On integral relations in economical cinematics.

Otomar Pankraz, Praha.

(Received February 11, 1939.)

In addition to my two papers 1. Zum Problem der Maßenfabrikation (Aktuárské vědy, Prague 1935.) and 2. Über eine Bedingung für den stationären wirtschaftlichen Güterkreislauf (Bull. intern. Acad. Tchèque des Sci. 1936.) I intend to deal in this communication with two integral relations concerning the cinematics of the economy of a territorial unit. We shall be therefore interested in the temporal development of the relations between the elements in a certain economic system. If we suppose that the economy in question is a system of economic elements in equilibrium, then it is possible to distinguish two sides: one consisting of goods (and services) and the other of money. The definition of an economic element depends on the object of the examination. In our study we shall understand ,,an economic element" to mean a place of production (i. e. factory etc.) and a consumption area served by it. The circulation of goods can be represented either for the whole system of economic elements or inside every element. We propose to study the second case.

I.

In order to find the numerical conditions for the equilibrium of the circulation of consumption goods in an economic element, we shall distinguish two parts of it:

a) Consumption. Every place of production has its own consumption area. Let us therefore denote by

$$A(t) \,.\, \mathrm{d}t$$

the number of units of consumption goods received from a certain place of production and consumed during the interval $(t, t + \mathrm{d}t)$, thus $A(t)$ is the rate of consumption.

b) Production. Let us denote by $l(t)$ the total production in the period $(0, t)$, i. e. the number of all units of consumption goods from the initial moment till the moment t.

For each of two moments $t_1 \leqq t_2$ we have $l(t_1) \leqq l(t_2)$. The function $l(t)$ need not be continuous, but if it has a derivation

$$l'(t) = \frac{\mathrm{d}l(t)}{\mathrm{d}t}$$

then always $0 \leqq l'(t)$. The function $l'(t)$ is, by definition, the rate of production.

The unit of consumption goods leaving the place of production is either immediately consumed or added to stock. Therefore we introduce the number $k(t)$, called the coefficient of immediate consumption, to denote that proportion of the quantity of consumption goods under production in the interval $(t, t + \mathrm{d}t)$, which is consumed in the same period. So $0 \leqq k(t) \leqq 1$ for every moment t and

$$\{1 - k(t)\} \, . \, \mathrm{d}l(t)$$

denotes the number of units of consumption goods put into stock during $(t, t + \mathrm{d}t)$. Let us call this „the stock corresponding to the interval $(t, t + \mathrm{d}t)$".

If we put

$$1 - k(t) = q(t),$$

then we can write this stock in the simple form

$$q(t) \, . \, \mathrm{d}l(t).$$

II.

Now we introduce the number $K(\xi, \tau) \, . \, \mathrm{d}\xi$ denoting the proportion of stock corresponding to the interval $(\tau, \tau + \mathrm{d}\tau)$ consumed during $(\xi, \xi + \mathrm{d}\xi)$.*) For the moments τ and ξ the relation $0 \leqq \tau \leqq \xi$ is valid.

For the function $K(\xi, \tau)$ we can establish two conditions:

1.

$$0 \leqq K(\xi, \tau)$$

for every ξ and τ.

2. From $q(\tau) \, . \, \mathrm{d}l(\tau)$ units of stock corresponding to $(\tau, \tau + \mathrm{d}\tau)$ there are

$$q(\tau) \, . \, \mathrm{d}l(\tau) \, . \, K(\xi, \tau) \, . \, \mathrm{d}\xi$$

units taken out during $(\xi, \xi + \mathrm{d}\xi)$ and therefore

*) The definition of the kernel $K(\xi, \tau)$ is based on the supposition that goods on stock are ear-marked. I agree with the criticism that this can be actually done only in few cases. But this gives a suggestion to new researches, in what manner can we bring the function K in touch with the capital theory.

$$q(\tau) \,.\, \mathrm{d}l(\tau) \,.\, \int_{\tau}^{t} K(\xi, \tau) \,.\, \mathrm{d}\xi$$

units taken out during the interval (τ, t) where $0 \leqq \tau \leqq \xi \leqq t$. But this quantity cannot be greater than the stock mentioned above, i. e.

$$q(\tau) \,.\, \mathrm{d}l(\tau) \,.\, \int_{\tau}^{t} K(\xi, \tau) \,.\, \mathrm{d}\xi \leqq q(\tau) \,.\, \mathrm{d}l(\tau).$$

Therefore for every τ and t the condition

$$\int_{\tau}^{t} K(\xi, \tau) \,.\, \mathrm{d}\xi \leqq 1$$

holds true.

