ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY A FYSIKY

ČÁST MATEMATICKÁ

Trigonometrický rozvoj $\mathfrak{K}(w, x, s) = \sum_{n=0}^{\infty} \frac{e^{2nx\pi i}}{(w+n)^s}$ a řad příbuzných.

Bedřich König, Nové Město na Moravě.

(Došlo dne 15. prosince 1937.)

Uvažujme nejprve trigonometrický rozvoj řady

$$Q(w, x, s, z) = \sum_{n=0}^{\infty} \frac{e^{2x\left(n+\frac{z}{\pi}\right)\pi i}}{(w + \frac{z}{\pi} + n)^s}. \tag{I}$$

V celé práci předpokládáme:

$$-\pi < z < \pi, \quad \Re w > 1,$$

buď $\Im x > 0$ a potom $\Re s > 0$, nebo $\Im x = 0$ a potom $\Re s > 1$. ($\Re$ = reálná část, $\Im$ = imaginární část.)

$$Q(w, x, s, z) = \sum_{K=-\infty}^{+\infty} c_K e^{iKz}. \tag{1}$$

$$c_K = \frac{1}{2\pi} \sum_{n=0}^{\infty} \int_{-\pi}^{+\pi} \frac{e^{2x\left(n+\frac{\alpha}{\pi}\right)\pi i}}{(w + \frac{\alpha}{\pi} + n)^s} e^{-iK\alpha}\, d\alpha =$$

$$= \frac{1}{2\pi\, \Gamma(s)} \sum_{n=0}^{\infty} \int_{t=0}^{\infty} \int_{\alpha=-\pi}^{+\pi} e^{-\left(w+\frac{\alpha}{\pi}+n\right)t + 2x\left(n+\frac{\alpha}{\pi}\right)\pi i - iK\alpha}\, t^{s-1}\, d\alpha dt =$$

$$= \frac{1}{2\Gamma(s)} \sum_{n=0}^{\infty} \int_{t=0}^{\infty} \int_{\alpha=-1}^{+1} e^{-t(n+\alpha) + 2x\pi i(n+\alpha) - K\pi i\alpha}\, e^{-wt}\, t^{s-1} d\alpha\, dt =$$

$$= \frac{1}{2\Gamma(s)} \sum_{n=0}^{\infty} \int_{t=0}^{\infty} \int_{\alpha=n-1}^{n+1} e^{-\alpha(t-2x\pi i) - K\pi i(\alpha-n)}\, e^{-wt}\, t^{s-1}\, d\alpha\, dt.$$

$$c_{2K}=\frac{1}{2\Gamma(s)}\int\limits_{t=0}^{\infty}\Big[\int\limits_{\alpha=0}^{1}e^{\alpha[t-\pi(2x-2K)i]}\mathrm{d}\alpha+2\int\limits_{\alpha=0}^{\infty}e^{-\alpha[t-\pi(2x-2K)i]}\mathrm{d}\alpha\Big]e^{-wt}t^{s-1}\mathrm{d}t$$

$$=\frac{1}{2\Gamma(s)}\int\limits_{t=0}^{\infty}\frac{e^{t-\pi(2x-2K)i}+1}{t-\pi(2x-2K)i}e^{-wt}t^{s-1}\,\mathrm{d}t.$$

Obdobně

$$c_{2K+1}=\frac{1}{2\Gamma(s)}\int\limits_{0}^{\infty}\frac{e^{t-\pi(2x-2K-1)i}-1}{t-\pi(2x-2K-1)i}e^{-wt}t^{s-1}\,\mathrm{d}t.$$

Takže obdržíme:

$$c_K=\frac{1}{2\Gamma(s)}\int\limits_{0}^{\infty}\frac{e^{t-\pi(2x-K)i}+(-1)^K}{t-\pi(2x-K)i}e^{-wt}t^{s-1}\,\mathrm{d}t. \tag{2}$$

Položíme-li $2x=\mathfrak{p}+i\mathfrak{y}$, dostaneme:

$$\underset{K=0,\pm1,\pm2,\dots}{c_K}=\frac{1}{2\Gamma(s)}\Big\{e^{\pi[\mathfrak{y}-(\mathfrak{p}-K)i]}\int\limits_{0}^{\infty}e^{-(w-1)t}t^{s-1}\frac{t+\pi\mathfrak{y}+\pi(\mathfrak{p}-K)i}{(t+\pi\mathfrak{y})^2+\pi^2(\mathfrak{p}-K)^2}\mathrm{d}t+$$

$$+(-1)^K\int\limits_{0}^{\infty}e^{-wt}t^{s-1}\frac{t+\pi\mathfrak{y}+\pi(\mathfrak{p}-K)i}{(t+\pi\mathfrak{y})^2+\pi^2(\mathfrak{p}-K)^2}\mathrm{d}t\Big\},\quad x\neq0. \tag{2^a}$$

