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MEAN VALUE THEOREM AND LAGRANGE SETS OF REAL FUNCTIONS

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0. Introduction

Let f: $[a, b] \to R$ (R — the real line, $[a, b] \subset R$) be a continuous function on the interval $[a, b]$ and differentiable on the interval (a, b). Then, according to Lagrange's mean value theorem ([13], p. 374, Theorem 3) there exists $\xi \in (a, b)$, such that

$$f'(\xi) = (f(b) - f(a))/(b - a).$$

In the connection with this fact a question can be raised to investigate some properties of the set $L(f)$ of all such reals $\xi \in (a, b)$, that

$$f'(\xi) = (f(y) - f(x))/(y - x) \tag{1}$$

where $a \leq x < \xi < y \leq b$. These reals will be called the Lagrange numbers of the function f and $L(f)$ will be called the Lagrange set of f. Analogously, let $L^*(f)$ be the set of all such $\xi \in (a, b)$, that

$$f'(\xi) = (f(y) - f(x))/(y - x) \tag{2}$$

where $a \leq x < y \leq b$ (without the assumption $x < \xi < y$). These reals will be called the generalized Lagrange numbers of f and $L^*(f)$ will be called the generalized Lagrange set of f. Obviously $L(f) \subset L^*(f)$.

The notion of Lagrange's numbers of the function f has been motivated by Problem 4 of [8], p. 323, according to which there are numbers $\xi \in (a, b)$ such that the expression of $f'(\xi)$ in the form (1) ($a \leq x < \xi < y \leq b$), or in the form (2) ($a \leq x < y \leq b$), is impossible. Put $a = -1$, $b = 1$, $f(x) = x^3$ for $x \in [-1, 1]$. Then $f'(x) = 3x^2$, and $f'(0) = 0$. Since the function f is increasing on the interval $[-1, 1]$, each ratio of the form $(f(y) - f(x))/(y - x)$, $a \leq x < y \leq b$, is positive, and we have $0 \notin L^*(f)$ (see [8], p. 323).

The present paper is devoted to the investigation of some properties of the sets $L(f)$, $L^*(f)$ for certain classes of functions. The first part deals with the basic properties of these sets. We will prove that $L(f) = (a, b)$ whenever f' is monotone. Some sufficient conditions will be given such than $\operatorname{Int} L(f) \neq \emptyset$ and $\operatorname{Int} L^*(f) \neq \emptyset$ Int A is the interior of A).

Further a class of functions f will be introduced, for which $L^*(f)$ is a residual subset of (a, b). The second part of the paper is devoted to the study of properties of sets $L(f)$ for convex functions.

1. Basic properties of sets $L(f)$ and $L^*(f)$

First we show that the introduced inclusion $L(f) \subset L^*(f)$ which holds for every function f continuous on $[a, b]$ and differentiable on (a, b) can be strict.

Example 1.1. Put $f(x) = x^2 - x^3$ for $x \in [-1, 0)$ and $f(x) = x^3 - x^2$ for $x \in [0,1]$. It is easy to see that $0 \in L^*(f) - L(f)$.

We shall frequently use the following simple result concerning the sets $L(f)$.

Proposition 1.1. Let f: $[a, b] \to R$ be a continuous function on $[a, b]$ and differentiable on (a, b). Then $L(f)$ is a dense set in (a, b).

Corollary 1.1. Let the assumptions of Proposition 1.1 be fulfilled. Then $L^*(f)$ is a dense set in (a, b).

Proof of Proposition 1.1. Let $(c, d) \subset (a, b)$. According to Lagrange's theorem there exists $\xi \in (c, d)$ such that

$$f'(\xi) = (f(d) - f(c))/(d - c).$$

Hence $\xi \in L(f)$, $\xi \in (c, d)$ and $(c, d) \cap L(f) \neq \emptyset$.

Let f: $I \to R$ (I is an interval) be a continuous function. In the paper [3] lower and upper bounds of the set of numbers of the form

$$(f(y) - f(x))/(y - x),\ x, y \in I,\ x < y$$

and lower and upper bounds of the Dini derivatives of the function f on I are investigated. The following result (Lemma 1.1) concerns this theme. Put

$$E = \{(x, y) \in R_2\colon a \leqq x < y \leqq b\},$$

$$H_f((x, y)) = H_f(x, y) = (f(y) - f(x))/(y - x)$$

for $(x, y) \in E$.

Note, that the function H_f: $E \to R$ is continuous on E. Since E is connected, $H_f(E)$ is an interval (it can be degenerated).

The following lemma is an easy consequence of the Lagrange's mean value theorem.

Lemma 1.1. Let the function f: $[a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b). Then

$$f'(L(f)) = H_f(E). \tag{3}$$

If f' is a continuous function on (a, b), then the set $L(f)$ is measurable with a simple topological structure. This is shown by the following theorem.

Theorem 1.1. Let a function f: $[a, b] \to R$ be continuous on $[a, b]$ and let f' be continuous on (a, b). Then the set $L(f)$ is an F_σ-set in (a, b).

Proof. Put for every $n \in N = \{1, 2, \ldots\}$ $F_n = \{\xi \in (a, b)$: $\exists x, y \in [a, b], x < \xi < < y, \xi - x \geq 1/n, y - \xi \geqq 1/n, f'(\xi) = (f(y) - f(x)/(y - x)\}$.

It is easy to verify that

$$L(f) = \bigcup_{n=1}^{\infty} F_n, \tag{4}$$

and it is sufficient to show that each of sets F_n, $n \in N$, is closed in (a, b). But it can be shown in the standard manner.

