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ON THE IMAGE OF TWO SETS OF POSITIVE OUTER LEBESGUE MEASURE IN R^n

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Abstract. We will prove that if A and B are subsets of R^n each having positive Lebesgue outer measure, and $f\colon R^n \times R^n \to R^n$ satisfies appropriate conditions, then $f(A \times B)$, the set of all vectors $f(a, b)$ with $a \in A$ and $b \in B$, is "full" in the sense of outer Lebesgue measure in some n-dimensional cube K. This result is related to theorems of Steinhaus, Smítal and Sander and doubly extends the $n = 1$, $f(x, y) = x + y$ case proved previously by the second author.

Introduction. A classical theorem in measure theory, called the Steinhaus Theorem (see [4], pg. 68), states that if A and B are Lebesgue measurable subsets of R^n, each having positive measure, then the set

$$A + B = \{a + b\colon (a, b) \in A \times B\}$$

contains an n-dimensional cube.

It follows from a theorem of Sander [9] that this result can be extended. Namely, if A, $B \in R^n$, A is Lebesgue measurable and $m(A) > 0$, $\bar{m}(B) > 0$, then $A + B$ contains an n-dimensional cube. Here m and $\bar{m}$ denote Lebesgue measure and outer Lebesgue measure respectively. A straightforward and elementary proof of this result can be found in [3].

The last mentioned result can not be extended, i.e. there exist sets A, $B \subset R^n$, such that $\bar{m}(A) > 0$ and $\bar{m}(B) > 0$ and such that $A + B$ contains no n-dimensional cube. To see this consider the vector subspace $C = \langle H \setminus \{h\} \rangle$, of co-dimension one, where H is a Hamel basis for R^n (over the rationals) and $h \in H$. If we set $A = B = C$, then, by Theorem 2 on page 255 in [4], $A + B$ contains no cube and $\bar{m}(A) = \bar{m}(B) = \bar{m}(C) = \infty$.

The following result, often called *Smítal's lemma* in the literature [5], has a wide range of applications.

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Smítal's Lemma. If $A, B \subset R^n$, $\bar{m}(A) > 0$ and B is dense in R^n, then for every n-dimensional cube I,

$$\bar{m}((A + B) \cap I) = m(I).$$

The proof of Smítal's lemma can be modified to obtain the following. If $\bar{m}(A) > 0$ and B is dense in some n-dimensional cube J, then there exists an n-dimensional cube I such that

$$\bar{m}((A + B) \cap I) = m(I).$$

In a previous paper, [6], the second author has investigated what can be said about the set $A + B$, if $A, B \subset R$ with $\bar{m}(A) > 0$ and $\bar{m}(B) > 0$. Notice that it is possible for both A and B to be nowhere dense and to contain no measurable subsets of positive measure. To see this suppose that M is a "Cantor-like" set of positive measure ([8], pg. 63) and N is a Bernstein set ([7], pg. 24). Then either $M \cap N$ or $M \cap N'$, where $N' = R \setminus N$, is a nowhere dense nonmeasurable subset of R, containing no measurable subsets of positive measure (see [7], pg. 24). Call the set with these properties C and set $A = B = C$.

In a similar manner one can construct a nowhere dense subset$C$, of R^n, containing no measurable subsets of positive measure and hence none of the results we have listed are applicable when considering the set $C + C$.

In [6] it was shown that if A and B are both subsets of the real line having positive outer measure, then there exists an interval K such that $\bar{m}((A + B) \cap K) = m(K)$, i.e. $A + B$ is full in K in the sense of outer measure. In our main result in this paper we show that if A and B are subsets of R^n each having positive outer measure, and f: $R^n \times R^n \rightarrow R^n$ satisfies appropriate conditions, then $f(A \times B)$, the set of all vectors $f(a, b)$ with $a \in A$ and $b \in B$, is full in the sense of outer Lebesgue measure in some n-dimensional cube K.

Results. We first prove the theorem just mentioned (our main result) in the introduction.

Theorem 1. Suppose that $f = (f_1, f_2, \ldots, f_n)$: $R^n \times R^n \rightarrow R^n$ satisfies the following conditions.

1. The $2n^2$ partial derivatives (n functions and $2n$ variables) exist and are continuous in some neighbourhood of $(x_0, y_0) \in R^n \times R^n$.

