2)}{\varrho}\,d(\eta^2)\right]\Delta\varphi^2 + \left[\frac{dN}{N}\,\frac{t\cos^2\varphi}{2\varrho}\,(1+\eta^2)\right]\Delta\lambda^2 +$$
$$+ \left[\frac{dN}{N}\,\frac{1}{3}\left(\frac{\cos\varphi}{\varrho}\right)^2 + \frac{1}{2}\left(\frac{t\cos\varphi}{\varrho}\right)^2 d(\eta^2)\right]\Delta\varphi\Delta\lambda^2 \qquad (8)$$
$$d\lambda = \left[-\frac{dN}{N}\right]\Delta\lambda + \left[-\frac{dN}{N}\,\frac{t(1-\eta^2)}{\varrho}\right]\Delta\varphi\Delta\lambda +$$
$$+ \left[-\frac{dN}{N}\,\frac{(2+3t^2)}{3\varrho^2}\right]\Delta\varphi^2\Delta\lambda + \left[\frac{dN}{N}\,\frac{1}{6}\left(\frac{t\cos\varphi}{\varrho}\right)^2 + \frac{1}{3\varrho^2}\,d(\eta^2)\right]\Delta\lambda^3.$$

Nazveme-li nyní součinitele v hranatých závorkách prvé rovnice A_1, A_2, A_3, A_4, druhé rovnice B_1, B_2, B_3, B_4, jsou obecné rovnice oprav pro převod zeměpisných souřadnic s jedné elipsoidické plochy na druhou

$$\begin{aligned} d\varphi &= A_1\Delta\varphi + A_2\,\Delta\varphi^2 + A_3\,\Delta\lambda^2 + A_4\,\Delta\varphi\Delta\lambda^2 \\ d\lambda &= B_1\,\Delta\lambda + B_2\,\Delta\varphi\Delta\lambda + B_3\,\Delta\lambda^3 + B_4\,\Delta\varphi^2\Delta\lambda \end{aligned} \qquad (9)$$

Hodnota dN/N má v čitateli rozdíl poloměrů příčné křivosti v centrálním bodě na obou elipsoidech (a bude tudíž při přechodu s elipsoidu Besselova na Hayfordův kladná); protože

$$\eta^2 = e'^2\cos^2\varphi, \quad \text{jest} \quad d(\eta^2) = d(e'^2)\cos^2\varphi \qquad (10)$$

a pro cestu s Besselova elipsoidu na Hayfordův jest

$$d(\eta^2) = +\,0{,}00004\,89514\cos^2\varphi.$$

Rovnice (9) představují tudíž jednoduchou transformační cestu pro převod geodetických souřadnic zeměpisných s jedné elipsoidické plochy na druhou. Pro číselný příklad zvolme úlohu již dříve počítanou. Jako centrální bod budiž na Besselově elipsoidu P_1, jehož $\varphi_1 = 57^0$, $\lambda_1 = 31^0$. Úlohou je převésti geodetické zeměpisné souřadnice bodu $P_2(\varphi_2, \lambda_2)$:

$\varphi_1 = 57^000'$	$\lambda_1 = 31^000'$
$\varphi_2 = 56^013'49{,}0218''$	$\lambda_2 = 32^022'06{,}0327''$
$\Delta\varphi = -\,46'10{,}9782''$	$\Delta\lambda = +\,1^022'06{,}0327''$
$= -\,2770{,}9782''$	$= +\,4926{,}0327''$

Pro centrální bod, jehož $\varphi = 57^0$, jest N (index B = elipsoid Besselův, index H = elipsoid Hayfordův):

$$\log N_B = 6{,}80566\;52708, \qquad \log N_H = 6{,}80574\;01529$$

čili

$$\frac{\mathrm{d}N}{N} = \frac{N_H - N_B}{N_B} = +\;0{,}00017\;24368$$

$$\mathrm{d}(\eta^2) = +\;0{,}00001\;45205$$

$$t = 1{,}53986 \qquad t^2 = 2{,}37117 \qquad \varrho = 206\;265$$

$$\eta^2 = 0{,}00199 \qquad 1 - \eta^2 = 0{,}99801 \qquad \cos^2\varphi = 0{,}29663$$