Analogously we can verify the following statement

Theorem 1.2. Let a function f: $[a, b] \to R$ be continuous on $[a, b]$ and let f' be continuous on (a, b). Then the set $L^*(f)$ is an F_σ-set in (a, b).

In the introduction it was shown that $L(f)$ can be a proper subset of the interval (a, b). The following statement gives simple conditions to the equality $L(f) = (a, b)$.

Theorem 1.3. Let f: $[a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b). Let f' be monotone on (a, b). Then:

If f' is strictly increasing on (a, b), then for each $c \in (a, b)$, one of following is true:

a) $$\frac{f(b) - f(a)}{b - a} = f'(c)$$

b) $$\frac{f(b) - f(a)}{b - a} > f'(c)$$

in which case there exists a real number β, $b > \beta > c > a$ for which

$$\frac{f(\beta) - f(a)}{\beta - a} = f'(c)$$

c) $$\frac{f(b) - f(a)}{b - a} < f'(c)$$

in which case there exists a real number α, $a < \alpha < c < b$ for which

$$\frac{f(b) - f(\alpha)}{b - \alpha} = f'(c)$$

Proof. (a) There exist points c in (a, b) for which $\dfrac{f(b) - f(a)}{b - a} = f'(c)$, by the Mean Value Theorem.

(b) Since f' is strictly increasing on (a, b), f is convex on (a, b). The equation of the line $T(x)$ which intersects the point $(a, f(a))$ and which runs parallel to the tangent line at $(c, f(c))$ is given by

$$T(x) = f'(c)(x - a) + f(a) \tag{5}$$

(see *picutre 1*). By the convexity of f on (a, b), the line $T(x)$ must intersect the graph of f at a unique point $(\beta, f(\beta))$, where $c < \beta < b$. Thus we have from (5)

$$T(\beta) = f(\beta) = f'(c)(\beta - a) + f(a)$$

hence

$$\frac{f(\beta) - f(a)}{\beta - a} = f'(c)$$

(c) We proceed analogously as in (b).

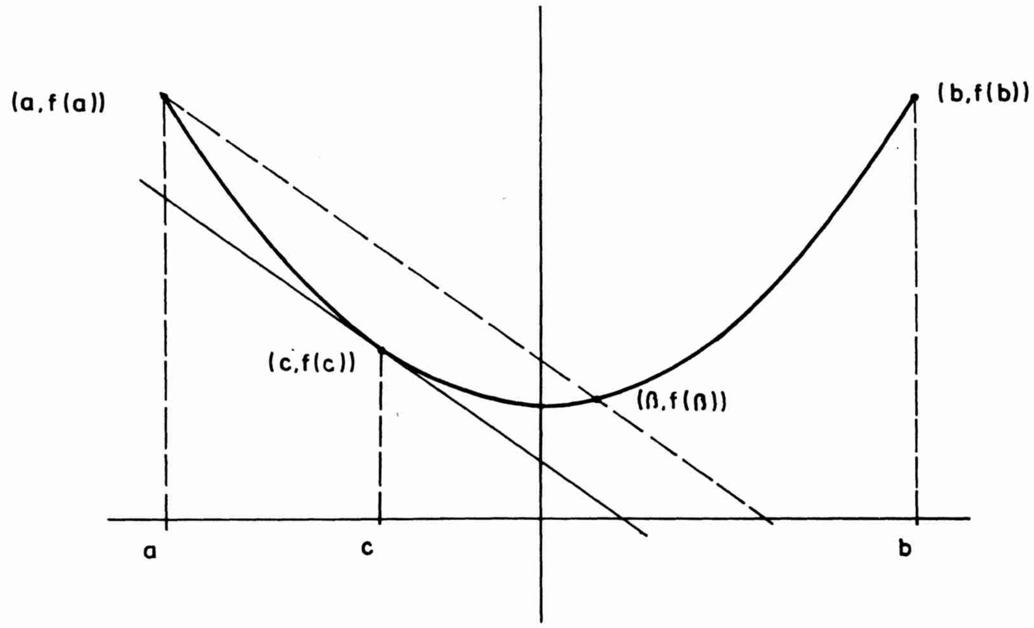

Fig. 1

An analogous result to (a), (b), (c) above applies symmetrically for f' strictly decreasing on (a, b), i.e. f concave.

Theorem 1.3 has an immediate consequence:

Theorem 1.4 Let a function f: $[a, b] \to R$ be continuous on $[a, b]$ and differentiable on (a, b). Let the function f' be strictly monotonic on (a, b). Then $L(f) = (a, b)$.

Corollary 1.2 (Lagrange sets for integrals) Let a function f: $[a, b] \to R$ be continuous on $[a, b]$ and either monotonic on (a, b) or constant on (a, b). Let $F(x) = \int_a^x f(t)\,dt$, $a \leqq x \leqq b$. Then $L(F) = (a, b)$.

Theorem 1.4 cannot be conversed. This is shown by the following example: Let $f(x) = x^3$ for $x \in [-2, 5]$ and $f(x) = x^2 + 16x + 20$ for $x \in [-15, -2)$. Thus $f'(x) = 3x^2$ for $x \in [-2, 5)$ and $f(x) = 2x + 16$ for $x \in (-15, -2)$. Note that f' is strictly increasing on $(-15, -2)$, strictly decreasing on $(-2, 0)$, and strictly increasing on $(0, 5)$. Note further that

$$f'(-2) = 12 = (f(5) - f(-9))/14$$

and

$$f'(0) = 0 = (f(\sqrt[3]{5}) - f(-15))/(\sqrt[3]{5} + 15)$$

According to these equalities and Theorem 1.4 we have $L(f) = (-15, 5)$. But f' is evidently neither strictly monotonic not constant on $(-15, 5)$.