2. $$\begin{vmatrix} D_1 f_1 \ldots D_n f_1 \\ \vdots \\ D_1 f_n \ldots D_n f_n \end{vmatrix} (x_0, y_0) \neq 0 \text{ and } \begin{vmatrix} D_{n+1} f_1 \ldots D_{2n} f_1 \\ \vdots \\ D_{n+1} f_n \ldots D_{2n} f_n \end{vmatrix} (x_0, y_0) \neq 0,$$

where $D_j f_i$ denotes the partial derivative of the function f_i with respect to the j^{th} variable, where $1 \leq j \leq 2n$ and $1 \leq i \leq n$. Suppose further that x_0 is a point of outer density of A and y_0 is a point of outer density of B, where $A, B \subset R^n$. Then there exists an n-dimensional cube I such that

$$\bar{m}(f(A \times B) \cap I) = m(I).$$

Proof. Without loss of generality (this is an easy exercise) we can assume that:

$$x_0 \in A \subset H_A \subset \bar{A} \qquad \text{and}$$

$$y_0 \in B \subset H_B \subset \bar{B},$$

where $\bar{A}$ and $\bar{B}$ denote the closures of A and B respectively and H_A and H_B are measurable outer covers of A and B respectively (i.e. H_A is measurable and contains A and C measurable and C a subset of $H_A \backslash A$ implies that $m(C) = 0$; and similarly for H_B). Furthermore, again without loss of generality, we may assume that $A \subset H$, $B \subset J$, where H and J are n-dimensional cubes centered at x_0 and y_0 respectively, whose measures are sufficently small to insure that:

3. The absolute values of both determinants in (2) are bounded away from zero and are bounded above on $H \times J$ and that for each $y \in J$, the function f_y, defined by the formula $f_y(x) = f(x, y)$ for each $x \in H$ is one-to-one.

By our hypotheses it is easy to see that f satisfies the conditions of Theorem (Satz) 3 on page 14 in [9]. Therefore, there exists an n-dimensional cube I such that $f(H_A \times H_B) \supset I$. The remainder of the proof will consist in showing that $f(A \times B)$ is full in the sense of outer measure in I.

We will proceed indirectly; i.e. assume that there exists a measurable set M of positive measure with $M \subset I \backslash f(A \times B)$. Let d be a fixed density point of M. There exist $a \in H_A$ and $b \in H_B$ such that $f(a, b) = d$. Given $\varepsilon > 0$, there exist $h_\varepsilon > 0$ and $b_\varepsilon \in B$ such that:

4. $$m(K(a, h_\varepsilon) \cap H_A) > (1 - \varepsilon)\, m(K(a, h_\varepsilon)$$

$$m(K(d, h_\varepsilon) \cap M) > (1 - \varepsilon)\, m(K(d, h_\varepsilon) \text{ and the}$$

Euclidean distance between b and b_ε is less than ε. Here $K(e, r)$ denotes, for each $e \in R^n$ and $r > 0$, the open ball in R^n with center e and radius r.

We will show that for properly chosen small positive numbers ε and ε':

5. $f(A \times \{b_\varepsilon\}) \cap (K(d, h_{\varepsilon'}) \cap M) \neq \emptyset$, which is a contradiction and hence $\bar{m}(f(A \times B) \cap I) = m(I)$.

To prove (5) we will first show that $d = f(a, b)$ is a point of outer density of the set $F(A \times \{b\})$. Again our argument is indirect. If d is not a point of outer density of $f(A \times \{b\})$, then there exists a strictly decreasing null sequence (r_n) of positive real numbers, a real number c, $c > 0$ and a sequence (G_n) of open sets in R^n such that:

6. $K_n \cap f(A \times \{b\}) \subset G_n \subset K_n$ and
$m(G_n) < (1 - c)\, m(K_n)$ for each natural number n,
where $K_n = K(d, r_n)$.

Therefore,

7. $m(K_n \setminus G_n) \geq cm(K_n)$.

For each n, let $T_n = f_b^{-1}(K_n \setminus G_n)$ and $S_n = f_b^{-1}(K_n)$, where, as before, $f_b(x) = f(x, b)$ for each $x \in H$.

From (3) and the Jacobian change of variable formula for multiple integrals (see [1], pg. 274) we have:

8. $m(T_n) \setminus m(S_n) = t_n m(K_n \setminus G_n)/s_n m(K_n)$ for each n, where (t_n) and (s_n) are sequences of positive real numbers, each bounded away from zero and bounded above.

From (7) and (8) and the fact that f_b^{-1} satisfies a Lipschitz condition (see [1], pg. 110) it follows that there exists a sequence of positive numbers (q_n), bounded above, such that:

9. $m(T_n)/m(Q_n) = m(T_n)/q_n m(S_n) \geq ct_n/q_n s_n$ for each n, where Q_n is the smallest open ball with center at a containing S_n.

Since $T_n \subset Q_n$, $T_n \cap A = \emptyset$ for each n and $\lim_{n \to \infty} m(Q_n) = 0$ (as $r_n \to 0$) we arrive at a contradiction of the fact that a is a density point of H_A. Therefore d is a point of outer density of $f(A \times \{b\})$. We remark that in connection with images of density points one should see [2].

By (4) and the fact that d is a point of outer density of $f(A \times \{b\})$ there exists an $\varepsilon' > 0$ such that:

10. $\bar{m}(f(A \times \{b\}) \cap K(d, h_{\varepsilon'}) > 3m(K(d, h_{\varepsilon'}))/4$.