Pro c_0 obdržíme též výraz:

$$c_0=\frac{1}{2\pi}\sum_{n=0}^{\infty}\int\limits_{-\pi}^{+\pi}\frac{e^{2x\left(n+\frac{\alpha}{\pi}\right)\pi i}}{\left(w+\frac{\alpha}{\pi}+n\right)^s}\mathrm{d}\alpha=\frac{1}{2}\int\limits_{0}^{1}\frac{e^{-2x\alpha\pi i}}{(w-\alpha)^s}\mathrm{d}\alpha+\int\limits_{0}^{\infty}\frac{e^{2x\alpha\pi i}}{(w+\alpha)^s}\mathrm{d}\alpha. \tag{2^b}$$

Pomocí $Q(w, x, s, z)$ obdržíme rozvoje následujících funkcí:

$$\mathfrak{R}(w,x,s)=\sum_{n=0}^{\infty}\frac{e^{2nx\pi i}}{(w+n)^s}=Q(w,x,s,0)=\sum_{K=-\infty}^{+\infty}c_K. \tag{II}$$

$$R(w,s)=\sum_{n=0}^{\infty}\frac{1}{(w+n)^s}=\mathfrak{R}(w,0,s)=\frac{1}{2(s-1)}\left\{\frac{1}{(w-1)^{s-1}}+\frac{1}{w^{s-1}}\right\}+$$

$$+\frac{1}{\Gamma(s)}\sum_{K=1}^{\infty}(-1)^K\left\{\int\limits_{0}^{\infty}e^{-(w-1)t}t^s\frac{dt}{t^2+\pi^2K^2}+\int\limits_{0}^{\infty}e^{-wt}t^s\frac{\mathrm{d}t}{t^2+\pi^2K^2}\right\}. \tag{III}$$

$$\left.\begin{aligned}\zeta(s) &= \sum_{n=1}^{\infty}\frac{1}{n^s} = \sum_{n=1}^{p-1}\frac{1}{n^s} + R(p, s) = \\ &= \sum_{n=1}^{p-1}\frac{1}{n^s} + \frac{1}{2(s-1)}\left\{\frac{1}{(p-1)^{s-1}} + \frac{1}{p^{s-1}}\right\} + \\ &+ \frac{1}{\Gamma(s)}\sum_{k=1}^{\infty}(-1)^k\left\{\int_0^{\infty} e^{-(p-1)t}\, t^s \frac{\mathrm{d}t}{t^2+\pi^2k^2} + \int_0^{\infty} e^{-pt}\, t^s \frac{\mathrm{d}t}{t^2+\pi^2k^2}\right\}.\end{aligned}\right\} \quad \text{(IV)}$$

$$L(x, s) = \sum_{n=1}^{\infty}\frac{e^{2nx\pi i}}{n^s} = \sum_{n=1}^{p-1}\frac{e^{2nx\pi i}}{n^s} + \mathfrak{R}(p, x, s). \quad \text{(V)}$$

$$\left.\begin{aligned}&U_1(w, x, s) = \sum_{n=0}^{\infty}\frac{\cos 2nx\pi}{(w+n)^s}, \quad U_2(w, x, s) = \sum_{n=0}^{\infty}\frac{\sin 2nx\pi}{(w+n)^s}, \\ &U_3(w, x, s) = \sum_{n=0}^{\infty}(-1)^n\frac{\cos 2nx\pi}{(w+n)^s}, \quad U_4(w, x, s) = \\ &\qquad = \sum_{n=0}^{\infty}(-1)^n\frac{\sin 2nx\pi}{(w+n)^s}\end{aligned}\right\} \; x \text{ reálné.} \quad \text{(VI)}$$

$$\left.\begin{aligned}U_1(w, x, s) &= \tfrac{1}{2}\{\mathfrak{R}(w, x, s) + \mathfrak{R}(w, -x, s)\}, \\ U_2(w, x, s) &= \frac{i}{2}\{\mathfrak{R}(w, -x, s) - \mathfrak{R}(w, x, s)\}, \\ U_3(w, x, s) &= \tfrac{1}{2}\{\mathfrak{R}(w, x+\tfrac{1}{2}, s) + \mathfrak{R}(w, \tfrac{1}{2}-x, s)\}, \\ U_4(w, x, s) &= \frac{i}{2}\{\mathfrak{R}(w, \tfrac{1}{2}-x, s) - \mathfrak{R}(w, x+\tfrac{1}{2}, s)\}.\end{aligned}\right\} \quad (3)$$