We showed (Proposition 1.1) that the set $L(f)$ is dense in (a, b). In connection with this fact the question arises how "rich" is the set $(a, b) - L^*(f)$. The following theorem shows that this set can be extremely rich from a topological point of view.

Theorem 1.5. There exists such a function g: $[a, b] \to R$, continuous on $[a, b]$ and differentiable on (a, b), that the set $(a, b) - L^*(g)$ is residual in (a, b) (i.e. its complement $L^*(g)$ is a set of the first Baire category in (a, b)).

Corollary 1.3. There is such a function g that each of the sets $L(g)$, $L^*(g)$ is a first Baire category set in (a, b).

Proof of Theorem 1.5. We will use a construction due to S. Marcus (see [1], the proof of Theorem 8, [2], p. 33).

Let $A_n > 0$ $(n = 1, 2, \ldots)$ and $\sum_{n=1}^{\infty} A_n < +\infty$. Let $\alpha, \beta, \in R$, $\alpha < \beta$ and let $\{a_1, a_2, \ldots, a_n\}$ be a countable set dense in $[\alpha, \beta]$. Put for $x \in [\alpha, \beta]$

$$F(x) = \sum_{n=1}^{\infty} A_n (x - a_n)^{1/3} \tag{6}$$

Then F is clearly continuous and strictly increasing on the interval $[\alpha, \beta]$.

In the paper [11] it is proved:

a) The function F has the finite positive derivative at each point $x \in [\alpha, \beta]$, $x \neq a_n$ $(n = 1, 2, \ldots)$, whenever the series

$$\sum_{n=1}^{\infty} A_n/(x - a_n)^{2/3} \tag{7}$$

is convergent.

b) The equality $F'(x) = +\infty$ holds at each point $x \in [\alpha, \beta]$, $x \neq a_n$ $(n = 1, 2, \ldots)$, whenever the series (7) is divergent.

c) $F'(a_n) = +\infty$ holds for every $n = 1, 2, \ldots$.

Since the function F is strictly increasing and continuous on $[\alpha, \beta]$, (see (6)) there exists $G = F^{-1}$ (the inverse function of F) defined on the interval $[A, B]$, where $A = F(\alpha)$, $B = F(\beta)$. The function G is continuous and strictly increasing on $[A, B]$.

The properties a) – c) of F' and the theorem on the derivative of an inverse function imply that the function G has a finite derivative at each $t \in (A, B)$ and that $G'(t_n) = 0$ holds for each $t_n = F(a_n)$ $(n = 1, 2, \ldots)$, where the set $\{t_1, t_2, \ldots, t_n, \ldots\}$ is dense in $[A, B]$. Since G is a strictly increasing function, the set of all such points t at which $G'(t) > 0$ is also dense in $[A, B]$.

Hence we have constructed such a function on the interval $[A, B]$, which is continuous, strictly increasing and differentiable on (A, B) and the set $\{t: G'(t) = 0\}$ and its complement are dense sets in (A, B).

Let φ: $[a, b] \to [A, B]$ be a homeomorphic map of the form

$$\varphi(t) = (t - a).(B - a)/(b - a) + A.$$

Define the function g: $[a,, b] \to R$ by $g(t) = G(\varphi(t))$.

It is easy to verify that the function g is strictly increasing and continuous on $[a, b]$, differentiable on (a, b), and both the set $Z_{g'} = \{x \in (a, b): g'(x) = 0\}$ and its complement are dense sets in (a, b).

The function g is strictly increasing on $[a, b]$ and every number of the form $(g(y) - g(x))/(y - x)$, $a \leq x < y \leq b$, is positive, hence

$$Z_{g'} \subset (a, b) - L^*(g). \tag{8}$$

Since g' is a Baire one function ([4], [10], [14], p. 81) $Z_{g'} = g'^{-1}(\{0\})$ is a G_δ-set in (a, b), hence $Z_{g'}$ is residual in (a, b) ([9], p. 49). According to (8) $(a, b) - L^*(g)$ is a residual set in (a, b).

Theorem 1.5 shows that the interior of the set $L^*(g)$ is void. Further some sufficient conditions, concerning f', will be given, such that $\operatorname{Int} L(f) \neq \emptyset$ (or $\operatorname{Int} L^*(f) \neq \emptyset$).

Recall the notion of quasicontinuity and somewhat continuity of a function (see [7], [5]).

Let X and Y be topological spaces. A function g: $X \to Y$ is said to be quasicontinuous at $x \in X$ if for each neighbourhood $V = V(g(x))$ of $g(x) \in Y$ and each neighbourhood $U = U(x)$ of $x \in X$ there is a non-void open set $G \subset U$ such that $g(G) \subset V$. A function g is said to be quasicontinuous on X, if it is quasicontinuous at each $x \in X$.

A function g: $X \to Y$ is said to be somewhat continuous (on X), if for any open set $G \subset Y$ the condition $g^{-1}(G) \neq \emptyset$ implies $\operatorname{Int} g^{-1}(G) \neq \emptyset$.

The quasicontinuity of g: $X \to Y$ implies its somewhatcontinuity. The inverse statement does not hold. If X is a real interval and $Y = R$, then there is a somewhat continuous function g: $X \to R$ that g is quasicontinuous at no point $x \in X$ (see [15]).