Součinitelé rovnic (9):

$$\begin{aligned}
A_1 &= -\;0{,}00015\;79452\\
A_2 &= -\;0{,}00000\;00001\;57790\\
A_3 &= +\;0{,}00000\;00001\;91308\;43311\\
A_4 &= +\;0{,}00000\;00000\;00000\;52077\\
B_1 &= -\;0{,}00017\;24368\\
B_2 &= -\;0{,}00000\;00012\;84755\\
B_3 &= -\;0{,}00000\;00000\;00012\;31242\\
B_4 &= +\;0{,}00000\;00000\;00000\;58889
\end{aligned}$$

Pro přehledný výpočet vyjádříme $\Delta\varphi$ a $\Delta\lambda$ v desetitisících vteřin jako jednotkách, čili $\Delta\varphi = -\;0{,}2771$, $\Delta\lambda = +\;0{,}4926$, takže v rovnicích (9) jest

$$\begin{aligned}
\Delta\varphi &= -\;0{,}2771 & \Delta\lambda &= +\;0{,}4926\\
\Delta\varphi^2 &= +\;0{,}0768 & \Delta\varphi\Delta\lambda &= -\;0{,}1365\\
\Delta\lambda^2 &= +\;0{,}2427 & \Delta\varphi^2\Delta\lambda &= +\;0{,}0378\\
\Delta\varphi\Delta\lambda^2 &= -\;0{,}0672 & \Delta\lambda^3 &= +\;0{,}1195
\end{aligned}$$

a rovnice pro transformaci na Hayfordův elipsoid s centrálním bodem $\varphi = 57^0$ jsou

$$\begin{aligned}
\mathrm{d}\varphi = &-\;1{,}5794\;\Delta\varphi & \mathrm{d}\lambda = &-\;1{,}7244\;\Delta\lambda\\
&-\;0{,}0158\;\Delta\varphi^2 & &-\;0{,}1285\;\Delta\varphi\Delta\lambda\\
&+\;0{,}0191\;\Delta\lambda^2 & &-\;0{,}0123\;\Delta\varphi^2\Delta\lambda\\
&+\;0{,}0005\;\Delta\varphi\Delta\lambda^2 & &-\;0{,}0006\;\Delta\lambda^3
\end{aligned}$$

Tím je úloha rozřešena a jak je patrno, vede k velmi jednoduchému počtu. Pro danou úlohu dostáváme tento výsledek:

transformací	$\mathrm{d}\varphi = +\;0{,}4410''$	$\mathrm{d}\lambda = -\;0{,}8322''$
geodetickým přenosem ...	$= +\;0{,}4410''$	$= -\;0{,}8322''$
redukčními vzorci (2)	$= +\;0{,}4409''$	$= -\;0{,}8322''$.

Jakmile bude rozhodnuto přepočísti trigonometrické sítě na jednotný elipsoid, bude míti způsob zde uvedený nesporný význam pro svou jednoduchost, nutnou při transformaci značného množství bodů.

Závěr.

Problém daný nadpisem článku je v podstatě kartografickou úlohou zobrazení elipsoidu na plochu druhého elipsoidu, při čemž jsou předpokládány relativně malé změny parametrů, takže lze užíti v konvergentních řadách koeficientů odvozených z diferenciálních poměrů. Zásadně by bylo třeba klásti požadavek, aby nové zobrazení bylo konformní, t. j. aby úhly zůstaly pokud možno nezměněny. Uvedené řešení pomocí polárních souřadnic (s pólem v centrálním bodě triangulace) není zobrazením konformním, nýbrž azimutální projekcí s neskreslenými délkami, které je charakterisováno minimálním skreslením délkovým, ale značným skreslením úhlovým. Největší délkové skreslení mají geodetické kružnice se středem v centrálním bodě, nejmenší skreslení pak její průvodiče, neboť zobrazením se jejich délka nemění. Rovněž azimuty těchto průvodičů (paprsků) zůstávají nezměněny. Tím se toto zobrazení podobá stereografické projekci, která však mimo to je konformní.