Using measurable outer covers, the Jacobian change of variable formula for multiple integrals, formula (10) and the properties of f it can be shown that there exists an $\varepsilon > 0$ such that:

11. $\bar{m}(f(A \times \{b_\varepsilon\}) \cap K(d, h_{\varepsilon'})) > m(K(d, h_{\varepsilon'})/4$.

Taking (10) and (11) together we get (5), completing the proof.

The following result is an immediate corollary of Theorem 1.

Corollary 2. If $A, B \subset R^n$ and $\bar{m}(A) > 0$, $\bar{m}(B) > 0$, then there exists an n-dimensional cube K such that $\bar{m}((A + B) \cap K) = m(K)$.

We will complete this paper with a series of remarks.

Remark 1. In our introduction we made use of the concept of a Bernstein set. A set B (in a topological space) is called a Bernstein set if both $B \cap F$ and $B' \cap F$ are non-empty for each uncountable closed set F. Here B' denotes the complement of B.

Remark 2. It is possible, using the Vitali Covering Lemma, as in [6] for the $n = 1$ case, to directly prove, without using Theorem 1 (or the Theorems of Steinhaus or Sander) Corollary 2. We will now present a new proof of the

Theorem of Steinhaus using Corollary 2. Namely, we will show that if A and B are measurable subsets of R^n, each having positive measure, then $A + B$ contains a cube. To prove this it is sufficient to assume that A and B are compact sets in R^n, each having positive measure. Then $A + B$ is compact and hence closed. We note that a closed subset of R^n is either nowhere dense or contains a cube. If $A + B$ is nowhere dense then then

$$m((A + B) \cap K) < m(K) \text{ for each cube } K,$$

which contradicts Corollary 2.

Remark 3. The approach of using the density topology, in the proof of Theorem 1 was suggested by Professor L. Zajček of Charles University in Prague. Theorem 1 can be proved without using the density topology and the Theorem of Sander (mentioned in our proof of Theorem 1). We have such a proof, which uses the Vitali Covering Lemma, but it is much more complicated than the proof presented here.

Remark 4. We wish to thank the referee for many helpful suggestions which shortened and improved this paper.

REFERENCES

1. Apostal, T., Mathematical Analysis, Addison Wesley, Reading Mass., 1957.
2. Ger, R., Kominek, Z., and Sablik, M., Generalized Smítal's lemma and a theorem of Steinhaus, Radovi Matematički, Vol. 1, No. 1, 1985, 101—119.
3. Kominek, Z. and Miller, H. I., Some remarks on a theorem of Steinhaus, Glasnik Matematički, Vol. 20 (40) (1985), 337—344.
4. Kuczma, M., An Introduction to the Theory of Functional Equations and Inequalities, Państwowe Wydawnictwo Naukowe, Katowice, 1985.
5. Kuczma, M. and Smítal, J., On measures connected with the Cauchy equation, Aequationes Math., 14 (1976), 421—427.
6. Miller, H. I., A theorem connected with results of Steinhaus and Smital, J. Math. Anal. and Appl., Vol. 124, No. 1, 27—32.
7. Oxtoby, J. C., Measure and Category, Springer Verlag, New York, 1970.
8. Royden, H. L., Real Analysis, Macmillan, Second Edition, New York, 1968.
9. Sander, W., Verallgemeinerung eines satzes von H. Steinhaus, Manusripta Math. 18 (1976), 25—42.

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РЕЗЮМЕ

ОБ ОБРАЗЕ ДВУХ МНОЖЕСТВ ПОЛОЖИТЕЛЬНОЙ ВНЕШНЕЙ МЕРЫ ЛЕБЕГА В R^n

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Если A, B — подмножества пространства R^n, каждое из которых имеет положительную внешнюю меру Лебега и если $f: R^n \times R^n \to R^n$ удовлетворяет подходящим условиям, $f(A \times B)$, т.е. множество всех векторов $f(a, b)$, где $a \in A$, $b \in B$ имеет полную внешнюю меру Лебега в некотором n — мерном кубе. Этот результат связан с теоремами Штейнхауса, Смитала и Сандерса и обобщает прежние результаты второго из авторов, касающиеся случая $n = 1$ и $f(x, y) = x + y$.

SÚHRN

O OBRAZE DVOCH MNOŽÍN KLADNEJ VONKAJŠEJ LEBESGUOVEJ MIERY V R^n

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Ak A a B sú podmnožiny priestoru R^n, z ktorých každá má kladnú vonkajšiu Lebesgueovu mieru, a ak $f: R^n \times R^n \to R^n$ spĺňa vhodné podmienky, tak $f(A \times B)$, t.j. množina všetkých vektorov $f(a, b)$, kde $a \in A$ a $b \in B$, má plnú vonkajšiu Lebesgueovu mieru v niektorej n-rozmernej kocke K. Tento výsledok súvisí s vetami Steinhausa, Smítala a Sandera a rozširuje staršie výsledky druhého z autorov, týkajúce sa prípadu $n = 1$ a $f(x, y) = x + y$.