Theorem 1.6. Let a function f: $[a, b] \to R$ be continuous on $[a, b]$ and let f' be quasicontinuous on (a, b). Then $\operatorname{Int} L(f) \neq \emptyset$.

Proof. If there is an open interval $I \subset (a, b)$ such that f' is constant on I, then $L(f) \supset I$ and the statement holds.

Further we shall suppose that f' is constant at no open interval.

Choose $\xi_0 \in L(f)$. It follows from the definition of $L(f)$ that there are x_0, y_0, $a \leq x_0 < \xi_0 < y_0 \leq b$, such that

$$f'(\xi_0) = (f(y_0 - f(x_0))/(y_0 - x_0).$$

We show that there are numbers x, y such that

$$x_0 < x < \xi_0 < y < y_0 \tag{9}$$

and

$$f'(\xi_0) \neq (f(y) - f(x))/(y - x).$$

Suppose, by the contradiction, that for every x and y fulfilling (9) we have $f'(\xi_0) = (f(y) - f(x))/(y - x)$.

Fix $y_1 \in (\xi_0, y_0)$. Then for each $x \in (x_0, \xi_0)$ we have

$$f'(\xi_0) = (f(y_1) - f(x))/(y_1 - x),$$

hence

$$f(x) = f(y_1) - f'(\xi_0)(y_1 - x).$$

The differentiation with respect to x gives $f'(x) = f'(\xi_0)$ for each $x \in (x_0, \xi_0)$. This is a contradiction with the assumption, that f' is constant at no open interval. Hence it is possible to choose x and y such that the inequalities (9) and

$$f'(\xi_0) \neq (f(y) - f(x))/(y - x)$$

are fulfilled. Then, according to the Lagrange theorem for a suitable $\xi_1 \in (x, y)$, we have

$$(f(y) - f(x))/(y - x) = f'(\xi_1).$$

Put $B = [x_0, x] \times [y, y_0]$. The set $B \subset E$ is compact and connected, and the function H_f is continuous on B. Therefore $H_f(B)$ is a bounded closed interval containing different points $f'(\xi_0)$ and $f'(\xi_1)$.

Suppose $f'(\xi_0) < f'(\xi_1)$ (if $f'(\xi_1) < f'(\xi_0)$ the proof is analogical). Since the function f' has the Darboux property, it gains on the interval J (with endpoints ξ_0 and ξ_1) all values between $f'(\xi_0)$ and $f'(\xi_1)$. Choose $c \in (f'(\xi_0), f'(\xi_1))$. Then there is a $d \in J$ such that $f'(d) = c$ and obviously $f'(d) \in \operatorname{Int} H_f(B)$.

Since f' is quasicontinuous at d, there exists a non-void open set $U \subset J \subset (x, y)$ such that $f'(U) \subset \operatorname{Int} H_f(B)$. For each $z \in U$ there are numbers s and t, $s \in [x_0, x]$, $t \in [y, y_0]$, such that $f'(z) = (f(t) - f(s))/(t - s)$. Hence we have $s \leqq x < z < y \leqq t$ and $z \in L(f)$. Therefore $U \subset L(f)$ and the theorem is proved.

The next theorem is a simple consequence of the above theorem.

Theorem 1.7. Let a function f: $[a, b] \to R$ be continuous and let f' be quasicontinuous on (a, b). Then the set $(a, b) - L(f)$ is nowhere dense in (a, b).

Proof. Let I be an open interval, $I \subset (a, b)$. Then f' is quasicontinuous on I. It follows from Theorem 1.6. that there is an open interval $I' \subset I$ such that $I' \subset \operatorname{Int} L(f)$.

The statement is an easy consequence of a well-known characterization of nowhere dense sets ([9], p. 37).

Corollary 1.4. Let a function f: $[a, b] \to R$ be continuous on $[a, b]$ and let f' be (R) — integrable on every closed interval $J \subset (a, b)$. Then the set $(a, b) - L(f)$ is nowhere dense in (a, b).

Proof. The integrability of f' on every closed subinterval of (a, b) implies that f' is quasicontinuous on (a, b) ([12]). The statement follows from Theorem 1.7.

In connection with Theorem 1.7 we shall construct a function f continuous on $[a, b]$, differentiable on (a, b) and such, that the set $(a, b) - L^*(f)$ is both nowhere dense and infinite.

Example 1.2. Choose $\alpha, \beta \in \mathrm{R}$, $\alpha < \beta$. Let $A_n > 0$ $(n = 1, 2, \ldots)$, $\sum_{n=1}^{\infty} A_n < +\infty$. Let $a_k \in (\alpha, \beta)$ $(k = 1, 2, \ldots)$, $a_1 > a_2 \ldots > a_k > \ldots$, $a_k \to \alpha$. Put

$$F(x) = \sum_{n=1}^{\infty} A_n (x - a_n)^{1/3}$$

for $x \in [\alpha, \beta]$. Using the method of the proof of Theorem 1.5 we can verify that the function F is strictly increasing and continuous on $[\alpha, \beta]$, further F has at every $x \neq a_n$ $(n = 1, 2, \ldots)$ a finite derivative $F'(x)$ and

$$F'(x) = 3^{-1} \cdot \sum_{n=1}^{\infty} A_n / (x - a_n)^{2/3}$$

and $F'(a_n) = +\infty$ $(n = 1, 2, \ldots)$.