Převod geodetických zeměpisných souřadnic na jinou elipsoidickou plochu uvedl prvý Helmert, který na rozdíl od zeměpisných souřadnic astronomických φ, λ, α, označil souřadnice geodetické B (Breite), L (Länge), A (Azimut). Nevhodnost transformačních rovnic, které obsahují délky geodetických čar s a jejich azimuty α byla vytčena již dříve. Bez hledání jiné cesty upravila geodetická služba německé branné moci rovnice Jordanovy (1) a Helmertovy*, ve kterých argument centrálního bodu je označen indexem 0, koncového bodu geodetické čáry indexem 1, $\varphi_m = \frac{1}{2}(\varphi_0 + \varphi_1)$.

Jordanovy rovnice:

$$\mathrm{d}\varphi = \left(-\Delta\varphi + \frac{\Delta\lambda^2}{2\varrho}\sin\varphi_0\cos\varphi_0\right)\frac{\mathrm{d}a}{a} + (\Delta\varphi\,(2 - 3\sin^2\varphi_m) + $$
$$+ \frac{\Delta\lambda^2}{2\varrho}\sin^3\varphi_m\cos\varphi_m)\,\mathrm{d}i \qquad (11)$$
$$\mathrm{d}\lambda = -\Delta\lambda\left(\frac{\mathrm{d}a}{a} + \sin^2\varphi_0\,\mathrm{d}i\right)\frac{\cos\varphi_0}{\cos\varphi_1}.$$

Helmertovy rovnice:

$$\begin{aligned} \mathrm{d}\varphi &= -\Delta\varphi\frac{\mathrm{d}a}{a} + (2\,\Delta\varphi\cos^2\varphi_m - p_5\sin^2\varphi_m),\\ \mathrm{d}\lambda &= -\Delta l\frac{\mathrm{d}a}{a} - q_5\sin^2\varphi_0\,\mathrm{d}i,\\ p_5 &= \Delta\varphi - \frac{\Delta\lambda^2}{4\varrho}\sin^2 2\varphi_m, \qquad q_5 = \Delta\lambda\frac{\cos\varphi_0}{\cos\varphi_1}. \end{aligned} \qquad (12)$$

*) R. Helmert: Veröffentlichungen des Kgl. Preußischen Geodät. Institutes, Berlin 1886, Lotabweichungen, Heft I.

Ani tyto rovnice nejsou nejvhodnějším tvarem, neboť téměř všechny členy obsahují proměnlivé argumenty pro různé body sítě. Prof. Vl. K. Hristov vyšel při svém řešení (rovněž azimutálním pomocí polárních souřadnic) z Jordanových rovnic (loc cit. str. 69—70), odtud vyjádřil $u = s \cos \alpha$, $v = s \sin \alpha$ řadou až do členů 3. řádu takto:

$$u = M\Delta\varphi + \tfrac{1}{2}t\,N\cos^2\varphi\,\Delta\lambda^2 + \tfrac{3}{2}\frac{\eta^2 t}{V^2}\,.\,M\,\Delta\varphi^2 + $$
$$+ \tfrac{1}{6}M\cos^2\varphi(1 - 3t^2)\,\Delta\varphi\,\Delta\lambda^2 \qquad (13)$$
$$v = N\cos\varphi\Delta\lambda - t\,M\cos\varphi\,.\,\Delta\varphi\,\Delta\lambda - \tfrac{1}{3}M\cos\varphi\,\Delta\varphi^2\,\Delta\lambda -$$
$$- \tfrac{1}{6}t^2\,N\cos^3\varphi\,\Delta\lambda^3.$$

Pomocí nich obdržel výrazy pro $d\varphi$, $d\lambda$ ve tvaru konvergentních řad, jejichž součinitelé byla pro pevný nulový bod (centrální bod triangulace) čísla konstantní.

