Conformal Invariants in Two Dimensions I.

Harry Levy, Urbana, Illinois.

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The conformal geometry of Riemannian spaces has been studied by Hlavatý, Cartan, T. Y. Thomas, Schouten, and others who have developed various algorithms for $n > 2$. In particular, Hlavatý[1]) has found the generalized conformal curvatures of a curve. But for $n = 2$ a single curve has no conformal invariants since any two surfaces are conformally equivalent. Here we shall obtain some invariants of two or more curves in the two dimensional case.

1. Let V_n and V'_n be two Riemannian spaces in conformal correspondence so that the coefficients of their linear elements

$$ds^2 = g_{\lambda\mu}\, dx^\lambda\, dx^\mu \qquad ds'^2 = g'_{\lambda\mu}\, dx^\lambda\, dx^\mu \tag{1}$$

can be taken in the relation

$$g'_{\lambda\mu} = \sigma g_{\lambda\mu} \quad (\lambda, \mu = 1, 2, \ldots, n). \tag{2}$$

The reciprocal elements $g^{\lambda\mu}$ and the determinant $g = |\, g_{\lambda\mu} \,|$ (which we assume to be different from zero) then satisfy

$$g'^{\lambda\mu} = \sigma^{-1} g^{\lambda\mu} \qquad g' = \sigma^n g. \tag{3}$$

The Christoffel symbols of the second kind, which we designate by $\left\{ {\lambda \atop \mu\nu} \right\}$ will satisfy the relation

$$\left\{ {\lambda \atop \mu\nu} \right\}' = \left\{ {\lambda \atop \mu\nu} \right\} + \sigma_\mu\, \delta_\nu^\lambda + \sigma_\nu\, \delta_\mu^\lambda - g_{\mu\nu}\, \sigma^\lambda, \tag{4}$$

where $\sigma^\lambda = g^{\lambda\mu}\, \sigma^\mu$ and $\sigma_\mu = \dfrac{1}{2} \dfrac{\partial}{\partial x^\mu} \log \sigma^2$).

We assume that V_n and V'_n are analytic so that $g_{\lambda\mu}$ and $g'_{\lambda\mu}$

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1) Zur Conformgeometrie, Akad. Wetensch. Amsterdam, Proc. 38 (1935), pp. 281, 738, 1006.

2) Cf. Eisenhart, Riemannian Geometry, Princeton University Press (1926), p. 89.

(and therefore σ too) are continuous and have continuous derivatives of all orders.

Let C be an analytic curve and C' its transform. The unit components of the tangent vectors, $\lambda_{1)}{}^{\mu}$ and $\lambda'_{1)}{}^{\mu}$ respectively, satisfy

$$\lambda'_{1)}{}^{\mu} = \sigma^{-\frac{1}{2}} \lambda_{1)}{}^{\mu} \tag{5}$$

and we obtain by a direction computation that the components of the principle normal (if defined) satisfy

$$k'_1 \lambda'_{2)}{}^{\mu} = \sigma^{-1} (k_1 \lambda_{2)}{}^{\mu} + \sigma_{\nu} \lambda_{1)}{}^{\nu} \lambda_{1)}{}^{\mu} - \sigma^{\mu}),\ ^{3)} \tag{6}$$

where k_1 and k'_1 are the first curvatures of C and C' respectively.

From equations (3) we have that $\sigma = (g'/g)^{1/n}$ and therefore that

$$\sigma_{\mu} = \frac{1}{2n} \frac{\partial}{\partial x^{\mu}} \log (g'/g). \tag{7}$$

If we eliminate σ from (6) by means of (7) we obtain that C^{μ} defined by

$$C^{\mu} = g^{1/n} \left\{ k_1 \lambda_{2)}{}^{\mu} + \frac{1}{2n} (g^{\mu\nu} - \lambda_{1)}{}^{\mu} \lambda_{1)}{}^{\nu}) \frac{\partial}{\partial x^{\nu}} \log g \right\} \tag{8}$$

is invariant under conformal transformations.

2. Henceforth we assume $n = 2$. In this case there is a unique direction normal to a given direction, and consequently we must have

$$\lambda'_{2)}{}^{r} = \sigma^{-\frac{1}{2}} \lambda_{2)}{}^{r} \qquad (r, s = 1, 2), \tag{9}$$

where the ambiguity in sign can be regarded as incorporated in $\sigma^{-\frac{1}{2}}$ itself. From (2) and (3) it then follows that

$$g'^{-\frac{1}{4}} \lambda'_{2)}{}^{r} g'_{rs} = g^{-\frac{1}{4}} \lambda_{2)}{}^{r} g_{rs} \tag{10}$$

and multiplying the right hand sides of (8) and (10) and contracting we obtain that the expression I_1, defined by

$$I_1 = g^{\frac{1}{4}} \left(k + \frac{1}{4} \lambda_{2)}{}^{r} \frac{\partial}{\partial x^{r}} \log g \right) \tag{11}$$

is invariant under conformal transformations.