Put $A = F(\alpha)$, $B = F(\beta)$. Then $A < B$. If $G = F^{-1}$, then G is strictly increasing and continuous on the interval $[A, B]$. Put $t_n = F(a_n)$ $(n = 1, 2, \ldots)$. Then G has a finite positive derivative at each $y \in (A, B)$, $y \neq t_n$ $(n = 1, 2, \ldots)$ and $G'(t_n) = 0$ $(n = 1, 2, \ldots)$. Obviously $t_n \to A$, $t_1 > t_2 > \ldots > t_n > \ldots$.

Note that for $x \in (A, B)$, $x \neq t_n$ $(n = 1, 2, \ldots)$ we have

$$|x - a_n|^{2/3} \leqq (\beta - \alpha)^{2/3} \qquad (n = 1, 2, \ldots)$$

and

$$F'(x) \geqq 3^{-1}(\beta - \alpha)^{-2/3} . \sum_{n=1}^{\infty} A_n = D > 0$$

This implies, using the theorem on the derivative of an inverse function, that for $t \in (A, B)$, $t \neq t_n$ $(n = 1, 2, \ldots)$ we have $G'(t) \leqq 1/D < +\infty$. Since $G'(t_n) = 0$ $(n = 1, 2, \ldots)$, G' is bounded on (A, B).

We prove that G' is continuous at each $t \neq t_n$ $(n = 1, 2, \ldots)$. Let $t_0 \neq t_n$ $(n = 1, 2, \ldots)$ and $t_0 = F(x_0)$. Since $a_1 > a_2 > \ldots$, $a_n \to \alpha$, there exists $\eta > 0$ such that $|a_k - x_0| \geqq 2\eta$ for every $k = 1, 2, \ldots$. Then the series

$$3^{-1} \sum_{n=1}^{\infty} A_n / (x - a_n)^{2/3} \qquad (= F'(x))$$

converges uniformly on the interval $I = (x_0 - \eta, x_0 + \eta)$ (this follows from the the Weierstrass M-test) and its sum, i.e. the function F' is continuous at x_0. This implies the continuity of G' at t_0. Put

$$\varphi(t) = ((t - a)/(b - a)).(B - A) + A \qquad (t \in [a, b])$$

and define $g(t) = G(\varphi(t))$ $(t \in [a, b])$.

It is easy to verify that the function g is continuous and strictly increasing on the interval $[a, b]$, that the function g' is continuous on (a, b), perhaps with the exception of points $\varphi(t_n)$ $(n = 1, 2, \ldots)$, and that g' is bounded on (a, b). This implies (R) — integrability of g' on $[a, b]$ and the set $(a, b) - L(g)$ is nowhere dense in (a, b) (see Corollary 1).

For points $u_n = \varphi^{-1}(t_n)$ $(n = 1, 2, \ldots)$ we have $u_n \in [a, b]$, $u_1 > u_2 > \ldots$, $u_n \to a$ and $g(u_n) = G(t_n) = 0$ $(n = 1, 2, \ldots)$. Since g is strictly increasing, we have $u_n \notin L^*(g)$. Hence $(a, b) - L^*(g)$ contains points u_n $(n = 1, 2, \ldots)$ and $(a, b) - - L^*(g)$ is an infinite nowhere dense subset of (a, b).

The non-voidness of Int $L^*(f)$ can be proved under a weaker condition than the quasicontinuity of f'. This is shown in the next theorem.

Theorem 1.8. Let a function f: $[a, b] \to R$ be continuous on $[a, b]$ and let f' be somewhat continuous on (a, b). Then Int $L^*(f) \neq \emptyset$.

Proof. If $H_f(E) = \{A\}$, then for a suitable $B \in R$ we have $f(t) = At + B$ for $t \in [a, b]$. Obviously $L^*(f) = (a, b)$ and the statement is in this case valid.

Let $H_f(E)$ be a non-degenerate interval. Then $\operatorname{Int} H_f(E) \neq \emptyset$. Choose $x_1 < y_1$ such that

$$(f(y_1) - f(x_1)) / (y_1 - x_1) \in \operatorname{Int} H_f(E)$$

Then there exists a $\xi_1 \in (x_1, y_1)$ such that

$$f'(\xi_1) = (f(y_1) - f(x_1)) / (y_1 - x_1) \in \operatorname{Int} H_f(E)$$

i.e. f'^{-1} $(\operatorname{Int} H_f(E))$ is a non-void set. It follows from somewhat continuity of f' that there is an open interval J_1 $(J_1 \neq \emptyset)$ such that $J_1 \subset f'^{-1}$ $(\operatorname{Int} H_f(E))$. From this we have $J_1 \subset L^*(f)$ and the theorem follows.

Remark 1.1 We proved Theorem 1.7 as a simple consequence of Theorem 1.6. One can conjecture that using Theorem 1.8 the following statement can be proved:

(V) If a function f: $[a, b] \to R$ is continuous on $[a, b]$ and f' is somewhat continuous on (a, b), then the set $(a, b) - L^*(f)$ is nowhere dense in (a, b).

We shall show that this idea is false. Indeed, the quasicontinuity of f' on (a, b) implies quasicontinuity of f' on every open interval $I \subset (a, b)$. This property is not implied by somewhat continuity. We shall show it in the following example.

Example 1.3 We shall construct a derivative which is somewhat continuous on $(-1, 1)$ and it is not somewhat continuous on $(0, 1)$.

