ČASOPIS PRO PĚSTOVÁNÍ MATEMATIKY A FYSIKY

ČÁST MATEMATICKÁ

On a Problem of Čech.

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Prof. Čech has introduced the following definition of local connectedness:

Def. A topologic space is said to be locally connected provided every finite covering by open sets contains a finite covering by connected sets.

He has proposed to us the question whether such a space is necessarily bicompact. We shall show, ultimately by a counter-example, that the answer to the question is in the negative. We shall also give a slight discussion of this quite interesting idea. We begin with the following simple

Theorem: For a regular topologic space S (in the sense of Hausdorff), local connectedness in the sense of Čech (above) implies local connectedness in the usual sense. That is, given any point x of our space and any neighborhood $U_x \supset x$, there exists an open connected set V_x, $x \subset V_x \subset U_x$.

Proof. Let U_x be any neighborhood of the point x and W any open set containing x such that $\overline{W} \subset U_x$, where $\overline{W}$ denotes the closure of W. Such a set exists, by the *regularity* of space. Now the two sets, $O_1 = U_x$ and $O_2 = S - \overline{W}$ form a finite covering by open sets of the space S. That is trivial. But, by our hypothesis, there must exist a finite set $M_1, M_2, \ldots, M_n$ of *connected* sets such that each of them belongs to O_1 *or* to O_2 and such that every point of space belongs to at least one of them.

Let j be any integer, $1 \leq j \leq n$, such that $\overline{M}_j \supset x$. Since W is open and contains x, it is clear that $W . M_j$ is not vacuous. Therefore M_j cannot belong to $O_2 = S - \overline{W}$, and we have the inclusion $M_j \subset O_1 = U_x$. Let $M = \sum_j M_j$ for all values of j such that $\overline{M}_j \supset x$. It is clear that M is connected, that it contains x (since at least one M_j *must* contain x), and is contained in U_x. Further, if $N = \sum_k \overline{M}_k$ for all values k such that $M_k \not\ni x$, then $S - N$ is open, contains x, and belongs to M. This means that x is an inner point of M. But now if we denote by M^* the component (i. e. the maximal connected subset) of U_x which contains x, the considerations above show that *every* point of this set is an inner point. Then we may take $V_x = M^*$ and our theorem is established.

*Theorem: If a regular topologic space S is locally connected in the sense of Čech, then it is compact.**)

Proof: Suppose there exists, in the space S, an infinite sequence $x_1, x_2, \ldots$, of points such that the set $X = \sum_1^\infty x_n$ has no limit point. Then X is closed, and $S - X$ is open. Since, in particular, no point x_n is a *limit point* of X it follows that $X - x_n$ is closed, and therefore by the *regularity* of space there exists, for every n, a neighborhood $U_n \supset x_n$ such that $\overline{U}_n \, . \, (X - x_n) = 0$. Let us write $V_1 = U_1$. There exists in U_2 a neighborhood V_2 of x_2 such that $V_1 \, . \, \overline{V}_2 = 0$, *otherwise* x_2 would be a limit point of U_1 which it is *not*, by construction. Similarly, if $V_1, V_2, \ldots, V_{n-1}$ have been defined, let V_n be a neighborhood of x_n, $x_n \subset V_n \subset U_n$, such that the intersection of V_n with $\sum_1^{n-1} \overline{U}_i$ is vacuous. It is clear that such a V_n exists because x_n is not a point of $\overline{U}_i$, for any $i \neq n$, and therefore not a point of any finite sum (necessarily closed) of these sets. At last, we take $O_1 = \sum_1^\infty V_n$, and take $O_2 = S - X$. This is a finite covering by open sets. Since $V_i \, . \, V_n = 0$, if i is fixed and $n \neq i$, it follows that a *connected* subset of O_1 containing x_i cannot contain any other point x_n. But this is true for every i, so that no connected subset of O_1 can contain as many as two points of X. No subset of O_2 contains *any* point of X. It is now trivial that O_1 and O_2 contain *no* finite covering by connected sets. This contradiction establishes the compactness.

We come now to the most interesting, perhaps, of these observations.

Theorem: If a compact topologic space S^* (not necessarily regular) is locally connected in the usual sense then it is locally connected in the Čech sense.

Proof. Let $U_1, U_2, \ldots, U_n$ be any finite covering of S^* by

*) This result was known to Prof. Čech.

open sets. For each point x and each U_i, let $C^i(x)$ denote the component of U_i containing x. Of the components $C^1(x)$, $x \subset S^*$, *let us retain those only which are not covered by* $\sum_2^n U_i$. Suppose that there are *infinitely* many *distinct* components of this sort and let C^1_1, $C^1_2, \ldots, C^1_n, \ldots$, denote some such infinite sequence. Then each C^1_n contains at least one point x_n such that x_n belongs to no U_i, $i \neq 1$. The set $X = \sum_1^\infty x_n$ has at least one limit point x, by the compactness of space. Now $x \not\subset U_i$, $i \neq 1$, for otherwise at least one $x_n \subset U_i$ because these sets are open. Therefore $x \subset U_1$. But $C^1(x)$ is open, from the local connectedness of space. Therefore at least *two* distinct points x_i and x_j belong to $C^1(x)$. Then $C^1(x_i)$ and $C^1(x_j)$ cannot be distinct. The contradiction shows that there exists a finite set of components of U_1, call them $K_1, K_2, \ldots, K_m$, such that together with $U_2, U_3, \ldots, U_n$ they form a finite covering by open sets of the space S^*. But now if we consider the components of U_2 we see, by the very argument above, that there must exist a finite set of these, call them $K_{m+1}, \ldots, K_{m'}$, such that:

$$K_1, K_2, \ldots, K_m, K_{m+1}, \ldots, K_{m'}, U_3, \ldots, U_n$$

is a finite covering of the space. It is clear that we can now replace U_3 by a finite set of components, enlarging the number of connected open sets, perhaps, but *certainly* diminishing the number that are not connected. In a finite number of steps we obtain a finite covering, $K_1, \ldots, K_{m'}, \ldots, K_N$, by open *connected* sets such that each K_i by its construction belongs to some U_j. This completes the proof.

We see now that any topologic space which is compact and locally connected, in the usual sense, but *not bicompact* furnishes a negative solution to the question proposed by Čech. As the simplest of such spaces, in a sense, we may recall the space S which consists of a set of points in (1 — 1) correspondance with all ordinal numbers of the first and second class such that between any two *consecutive* members of this class there is interpolated a „linear“ segment. That is, each point of S corresponds uniquely to a coordinate (τ, t) where τ is a number of the first or second ordinal class and $0 \leq t < 1$. The points are *linearly* ordered by the convention that (τ, t) *precedes* (τ', t') if $\tau < \tau'$ *or* if $\tau = \tau'$ and $t < t'$. A generic *open* set is the set of points *between* two distinct points, not including these.

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