It will be convenient to write μ^r for $\lambda_{2)}{}^{r}$ and simply λ^r for $\lambda_{1)}{}^{r}$ and

$$\varphi = g^{\frac{1}{4}}, \qquad \varphi_r = \frac{\partial}{\partial x^r} \varphi$$

so that (11) may be written

$$I_1 = \varphi k + \varphi_r \mu^r. \tag{12}$$

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[3]) Cf. Modesitt, Some Singular Properties of Conformal Transformations between Riemannian Spaces, Am. Journ. of Math., 1938.

We obtain a sequence of functions I_α ($\alpha = 1, 2, 3, \ldots$) from (12) by differentiation with respect to the arc s of the given curve and multiplication by φ,

$$I_\alpha = \varphi \frac{\mathrm{d}}{\mathrm{d}s} I_{\alpha-1} \qquad (\alpha = 2, 3, 4, \ldots) \tag{13}$$

and clearly I_α is (for all values of α) invariant under conformal transformations. We can readily obtain by means of the Frenet equations that the values of I_α for $\alpha = 2, 3$ are given by

$$I_2 = \varphi^2 \frac{\mathrm{d}}{\mathrm{d}s} k + \varphi \varphi_{rs} \mu^r \lambda^s, \tag{14}$$

$$\begin{aligned} I_3 = \varphi^3 \frac{\mathrm{d}^2}{\mathrm{d}s^2} k + 2\varphi^2 \varphi_r \lambda^r \frac{\mathrm{d}}{\mathrm{d}s} k + \varphi^2 \varphi_{rs} (\mu^r \mu^s - \lambda^r \lambda^s) k + \\ + \varphi^2 (\varphi_{rst} + \varphi \varphi_t \varphi_{rs}) \lambda^r \mu^s \lambda^t, \end{aligned} \tag{15}$$

where $\varphi_{r\ldots st}$ is obtained from $\varphi_{r\ldots s}$ by the formal process of covariant differentiation with respect to the $g's$,

$$\varphi_{rs} = \frac{\partial}{\partial x^s} \varphi_r - \varphi_u \begin{Bmatrix} u \\ rs \end{Bmatrix}$$

etc.

We can prove by induction that I_α, for an arbitrary value of α, has an expansion of the following form

$$\begin{aligned} I_\alpha = {} & \varphi^\alpha k^{(\alpha-1)} + a_\alpha \varphi^{\alpha-1} \varphi_r \lambda^r k^{(\alpha-2)} \\ & + \{b_\alpha \varphi^{\alpha-2} (\varphi_r \lambda^r)^2 + (c_\alpha + 1) \varphi^{\alpha-1} \varphi_{rs} \lambda^r \lambda^s + \varphi^{\alpha-1} \varphi_{rs} \mu^r \mu^s \\ & + \mathrm{d}_\alpha \varphi^{\alpha-1} \varphi_r \mu^r k\} k^{\alpha-1} + * \end{aligned} \tag{16}$$

where $a_\alpha, b_\alpha, c_\varphi, d_\alpha$ are the constants given by

$$\begin{aligned} a_\alpha &= \frac{1}{2} (\alpha - 2)(\alpha + 1), & b_\alpha &= \frac{1}{4!} (\alpha - 1)(\alpha - 2)(\alpha - 3)(3\alpha + 4), \\ c_\alpha &= \frac{1}{3!} \alpha (\alpha + 1)(\alpha - 4), & d_\alpha &= \frac{1}{3!} (\alpha - 3)(\alpha^2 - 4), \end{aligned} \tag{17}$$

and the * represents a polynomial in $\lambda^r, \mu^r, \varphi, \varphi_r, \ldots, \varphi_{r_1 \ldots r_\alpha}$, $k, k', k'', \ldots, k^{(\alpha-4)}$.

We can prove further by induction by means of (13) that I_α regarded as a function of $\mu^r, k, k', \ldots, k^{(\alpha-1)}$ is an odd function of these variables taken together, that is

$$I_\alpha (-\mu^r, -k, -k', \ldots, -k^{(\alpha-1)}) = -I_\alpha (\mu^r, k, k', \ldots, k^{(\alpha-1)}).$$

3. Let C_i ($i = 1, 2, 3$) be three curves concurrent in a point P_0 of the surface V_2, and let $\lambda_{i)}{}^n$ be the contravariant components of their unit tangent vectors and $\mu_{i)}{}^r$ of their unit normal vectors.