Let H be the function constructed in the paper [6]. For every $x \in (0, 1)$ we have $-1 < H'(x) < 1$ and each of the sets $\{x \in (0, 1)$: $H'(x) > 0\}$, $\{x \in (0, 1)$: $H'(x) < 0\}$ is dense in $(0, 1)$. We can also suppose that $H(0) = 0$ and $H'(0) = 0$. We now construct a function f on the interval $[-1, 0]$ in such a way that it is continuous, $f(-1) = 1$, $f(-1/2) = 0$, $f(0) = 0$ and $f(-1/4) = -1$ and it is linear on each of the intervals $[-1, -1/2]$, $[-1/2, -1/4]$ and $[-1/4, 0]$. Put $F(x) = \int_{-1}^{x} f(t)\,dt$ for $x \in (-1, 0]$. Then obviously $F(0) = 0$, $F'(0) = 0$ $(= f(0))$.

Define the function G: $(-1, 1) \to R$ in the following way: $G(x) = F(x)$ for $-1 < x \leqq 0$, $G(x) = H(x)$ for $0 < x < 1$. Obviously $G(0) = F(0) = 0$ and $g(x) = G'(x) = f(x)$ for $-1 < x \leqq 0$, $g(x) = G'(x) = H'(x)$ for $0 < x < 1$ and $-1 \leqq g(x) \leqq 1$ for every $x \in (-1, 1)$. The function g is continuous on $(-1, 0]$ and it takes on every value from the interval $[-1, 1]$. Consequently the function $g = G'$ is somewhat continuous on $(-1, 1)$. But it is not somewhat continuous on $(0, 1)$. This follows from the fact that $g|(0, 1) = H'$, $H'^{-1}((0, 1)) \neq \emptyset$ and $\operatorname{Int} H'^{-1}((0, 1)) = \emptyset$ because the set $\{x \in (0, 1)$: $H'(x) < 0\}$ is dense in $(0, 1)$.

Using Theorem 1.8 we can prove the following result, which is weaker than the statement (V).

Theorem 1.9 Let the function f: $[a, b] \to R$ be continuous on $[a, b]$ and let

$f|I$ be somewhat continuous for every open interval $I \subset (a, b)$. Then the set $(a, b) - L^*(f)$ is nowhere dense in (a, b).

Proof of Theorem 1.9 is analogical to the proof of Theorem 1.7 therefore can be omitted.

In what follows we shall give another condition for $\operatorname{Int} L^*(f) \neq \emptyset$.

Theorem 1.10 Let a function f: $[a, b] \to R$ be continuous on $[a, b]$ and let f' be continuous almost everywhere in (a, b). Then $\operatorname{Int} L^*(f) \neq \emptyset$.

Proof. If $H_f(E)$ is a singleton, then $L^*(f) = (a, b)$ and the assertion holds (see the proof of Theorem 1.8).

Further we can suppose that $H_f(E)$ is a non-degenerate interval. Let u, $v \in H_f(E)$, $u < v$, hence $(u, v) \subset H_f(E)$. Put $B = \{x \in (a, b)\colon u < f'(x) < v\} = f'^{-1}((u, v))$. It follows from Lemma 1.1 that $f'(L(f)) = H_f(E)$ and

$$(u, v) \subset H_f(E) = f'(L(f)) \tag{10}$$

Choose $t_0 \in (u, v)$. Then (see (10)) there exists $\xi_0 \in L(f)$ such that $t_0 = f'(\xi_0)$. Hence $\xi_0 \in f'^{-1}((u, v)), f'^{-1}((u, v)) \neq \emptyset$. It follows from the Denjoy's property of the derivative (see [4]) that $\lambda(B) > 0$ (λ — the Lebesgue measure on the real line). According to the assumption, there is a continuity point x_0 of f' such that $x_0 \in B$. Hence $f'(x_0) \in (u, v)$ and f' is continuous at x_0. Consequently there exists a neighbourhood U of x_0 such that $f'(U) \subset (u, v) \subset H_f(E)$ and for each $\xi \in U$ the number $f'(\xi)$ has the form

$$f'(\xi) = (f(y) - f(x)) / (y - x), \qquad a \leqq x < y \leqq b$$

So we have $U \subset L^*(f)$ and $\operatorname{Int} L^*(f) \neq \emptyset$.

From Theorem 1.10 we get

Theorem 1.11 Let f: $[a, b] \to R$ be continuous on $[a, b]$ and let f' be almost everywhere continuous in (a, b). Then the set $(a, b) - L^*(f)$ is nowhere dense in (a, b).

2. Lagrange's sets of convex functions

In this part of the paper let (a, b) stand for an open real interval, $-\infty \leq a < b \leq +\infty$. In the introduction Lagrange's sets have been defined for functions continuous on the compact interval $[a, b]$ and differentiable on (a, b). This definition can be formulated, in a natural way, also for functions defined on (a, b), $-\infty \leq a < b \leq +\infty$. Note, that for such functions, the assertion of Proposition 1.1 concerning of the density of the set $L(f)$ in (a, b), is valid (and $L(f) \subset L^*(f)$).

Recall some basic notions of the theory of convex functions.

Definition 2.1 A function g: $(a, b) \to R$ is said to be strictly convex on (a, b), if for every two points x, $y \in (a, b)$, $x < y$,

$$g(sx + (1 - s)y) < sg(x) + (1 - s)g(y)$$

holds for each s, $0 < s < 1$ (see [1], p. 57).

A strictly concave function is defined analogously.