Pro azimutální transformaci podle Helmertova způsobu odvodil H. Bodenmüller výrazy pro meridianovou konvergenci a poloosy a, b Tissotovy indikatrix (Mitteilungen des Chefs des Kriegs — Karten — und Vermessungswesens 1944, str. 305—306):

$$a = 1 + \tfrac{1}{3}\cos^2\varphi_0\left[\frac{da}{a} - \cos^2\varphi_0(1 - t_0^2)\,di\right]\Delta\lambda^2,$$
$$b = 1 + \tfrac{1}{3}\left[\frac{da}{a} - \cos^2\varphi_0\,(3 - t_0^2)\,di\right]\Delta\varphi^2, \qquad (14)$$

kde opět hodnotám φ a t přísluší argumenty nulového bodu. Z rovnic je ihned zřejmé, že i pro relativně značné $\Delta\varphi$ a $\Delta\lambda$ jsou a a b nepatrné. Z Tissotovy rovnice

$$\sin\omega = \frac{a - b}{a + b}$$

jest maximální hodnota úhlového zkreslení 2ω, kterou obdržíme již snadno:

$$2\omega'' = \tfrac{1}{3}\left\{\left[\frac{da}{a} - \cos^2\varphi_0(3 - t_0^2)\,di\right]\frac{\Delta\varphi^2}{\varrho''} - \right.$$
$$\left. - \cos^2\varphi_0\left[\frac{da}{a} - \cos^2\varphi_0(1 - t_0^2)\,di\right]\right\}\frac{\Delta\lambda^2}{\varrho''}. \qquad (15)$$

Při větších vzdálenostech jest hodnota úhlového zkreslení dosti značná; v naší úloze pro

$$\frac{da}{a} = + 0{,}00015$$
$$di = - 0{,}00002,$$
$$\Delta\varphi = 2770'',$$

$$\Delta\lambda = 4926'',$$
$$\varphi_0 = 57°,$$
$$2\omega = 0{,}03''.$$

Při vzdálenosti 1000 km dosáhne 2ω dokonce úhlovou přesnost měření $\pm$ 0,5″. Z daných předpokladů „polární" transformace (jak lze nazvati princip Helmertova převodu geodetických zeměpisných souřadnic) plyne, že nezměněnými zůstávají pouze azimuty geodetických čar procházejících „pólem", t. j. centrálním bodem. Z těchto důvodů se poledníky a rovnoběžky nezobrazují na druhé elipsoidické ploše jako takové, a jejich obrazy mají meridianovou konvergenci γ:

$$\operatorname{tg}\gamma = -\sin\varphi_0\left[\frac{\mathrm{d}a}{a} + \sin^2\varphi_0\,(1 - \tfrac{1}{2}\eta_0{}^2 + \tfrac{1}{2}\eta_0 t_0{}^2)\,\mathrm{d}i\right]\Delta\lambda - $$
$$- \tfrac{1}{3}\cos\varphi_0\,(4 + 3t_0{}^2)\left[\frac{\mathrm{d}a}{a} + \sin^2\varphi_0\,\mathrm{d}i\right]\Delta\varphi\,\Delta\lambda \qquad (16)$$

Význam převodu geodetických zeměpisných souřadnic s jedné elipsoidické plochy na druhou se uplatní prakticky zejména tehdy, když je třeba spojiti dvě triangulace, z nichž každá je počítána na jiném elipsoidu, a kromě toho zpravidla v jiné zobrazovací rovině.