In order to establish equilibrium between production on one side and consumption on the other it is necessary that the condition

$$A(t) \,.\, \mathrm{d}t = k(t) \,.\, \mathrm{d}l(t) + \mathrm{d}t \,.\, \int_{0}^{t} K(t, \tau) \,.\, \mathrm{d}l(\tau)$$

or

$$A(t) = k(t) \,.\, l'(t) + \int_{0}^{t} K(t, \tau) \,.\, l'(\tau) \,.\, \mathrm{d}\tau \qquad \text{(I)}$$

should hold true. We also assume that the beginning of production and of consumption occur generally at the same moment, which is without any restriction of our problem. The condition (I) is an equation for the rate of production. The functions $A(t)$ and $k(t)$ are supposed to be known, but $K(t, \tau)$ we do not usually know, and therefore our problem involves a further equation.

III.

Instead of the function $K(t, \tau)$ we know the function $H(t, \tau)$, which we introduce with this definition: $H(t, \tau) \,.\, \mathrm{d}t$ is the number denoting the proportion of stock corresponding to the interval $(\tau, \tau + \mathrm{d}\tau)$ and being left on hand during (τ, t), which is consumed during $(t, t + \mathrm{d}t)$.

By means of the function $H(t, \tau)$ we can analyse the structure of the stock on hand at the moment t. This stock is composed of the units of consumption goods arriving from different stocks $q(\tau) \,.\, \mathrm{d}l(\tau)$ corresponding to $(\tau, \tau + \mathrm{d}\tau)$. During (t, τ) there are not consumed

$$\{1 - \int_{\tau}^{t} K(\xi, \tau) \,.\, \mathrm{d}\xi\} \,.\, q(\tau) \,.\, \mathrm{d}l(\tau)$$

units of goods, and therefore this quantity is a part of the stock actually on hand at the moment t, from which in $(t, t + \mathrm{d}t)$

$$H(t, \tau) \,.\, \mathrm{d}t \,.\, \{1 - \int_{\tau}^{t} K(\xi, \tau) \,.\, \mathrm{d}\xi\} \,.\, q(\tau) \,.\, \mathrm{d}l(\tau)$$

units of goods are taken out. But by the definition of the function $K(t, \tau)$ this quantity must be equal to

$$K(t, \tau) \,.\, \mathrm{d}t \,.\, q(\tau) \,.\, \mathrm{d}l(\tau).$$

Therefore the relation

$$H(t, \tau) \,.\, \{1 - \int_{\tau}^{t} K(\xi, \tau) \,.\, \mathrm{d}\xi\} = K(t, \tau) \qquad \text{(II)}$$

should be true. This is the second equation which we need in our problem.

Form the definition of the function $H(t, \tau)$ we can easily establish these two of its properties:

1. For every t and τ is $0 \leqq H(t, \tau)$.
2. If T is the last moment of consumption (it can be $T = \infty$), then for every t

$$\int_{t}^{T} H(\sigma, t) \,.\, \mathrm{d}\sigma \leqq 1.$$

IV.

As a simple example let us put $A =$ constant, $k =$ constant $>$ > 0, $K =$ constant, $l(0) = 0$.

Then from (I) we get the differential equation

$$A = k \,.\, l'(t) + K \,.\, l(t),$$

which has the solution

$$l(t) = \frac{A}{K}\left(1 - e^{-\frac{K}{k}\cdot}\right).$$

From the equation (II) it follows

$$H(t, \tau) = \frac{K}{1 - K \,.\, (t - \tau)}.$$

From this we can establish that for $t \to \infty$ there is