Remark 2.1 Elementary properties of strictly convex functions are given in [1], pp. 54–69. In the sequel we shall use the following characterizations of strictly convex functions

a) A function g is strictly convex on (a, b) if and only if the function $\varphi(x, y) = (g(y) - g(x))/(y - x)$ $(x \neq y)$ is strictly increasing at each of its arguments x and y,

b) If g is differentiable on (a, b), then g is strictly convex on (a, b) if and only if g' is strictly increasing on (a, b) (and, since g' has the Darboux property, it is continuous on (a, b)). Moreover, the convexity of g implies its continuity and continuity of every function

$$\varphi_y(x) = \varphi(x, y), \ \varphi_x(y) = \varphi(x, y)$$

c) If the second derivative of a function g is finite on (a, b), then g is strictly convex (concave) on (a, b) if and only if $g'' \geq 0$ on (a, b) and the set $\{x \in (a, b)$: $g''(x) > 0\}$ is dense in (a, b) (if $g'' \leq 0$ on (a, b) and the set $\{x \in (a, b)$: $g''x) < 0\}$ is dense in (a, b)).

The following statement is a consequence of Theorem 1.4.

Theorem 2.1. Let f be strictly concave differentiable on (a, b). Then $L(f) = (a, b)$.

Corollary 2.1. Let f be strictly concave differentiable on (a, b). Then $L(f) = (a, b)$.

Proof. It is sufficient to apply Theorem 2.1. to the function $g = -f$.

Corollary 2.2. Let P be a polynomial. Then the set $L(P)$ contains all but a finite number of points from (a, b).

Proof. If P is a linear or quadratic polynomial, then obviously $L(P) = (a, b)$. Suppose $\deg(P) > 2$. Then P'' is a polynomial of degree at least one. Hence the set

$K = \{x \in (a, b)$: $P''(x) = 0\}$ is finite. Obviously

$$(a, b) - K = \bigcup_{j=1}^{m} I_j. \tag{11}$$

where I_j $(j = 1, 2, \ldots, m)$ are mutually disjoint intervals and on each of them either $P'' < 0$, or $P'' > 0$. It follows from the part c) of Remark 2.1 that P is on I_j either strictly convex, or strictly concave. According to Theorem 2.1 we have

$I_j \subset L(P)$ $(j = 1, 2, \ldots, m)$, hence

$$\bigcup_{j=1}^{m} I^j \subset L(P). \tag{12}$$

Relations (11) and (12) imply that $(a, b) - K \subset L(P)$, $(a, b) - L(P) \subset K$ and the statement of the theorem follows.

The next statement is an easy consequence of Theorem 1.7 (see Remark 1.1).

Theorem 2.2. If the second derivative of a function $f\colon (a, b) \to R$ is continuous, then the set $(a, b) - L(f)$ is nowhere dense in (a, b).

Further, let $C^2(0, 1)$ stand for the family of all such functions $f\colon [0, 1] \to R$, that the second derivative f'' is finite and continuous (in the endpoints 0 and 1 we suppose the one-sided differentiability). Put $\| g \|_1 = \sup\{|g(t)| : 0 \leq t \leq 1\}$ for $g \in C^2(0, 1)$. The family $C^2(0, 1)$ can be endowed with the norm $\|f\|$ defined by:

$$\|f\| = \|f\|_1 + \|f'\|_1 + \|f''\|_1$$

The family $C^2(0, 1)$ with the above introduced norm is a Banach space.

In the following we shall deal with the structure of the space $C^2(0, 1)$ from the point of view of Lagrange's sets of functions. We show that a typical function in $C^2(0, 1)$ has its Lagrange's set of the Lebesgue measure one.

First we shall state the following simple result.

Lemma 2.1. The family of all polynomials is dense in $C^2(0, 1)$.

The proof of Lemma 2.1. is an easy consequence of the Weierstrass Approximation Theorem.

Remark 2.2. It is easy to verify that the family of all polynomials of the degree greater than 2 is also dense in $C^2(0, 1)$. Note, that for $f \in C^2(0, 1)$ the set $L(f)$ is a F_σ-set (and hence measurable) — see Theorem 1.1.

Theorem 2.3. The family $\mathscr{F}$ of all functions $f \in C^2(0, 1)$ with $\lambda(L(f)) = 1$ is residual in the Banach space $C^2(0, 1)$.

Proof. Let $\mathscr{P}$ stand for the family of all polynomials of the degree greater than 2. If $P \in \mathscr{P}$, then the set

$$K_P = \{x \in [0, 1]: P''(x) = 0\}$$

is finite and $\lambda(K_P) = 0$. It follows from the continuity of the polynomial P that for every $n \in N$ there exists $\eta_n = \eta_n(P)$, $0 < \eta_n < 1/n$, such that for the closed set

$$D_P = \{x \in [0, 1] : |P''(x)| \leq \eta_n(P)\}$$

we have

$$\lambda(D_P) < 1/n. \tag{13}$$

Put $\mathscr{A}_n = \bigcup_{P \in \mathscr{P}} \{f \in C^2(0, 1): \|f - P\| < \eta_n(P)\}$.

Each of the sets $\mathscr{A}_n$ is open and hence $\mathscr{A} = \bigcap_{n=1}^{\infty} \mathscr{A}_n$ is a G_δ-set containing the set $\mathscr{P}$ (see Remark 2.2). Consequently $\mathscr{A}$ is a G_δ-set dense in $C^2(0, 1)$ and hence residual in $C^2(0, 1)$ (see [9], p. 49).