Choose a positive direction of rotation about P_0 and let Θ_i be the directed angle from μ_{i+1} to μ_{i+2}.[4])

Three directions at a point in two dimensions are necessarily linearly dependent, and it follows directly that the coefficients of dependence for the normal directions are the sines of their angles, so that

$$\mu_{1)}{}^{r} \sin \Theta_1 + \mu_{2)}{}^{r} \sin \Theta_2 + \mu_{3)} \sin \Theta_3 = 0. \tag{18}$$

If we write (12) for each of the three curves and evaluate at their common point P, multiply each of these three equations by the corresponding $\sin \Theta_i$ and add, the terms in μ drop out by virtue of (18) and we obtain

$$\Sigma I_1 \sin \Theta = \varphi \, \Sigma k \sin \Theta. \tag{19}$$

The left hand side is invariant under conformal transformations and the right under coordinate transformations. We have consequently the following theorem:

If three curves C_i on a surface are concurrent and if K_i are the values, at their common point, of their (geodesic) curvatures and $\Theta_1, \Theta_2, \Theta_3$ the directed angles from the normals to C_2, C_3, C_1 to the normals to C_3, C_1, C_2 respectively, then $\sqrt[4]{g}\,(k_1 \sin \Theta + k_2 \sin \Theta_2 + k_3 \sin \Theta_3)$ is an absolute conformal invariant.

This theorem has a particularly simple geometric interpretation in the Euclidean plane; if we assume that $k_1 k_2 k_3 \neq 0$, the sum with which we are concerned differs by a factor from

$$R_2 R_3 \sin \Theta_1 + R_1 R_3 \sin \Theta_2 + R_1 R_2 \sin \Theta_3,$$

where the $R's$ are the radii of curvature, reciprocals of the $K's$ and we observe that this expression represents the area of the triangle formed by the three centers of curvature. In particular, it then follows that if the centers of curvature for a common point of three curves in the Euclidean plane are collinear, they remain collinear under conformal transformations of the plane into itself.

The above results can be extended by writing in place of (18) the linear equations satisfied by the λ's and by the products λ_μ. Let us assume for the sake of definiteness that the positive direction of rotation has been chosen so that the angle from $\lambda_{1)}$ to $\mu_{1)}$ is $+\frac{\pi}{2}$. Then Θ_3 and $-\Theta_2$ are the directed angles from $\mu_{1)}$ to $\mu_{2)}$ and μ_3 respectively and we have that

[4]) It is of course understood that by $\mu_{k)}$ we mean μ_{k-3} if $k > 3$.

$$\begin{aligned} \lambda_{2)}{}^r &= e\,(\cos\Theta_3\,\lambda_{1)}{}^r + \sin\Theta_3\,\mu_{1)}{}^r), \\ \lambda_{3)}{}^r &= \bar{e}\,(\cos\Theta_2\,\lambda_{1)}{}^r - \sin\Theta_2\,\mu_{1)}{}^r), \\ \mu_{2)}{}^r &= -\sin\Theta_3\,\lambda_{1)}{}^r + \cos\Theta_3\,\mu_{1)}{}^r, \\ \mu_{3)}{}^r &= \sin\Theta_2\,\lambda_{1)}{}^r + \cos\Theta_2\,\mu_{1)}{}^r, \end{aligned} \tag{20}$$

where $e(\bar{e})$ is ± 1 according as $\lambda_{2)}, \mu_{2)}$ $(\lambda_{3)}, \mu_{3)})$ and $\lambda_{1)}, \mu_{1)}$ have the same or opposite orientation.

It follows by direct computation, by means of (20), that

$$\sin\Theta_1\,\lambda_{1)}{}^r + e\sin\Theta_2\,\lambda_{2)}{}^r + \bar{e}\sin\Theta_3\,\lambda_{3)}{}^r = 0 \tag{21}$$

and, denoting by S the symmetric part, that is

$$S\,\{\sin 2\Theta_1\,\lambda_{1)}{}^r\mu_{1)}{}^s + e\sin 2\Theta_2\,\lambda_{2)}{}^r\mu_{2)}{}^s + \bar{e}\sin\Theta_3\,\lambda_{3)}{}^r\mu_{3)}{}^s\} = 0. \tag{22}$$

We can now write (14) for each of the three curves, multiply each I_2 by the appropriate $\pm\sin 2\Theta$ and add the three products. The terms in $\lambda\mu$ drop out by virtue of (22) leaving

$$\begin{aligned} &I_2{}^{(1)}\sin 2\Theta_1 + e\,I_2{}^{(2)}\sin 2\Theta_2 + \bar{e}\,I_2{}^{(3)}\sin 2\Theta_3 = \\ &\qquad = \varphi^2\,(k'_1\sin 2\Theta_1 + e\,K'_2\sin 2\Theta_2 + \bar{e}\,K'_3\sin 2\Theta_3). \end{aligned}$$

Hence we have the theorem:

If three curves C_i on a surface have a common point and if k'_i is the value there of the derivative of the (geodesic) curvature of C_i with respect to its arc, if Θ_i is the directed angle from the normal to C_{i+1}[4] *to the normal to C_{i+2} and e and $\bar{e}$ are ± 1 (as above defined), then $\sqrt{g}\,(k'_1\sin 2\Theta_1 + e\,k'_2\sin 2\Theta_2 + \bar{e}\,k'_3\sin 2\Theta_3)$ is invariant under conformal maps of the surface.*[5]

4. In the determination of the curvature of a curve an ambiguity may arise in the choice of sign. Let us assume that we take the (geodesic) curvature as always non-negative and that thereby a positive normal is determined in accordance with the Frenet equations.[6] If we introduce normal coordinates at P_0 and find the expansion of the coordinates of the points of a curve in a neighborhood of P_0 we see that the given curve and its positive principal normal lie on the same side of the geodesic tangent at P_0 to the curve. Hence if two curves are tangent and lie on the same side of their common tangent geodesic they have the same principal normal whereas if they lie on opposite sides of their common tangent geodesic their normals are directed oppositely.

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[5]) If a conformal transformation of V_2 into itself interchanges C_1 and C_2 and leaves C_3 invariant this and the preceding theorem reduce to special theorems obtained by Kasner in the plane. Cf. Geometry of Conformal Symmetry, Annals of Math., 2nd series, vol. 38 (1937) pp. 876—877.

[6]) See, for example, Eisenhart, l. c., p. 106.

Suppose C_2 and C_3 are tangent to each other but not to C_1 and that their tangent vectors have the same direction. Then e and $\bar{e}$ of equations (20) will have the same or opposite signs according as C_2 and C_3 lie on the same or opposite sides of their common tangent geodesic. But in either case $\sin \Theta_1 = 0$ and $\sin 2\Theta_2 = -\sin 2\Theta_3$ while $\sin \Theta_2 = \mp \sin \Theta_3$. The preceding theorem reduces to the following special case:

$\sqrt[4]{g}\,(k_1 \mp k_2)$ and $\sqrt{g}\,(k'_1 \mp k'_2)$ are absolute conformal invariants of two tangent curves where the $\mp$ is to be taken according as the curves lie on the same or opposite sides of their common tangent geodesic.

The derivation of the result that $\sqrt[4]{g}\,(k_1 \mp k_2)$ and $\sqrt{g}\,(k'_1 \mp k'_2)$ are absolute conformal invariants was communicated to the writer by Professor Hlavaty who obtained it by methods only different from the above. Others have obtained somewhat similar results.[7])

If two curves are tangent and have contact of order $h > 1$ either $k_1 = k_2,\ k'_1 = k'_2, \ldots, k_1^{(h-2)} = k_2^{(h-2)}$ and they lie on the same side of their common geodesic tangent so that the positive direction of their normals coincide or $k_1 = k_2 = k'_1 = \ldots = k_2^{(h-2)} = 0$ in which case the positive directions of their normals need not coincide. It follows from (16) and the remarks in the paragraph following (17) that the functions I_α for $\alpha = 1, 2, \ldots, h-1$ formed for one curve are equal, except possibly for sign, to the corresponding functions for the other curve and that the functions I_h differ only in the first term so that

$$I_h^{(1)} \mp I_h^{(2)} = \varphi^h\,(k_1^{(h-1)} \mp k_2^{(h-1)}),$$

where the $\mp$ sign is to be taken according as the normals coincide or are oppositely directed. Hence we have the result:

If two curves have contact of order h, $\sqrt[4]{g^h}\,(k_1^{(h-1)} \mp k_2^{(h-1)})$ is an absolute conformal invariant.

Equations (14) yield one final result. If C and C' are two curves intersecting orthogonally, the pairs of directions $\lambda_{1)}, \mu_{1)}$

[7]) Cf. Kasner, Conformal Geometry, Proceedings, Fifth Int. Cong. 2 (1912) p. 81; Ostrowski, Berührungsmaße, nullwinklige Kreisbogendreiecke und die Modulfigur, Jahresb. Deut. Math.-Verein. **44** (1934) p. 56; and Kasner and Comenetz, Conformal Geometry of Horn Angles, Proceedings, Nat. Ac. Sciences (Washington) **22** (1936) p. 303. These writers restrict themselves to conformal transformations of the Euclidean plane into itself and naturally find only the relative invariants $k_1 - k_2$ and $k'_1 - k'_2$. Comenetz, Conformal Geometry on a Surface, Bull. Am. Math. Soc. **42** (1936) p. 806 extends the results of Kasner and Comenetz to surfaces and notes that $(k'_1 - k'_2)/(k_1 - k_2)^2$ is an absolute invariant.