$$\left.\begin{aligned}T_1(w, x, z, s) &= \sum_{n=0}^{\infty} e^{2nz\pi i}\frac{\cos 2nx\pi}{(w+n)^s}, \\ T_2(w, x, z, s) &= \sum_{n=0}^{\infty} e^{2nz\pi i}\frac{\sin 2nx\pi}{(w+n)^s}, \\ T_3(w, x, z, s) &= \sum_{n=0}^{\infty}(-1)^n e^{2nz\pi i}\frac{\cos 2nx\pi}{(w+n)^s}, \\ T_4(w, x, z, s) &= \sum_{n=0}^{\infty}(-1)^n e^{2nz\pi i}\frac{\sin 2nx\pi}{(w+n)^s}.\end{aligned}\right\} \quad \text{(VII)}$$

V řadách (VII) jest buď $\Im(z \pm x) > 0$ a potom $\Re s > 0$, nebo $\Im(z \pm x) = 0$ a $\Re s > 1$.

$$\left.\begin{aligned}
T_1(w, x, z, s) &= \tfrac{1}{2}\{\mathfrak{K}(w, z + x, s) + \mathfrak{K}(w, z - x, s)\},\\
T_2(w, x, z, s) &= \frac{i}{2}\{\mathfrak{K}(w, z - x, s) - \mathfrak{K}(w, z + x, s)\},\\
T_3(w, x, z, s) &= \tfrac{1}{2}\{\mathfrak{K}(w, z + x + \tfrac{1}{2}, s) + \mathfrak{K}(w, z - x + \tfrac{1}{2}, s),\\
T_4(w, x, z, s) &= \frac{i}{2}\{\mathfrak{K}(w, z - x + \tfrac{1}{2}, s) - \mathfrak{K}(w, z + x + \tfrac{1}{2}, s)\}.
\end{aligned}\right\} \quad (4)$$

Integrály vyskytující se v trigonometrických rozvojích uvedených funkcí jsou tvaru:

$$\mathfrak{I} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-wt}\, t^{s-1} \frac{t}{(t + \pi\mathfrak{y})^2 + v^2}. \quad (5)$$

Speciálním případem (pro $\mathfrak{y} = 0$) integrálu $\mathfrak{I}$ jest

$$\mathfrak{I}_1 = \frac{1}{\Gamma(s)} \int_0^\infty e^{-wt}\, t^{s-1} \frac{t}{t^2 + v^2}, \quad (6)$$

kterým se zabýval M. Lerch v Časopise pro pěst. matem. a fysiky. XLIX, str. 31—37, str. 81—88.

Použijeme-li vztahu

$$\frac{1}{v^2 + \log^2(1 + z)} = \sum_{\nu=0}^{\infty} \frac{\mathfrak{a}_\nu}{\nu!} z^\nu, \text{[1]} \quad (a)$$

obdržíme:

$$\frac{1}{v^2 + t^2} = \sum_{\nu=0}^{\infty} \frac{\mathfrak{a}_\nu}{\nu!} (e^{-t} - 1)^\nu, \quad (a')$$

$$\mathfrak{I}_1 = \sum_{\nu=0}^{\infty} \frac{\mathfrak{a}_\nu}{\nu!} \Delta^\nu \frac{1}{w^s}, \quad \Delta w = 1. \quad (7)$$

Čísla $\mathfrak{a}_\nu$ vypočteme z rovnic:

$$\frac{1}{v - \log(1 + z)} = \sum_{\nu=0}^{\infty} \frac{C_\nu(v)}{\nu!} z^\nu, \text{[2]} \; v \text{ kladné} > \log 2, \quad (b)$$

$$\frac{1}{-iv - \log(1 + z)} - \frac{1}{iv - \log(1 + z)} = \frac{2iv}{v^2 + \log^2(1 + z)};$$

čili

$$\mathfrak{a}_\nu = \frac{C_\nu(-iv) - C_\nu(iv)}{2iv}. \quad (c)$$

[1]) Časopis pro pěst. mat. a fys. 49 (1919), 35—36.
[2]) Časopis pro pěst. mat. a fys. 48 (1918), 313—317.