Choose $f \in \mathscr{A}$. Then $f \in \mathscr{A}_n$ holds for each $n \in N$. It follows from the definition of $\mathscr{A}_n$ that there is a polynomial P of the degree greater than 2 such that

$$\|f - P\| < \eta_n(P). \tag{14}$$

Put $M_P = [0, 1] - D_P$. Then M_P is open and it is the set of all $x \in [0, 1]$ fulfilling

$$|P''(x)| > \eta_n(P). \tag{15}$$

The relation (13) implies

$$\lambda(M_P) > 1 - 1/n. \tag{16}$$

If $x_0 \in M_P$, then we have from (15) that either $P''(x_0) > \eta_n(P)$ or $-P''(x_0) > > \eta_n(P)$. Suppose $P''(x_0) > \eta_n(P)$ (if $-P''(x_0) > \eta_n(P)$ we can proceed analogously). Since P'' is continuous at x_0, there exists an open interval I, containing x_0, such that

$$P''(x) > \eta_n(P) \tag{17}$$

holds for every $x \in I$.

From (14) we have for every $x \in [0, 1]$

$$|f''(x) - P''(x)| < \eta_n(P). \tag{18}$$

Hence, relations (17) and (18) imply

$f''(x) > P''(x) - \eta_n(P) > 0$ for every $x \in I$. The function $f|I$ is strictly convex, $I \subset L(f)$ and $x_0 \in L(f)$.

We showed $M_P \subset L(f)$. According to (16) we have $\lambda(L(f) \geq \lambda(M_P) > 1 - - 1/n$ $(n = 1, 2, \ldots)$, and hence $\lambda(L(f)) = 1$.

We have proved that $\lambda(L(f)) = 1$ for every function $f \in \mathscr{A}$. It follows from the definition of $\mathscr{F}$ that $\mathscr{F} \supset \mathscr{A}$. Consequently $\mathscr{F}$ is a residual subset of $C^2(0, 1)$.

In connection with Theorem 2.3 one can raise a natural question if there is a function $f \in C^2(0, 1)$, such that $\lambda(L(f)) < 1$. The following example gives an affirmative answer to this question.

Example 2.1. Let $A \subset [0, 1]$ be a nowhere dense set with $\lambda(A) > 0$. Suppose that $A = [0, 1] - \bigcup_{n=1}^{\infty} I_n$, where $\{I_n\}_{n=1}^{\infty}$ is a countable family of mutually disjoint open intervals.

Let q_n be defined on the interval $I_n = (a_n, b_n)$ $(n = 1, 2, \ldots)$ in the following way:

$$q_n(x) = x - a_n \text{ for } x \in (a_n, a_n + (b_n - a_n)/4),$$

$$q_n(x) = -x + (b_n + a_n)/2 \text{ for } x \in [a_n + (b_n - a_n)/4,\ b_n - (b_n - a_n)/4],$$

and $q_n(x) = x - b_n$ for $x \in (b_n - (b_n - a_n)/4, b_n)$.

Then $\int_{a_n}^{x} q_n(t)\,\mathrm{d}t \geq 0$ holds for each $x \in [a_n, b_n]$ and $\int_{a_n}^{x} q_n(t)\,\mathrm{d}t = 0$ if and only if $x = a_n$, or $x = b_n$.

Define the function g: $[0, 1] \to R$ in the following way:

$g(x) = 0$ for $x \in A$ and $g(x) = q_n(x)$ for $x \in I_n$ $(n = 1, 2, \ldots)$.

Then the function p: $[0, 1] \to R$, $p(x) = \int_{0}^{x} g(t)\,\mathrm{d}t (x \in [0, 1])$ is nonnegative on $[0, 1]$ and $p(x) = 0$ if and only if $x \in A$, $x = 0$ or $x = 1$.

For the function P: $[0, 1] \to R$, $P(x) = \int_{0}^{x} p(t)\,\mathrm{d}t$ $(x \in [0, 1])$ we have $P'(x) = = p(x)$ and $P''(x) = g(x)$ $(x \in (0, 1))$. Moreover, since A is nowhere dense, the function P is strictly increasing and

$$(P(y) - P(x))/(y - x) > 0$$

holds for every $x, y \in [0, 1]$, $x < y$. So we have

$$A \subset [0, 1] - L(P)$$

and

$$L(P) \subset [0, 1] - A.$$

Consequently $\lambda(L(P)) < 1$.

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РЕЗЮМЕ

ТЕОРЕМА О СРЕДНЕМ И МНОЖЕСТВО ЛАГРАНЖА ДЕЙСТВИТЕЛЬНЫХ ФУНКЦИЙ

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Пусть f: $[a, b] \to R$ (R — вещественная прямая, $[a, b] \subset R$) функция непрерывна на отрезке $[a, b]$ и дифференцируема на интервале (a, b). Множеством Лагранжа от функции f называется непустое множество $L(f)$ всех тех $\xi \in (a, b)$ для которых

$$f'(\xi) = (f(y) - f(x)) / (y - x)$$

и $a \leq x < \xi < y \leq b$. В работе рассматриваются свойства множества $L(f)$ и некоторых его обобщений.

SÚHRN

VETA O STREDNEJ HODNOTE A LAGRANGEOVE MNOŽINY REÁLNYCH FUNKCIÍ

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Nech f: $[a, b] \to R$ (R — reálna priamka, $[a, b] \subset R$) je funkcia spojitá na intervale $[a, b]$ a diferencovateľná na intervale (a, b). Lagrangeovou množinou funkcie f budeme nazývať neprázdnu množinu $L(f)$ všetkých tých $\xi \in (a, b)$ s vlastnosťou

$$f'(\xi) = (f(y) - f(x)) / (y - x),$$

kde $a \leq x < \xi < y \leq b$. Práca je venovaná štúdiu vlastností množiny $L(f)$ ako aj niektorých jej zovšeobecnení.