## Werk

**Label:** Article

**Jahr:** 1991

**PURL:** https://resolver.sub.uni-goettingen.de/purl?312901348_58-59|log5

UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LVIII—LIX

# AN ERGODIC PROPERTY OF BOCHNER INTEGRABLE FUNCTIONS

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In this paper we are interested by ergodic properties of a subset of Bochner integrable functions (see [1]). We shall consider integrable functions with values in a regular boundedly $\sigma$-complete vector lattice.

Throughout the paper, R will denote the set of all real numbers, $N$ the set of all positive integers, $Y$ a regular boundedly $\sigma$-complete vector lattice (see [5]) and $Y^+$ the set $\{y \in Y: y > 0\}$. We shall denote the r-convergence of a sequence $(y_n)_{n \in N}$ of elements of $Y$ by $\lim\limits_{n \to \infty} y_n = y$.

Let $X$ be a non-empty set. We say that a sequence of functions $(f_n)_{n \in N}, f_n$: $X \to Y$, uniformly $r$-converges to $f$ iff there exists $u \in Y^+$ such that the following condition holds: given $\varepsilon \in R^+$, we can find $n_0 \in N$ such that, for each $n \geqq n_0$ and $x \in X$, we have the inequality: $|f_n(x) - f(x)| \leqq \varepsilon u$.
We shall denote by $u - \lim\limits_{n \to \infty} f_n = f$ the uniform $r$-convergence of a sequence of functions $(f_n)_n$ to $f$.

Let $(X, \mathscr{S}, P)$ be a probability space. A function $f$: $X \to Y$ is said to be a simple integrable function if there are pairwise disjoint sets $A_1, \ldots, A_n \in \mathscr{S}$ and elements $a_1, \ldots, a_n \in Y$ such that

$$f = \sum_{i=1}^{n} a_i \chi_{A_i}.$$

The element $I(f)$ defined by

$$I(f) = \sum_{i=1}^{n} a_i P(A_i)$$

is called the integral of the function $f$. If $(f_n)_n$, $(g_n)_n$ are sequences of simple integrable functions such that

$$u - \lim_{n \to \infty} f_n = u - \lim_{n \to \infty} g_n = f,$$

then there is $c \in Y$ such that

$$\lim_{n \to \infty} I(f_n) = \lim_{n \to \infty} I(g_n) = c. \tag{1}$$

The value $c$ from (1) is called the integral of the function $f$ and we shall denote it $I(f)$, too.
We denote

$F_1 = \{f\colon X \to Y$; there are simple integrable functions $f_n$ such that $u - \lim_{n \to \infty} f_n = f\}$,

$F = \{f\colon X \to Y$; there is $g \in F_1$ such that $f = g$ a.e.$\}$.

We define for $f \in F$

$$I(f) = I(g).$$

The value $I(f)$ is called the integral of the function $f$, too, and the family $F$ denote the family of integrable functions.

The following proposition is evident.

**Proposition**. If $f, g \in F$, $c \in R$, then $f + g$, $cf$, $|f|$, $sup\{f, g\}$, $inf\{f, g\}$ are integrable functions and

$I(f + g) = I(f) + I(g)$,

$I(cf) = cI(f)$

$|I(f)| \leqq I(|f|)$.

If $f \leqq g$ then $I(f) \leqq I(g)$. Further, if $T$ is a measurable $P$-preserving transformation and $f \in F$, then $f \circ T \in F$, too.

Recall that $(X, \mathcal{S}, P, T)$ is called an ergodic system if $(X, \mathcal{S}, P, T)$ is a dynamical system and T is an ergodic transformation. Put

$$S_n(f, x) = \frac{1}{n} \sum_{i=0}^{n-1} f(T^i x)$$

for $f \in F$, $x \in X$, $n \in N$.

**Theorem**. Let $(X, \mathcal{S}, P, T)$ be an ergodic system and $f \in F$. Then

$$\lim_{n \to \infty} S_n(f, x) = I(f) \text{ a.e.}$$

**Proof**. (i) For a simple $Y$-valued function the proof of this theorem follows from the individual ergodic theorem for a real valued function.

(ii) Let $f \in F_1$ be arbitrary, then there is a sequence of simple $Y$-valued measurable functions $(f_k)_k$ and $u \in Y^+$ such that for each $\varepsilon \in R^+$ there is $k_o$ such that for each $k \geqq k_o$ and $x \in X$

$$f_k(x) - \varepsilon u \leqq f(x) \leqq f_k(x) + \varepsilon u. \tag{2}$$

By Theorem 5 in [2]

$$\lim_{n \to \infty} S_n(f, x) = f^*(x), f^* \in F \text{ and } I(f) = I(f^*).$$

From (2) we have

$$|f^*(x) - I(f_k)| \leqq \varepsilon u \quad \text{a.e.}$$

for $k \geqq k_o$, i.e. $f^*(x) = w \in Y$ a.e. Since $u - \lim_{n \to \infty} f_n = f$, so $w = I(f)$ and therefore Theorem holds for $f \in F_1$.

(iii) Let $f \in F$ be arbitrary. Then there are $g \in F_1$ and $A \in \mathscr{S}$ with $P(A^c) = 0$ such that $f(x) = g(x)$ for $x \in A$. Put

$$B^c = \bigcup_{i=0}^{\infty} A^c, \ h = g\chi_B,$$

then $P(B^c) = 0$ and $I(h) = I(f)$. According to (ii) we have $C \in \mathscr{S}$ with $p(C^c) = 0$ and for each $x \in C$

$$\lim_{n \to \infty} S_n(h, x) = I(f)$$

and therefore

$$\lim_{n \to \infty} S_n(f, x) = I(f)$$

for $x \in B \cap C$.

The theorem is proved.

**Remark**. An individual ergodic theorem in vector lattices is proved in [2] (see Theorem 5).

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Received: 20. 12. 1987

## РЕЗЮМЕ

### ЭРГОДИЧЕСКИЕ СВОЙСТВО БОХНЕРОВСКИХ ИНТЕГРИРУЕМЫХ ФУНКЦИИ

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В этой статии мы занимаемся одним эргодическим свойством функции со значениями в регулярной упорядоченной $\sigma$-полной структуре.

## SÚHRN

### ERGODICKÁ VLASTNOSŤ BOCHNEROVSKÝCH INTEGROVATEĽNÝCH FUNKCIÍ

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Článok sa zaoberá ergodickou vlastnosťou funkcie s hodnotami v regulárnom usporiadanom $\sigma$-úplnom vektorovom zväze.