$$
\left.\begin{aligned}
&\mathfrak{a}_0 = \frac{1}{v^2},\ \mathfrak{a}_1 = 0,\ \mathfrak{a}_2 = -\frac{2!}{v^4},\ \mathfrak{a}_3 = \frac{3!}{v^4},\ \mathfrak{a}_4 = \frac{4!}{v^6} - \frac{2\,.\,11}{v^4},\\
&\mathfrak{a}_5 = -\frac{2\,.\,5!}{v^6} + \frac{100}{v^4},\ \mathfrak{a}_6 = -\frac{6!}{v^8} + \frac{17\,.\,5!}{v^6} - \frac{2^2\,.\,137}{v^4},\\
&\mathfrak{a}_7 = \frac{7!}{v^8} - \frac{5!\,147}{v^6} + \frac{4!\,147}{v^4},\ \mathfrak{a}_8 = \frac{8!}{v^{10}} - \frac{6!\,322}{v^8} + \frac{4!\,967\,.\,7}{v^6} -\\
&\quad - \frac{4!\,121\,.\,9}{v^4},\\
&\mathfrak{a}_9 = -\frac{4\,.\,9!}{v^{10}} + \frac{9\,.\,9!}{v^8} - \frac{2^7.\,13\,397}{v^6} + \frac{4!\,761\,.\,3\,.\,4}{v^4},\ \text{atd.}
\end{aligned}\right\}\quad \text{(d)}
$$

Pro integrál (5) obdržíme, užitím rovnice (a′),

$$
\left.\begin{aligned}
\mathfrak{J} &= \frac{1}{\Gamma(s)}\sum_{\nu=0}^{\infty}\frac{\mathfrak{a}_\nu}{\nu!}\int_0^\infty e^{-wt}\,t^{s-1}\,(e^{-(t+\pi\mathfrak{y})}-1)^\nu\,dt =\\
&= \sum_{\nu=0}^{\infty}\frac{\mathfrak{a}_\nu}{\nu!}\,\Delta^\nu\left(e^{-\pi\mathfrak{y}z}\frac{1}{(w+z)^s}\right),\ \Delta z = 1,\ z = 0;
\end{aligned}\right\}\quad (8)
$$

$$
\left.\begin{aligned}
\Delta^\nu\left\{e^{-\pi\mathfrak{y}z}\frac{1}{(w+z)^s}\right\} &= \frac{e^{-\pi\mathfrak{y}\nu}}{(w+\nu)^s} - \binom{\nu}{1}\frac{e^{-\pi\mathfrak{y}(\nu-1)}}{(w+\nu-1)^s} + - \dots\\
&\dots + (-1)^{\nu-1}\frac{\nu e^{-\pi\mathfrak{y}}}{(w+1)^s} + (-1)^\nu\frac{1}{w^s}.
\end{aligned}\right\}\quad (9)
$$

Při výpočtech $\zeta(s)$, $R(w, s)$, $\mathfrak{R}(w, x, s)$ s $\Re x = 0$ dle (2^a) a (8) přijdeme k řadám:

$$
\left.\begin{aligned}
\zeta(2n) &= \sum_{k=1}^{\infty}\frac{1}{k^{2n}} = \frac{B_n}{(2n)!}\,2^{2n-1}\,\pi^{2n},\text{[3]}\\
S(2n) &= \sum_{k=1}^{\infty}\frac{(-1)^k}{k^{2n}} = -\frac{2^{2n-1}-1}{2^{2n-1}}\,\zeta(2n) = -\frac{2^{2n-1}-1}{(2n)!}\,\pi^{2n}B_n.
\end{aligned}\right\}\quad (10)
$$

V rovnicích (10) jest n celé, > 0, B_n jsou čísla Bernoulliova: $B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$, $B_3 = \frac{1}{42}$, $B_4 = \frac{1}{30}$, atd.

Abychom ukázali užitečnost vypočtených vzorců, určeme $\zeta(3)$, kladouce v (III) a (IV) $s = 3$, $p = w = 20$; tedy

$$
\zeta(3) = \sum_{n=1}^{19}\frac{1}{n^3} + \frac{1}{4}\left(\frac{1}{19^2} + \frac{1}{20^2}\right) + \sum_{k=1}^{\infty}\mathfrak{a}_k =
$$

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[3]) Viz na př. Serret-Scheffers: Lehrbuch der Differrential u. Integral Rechnung, II Teil, 4. u. 5. Auflage, str. 251.