Při spojování dvou triangulací přes zeměpisné souřadnice bude však třeba pomocí řady identických bodů „včlenit" prvou síť do druhé. Z Helmertových diferenciálních rovnic

$$\begin{aligned} \delta\varphi_i &= -\,p_1\delta\varphi_0 + p_5 k - p_4\delta\alpha_0 \\ \delta\lambda_i &= -\,q_1\delta\varphi_0 + q_5 k - q_4\delta\alpha_0 + \delta\lambda_0, \end{aligned} \qquad (17)$$

které se sestaví pro každý identický bod, pro který jsou známé původní geodetické zeměpisné souřadnice na ploše druhého elipsoidu i souřadnice s prvého elipsoidu na ni transformované (jejich rozdíl jest $\delta\varphi$, $\delta\lambda$), se vypočtou konstanty:

$$\left.\begin{array}{l}\text{rovnoběžný posun nulového bodu } \delta\varphi_0,\ \delta\lambda_0 \\ \text{stočení nulového azimutu } \delta\alpha_0 \\ \text{koeficient délkového skreslení } k\end{array}\right\}\ \text{ve vteřinách,}$$

když

$$-\,p_1 = \frac{M_0}{M_i}\cos\Delta\lambda,$$

$$p_5 = \Delta\varphi - \frac{1}{4\varrho''}\,\Delta\lambda^2\sin 2\varphi_m, \qquad (18)$$

$$-\,p_4 = -\frac{N_0}{M_0}\cos\varphi_0\,\frac{\Delta\lambda}{\varrho''},$$

$$-q_1 = \frac{\Delta\lambda}{N_0} \operatorname{tg} \varphi_i \frac{M_0}{\varrho^2},$$

$$q_5 = \Delta\lambda \cos \varphi_0 \sec \varphi_i,$$

$$-q_4 = \frac{\Delta\lambda}{\varrho''} p_5 \sec \varphi_i.$$

Pro převod celé sítě trigonometrických bodů (kterých jsou tisíce i desetitisíce) by naznačený způsob byl příliš zdlouhavý a nákladný. Proto se spokojíme s převodem jen bodů prvého, nejvýš druhého řádu o délkách stran do 20 km. Body uvnitř trojúhelníku transformujeme jednoduchým způsobem affinní transformace (viz Kästner: „Eine affine Übertragung ...“ v Zeitschrift für Vermessungswesen 1933, str. 225).

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Transformation des coordonnées géographiques sur l'ellipsoïde international.

(Extrait de l'article précédent.)

La première partie contient les plus nouveaux résultats des calculs du corps terrestre et les dimensions, qu'on emploie jusqu'ici dans les travaux européens de cartographie. Ensuite est traité la transmission des coordonnées géodésiques géographiques d'un ellipsoïde sur un autre. L'auteur réduit tout d'abord les équations de Jordan (1) en des équations plus simples (2), qu'il emploie pour le calcul du problème.

Le calcul commence par la transformation des équations (4) sur la base desquelles sont formées les équations inverses (5). Les équations (4) sont abrégées sous la forme (6) et par différentiation on obtient les équations (7). Par la substitution $u = s \cos \alpha$, $v = s \sin \alpha$ on transforme les équations (5) en (7) et on obtient la forme finale des équations de transformation exprimées généralement par (9).

Dans la conclusion l'auteur explique le sens pratique du problème et il introduit les formules de Jordan et de Helmert sous une nouvelle forme. Il indique la façon, qui a été employée pour le même problème par V. K. Hristov et H. Bodenmüller. Surtout sont importantes les équations de la déformation angulaire après la transformation (15) et les équations de la convergence méridienne (16). Pour la jonction des différentes triangulations on emploie les équations (17) d'après Helmert, quoiqu'il en existent d'autres façons de réunion et de transformation à l'aide d'une rangée de points identiques.