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OBSERVABLES IN FUZZY QUANTUM POSETS

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1. Introduction

A fuzzy analogue of a random variable of a classical probability space is an observable. If $(\Omega, \mathscr{S})$ is a measurable space and ξ: $\Omega \to R$, then the measurability of ξ means that $\xi^{-1}(E) \in \mathscr{S}$, S, for any $E \in B(R)$, where $B(R)$ is the Borel σ-algebra of the real line R. The mapping x: $B(R) \to \mathscr{S}$ defined by $x(E) := \xi^{-1}(E)$, $E \in B(R)$, is a σ-homomorphism, called an observable of $\mathscr{S}$. It is well known that there is a one-to-one correspondence [8] between random variables of $(\Omega, \mathscr{S})$, ξ, and σ-homomorphisms, x, of $(\Omega, \mathscr{S})$ which is given by the formula $x(E) = \xi^{-1}(E)$, $E \in B(R)$. The concept of an observable as an homomorphism is also accepted in quantum logic theory [12] as well as in some models of fuzzy sets [10, 5].

In the present paper, we give some characterizations of observables of fuzzy quantum posets via pointwisely defined mappings, and the compatibility problem is solved, too. These results extend those in [2, 3, 5, 7].

2. Observables

We recall [4, 9] that a couple (Ω, M) is a fuzzy quantum poset if Ω is a nonvoid set, and M is a subset of $[0, 1]^{\Omega}$ such that

(i) if $1(\omega) = 1$ for any $\omega \in \Omega$, then $1 \in M$;

(ii) if $a \in M$, then $a^{\perp} := 1 - a \in M$;

(iii) if $1/2(\omega) = 1/2$ for any $\omega \in \Omega$, then $1/2 \notin M$;

(iv) if $\{f_i\} \subset M$, $\min(f_i, f_j) \leq 1/2$, for $i \neq j$, then $\bigcup_i f_i) = \sup_i f_i \in M$.

In an analoguos way as a fuzzy union, $\cup$, we define a fuzzy intersection $\bigcap_i a_i$: $\inf_i a_i$ of a sequence of fuzzy sets $\{a_i\}$ of M. The element $a^\perp$ is said to be the fuzzy complement of a fuzzy set a. Two elements a and b of M are said to be fuzzy orthogonal and we write $a \perp_F b$ iff $a \cap b \leq 1/2$. It is evident that $a \perp_F b$ iff $\{a > 1/2\} \cap \{b > 1/2\} = \emptyset$, where $\{a > 1/2\} = \{\omega \in \Omega: a(\omega) > 1/2\}$, etc. Moreover, we write $a \perp b$ iff $a \leq b^\perp$. It is clear that if $a \perp b$, then $a \perp_F b$, and the converse does not hold in general. In addition, if $a \cup a^\perp = b \cup b^\perp$, then $a \perp_F b$ iff $a \perp b$.

It is evident that, with respect to the natural ordering $\leq$ on M defined via $a \leq b$ iff $a(\omega) \leq b(\omega)$ for each $\omega \in \Omega$, M is a poset with the minimal and maximal elements 0 and 1, respectively, and with an orthogonality $\perp: a \mapsto a^\perp$, a M, such that (i) $a \cup a^\perp \leq 1$, $a \cap a^\perp \geq 0$ for any $a \in M$; (ii) $(a^\perp)^\perp = a$ for any $a \in M$; (iii) if $a \leq b$, then $b^\perp \leq a^\perp$.

We say that a mapping x from the Borel sets $B(R)$ of the real line R into M is an observable of (Ω, M) if

(i) $x(E^c) = x(E)^\perp$ for any $E \in B(R)$;

(ii) $x(E) \perp_F x(F)$ whenever $E \cap F = \emptyset$, E, $F \in B(R)$;

(iii) $x\left(\bigcup_i E_i\right) = \bigcup_i x(E_i)$ if $E_i \cap E_j = \emptyset$ for $i \neq j$, $\{E_i\} \subset B(R)$.

A simple example of an observable is a mapping x_a, where a is a fixed fuzzy set of M, defined via

$$x_a(E) = \begin{cases} a \cup a^\perp & \text{if } 0, 1 \in E \\ a^\perp & \text{if } 0 \in E,\ 1 \notin E \\ a & \text{if } 0 \notin E,\ 1 \in E \\ a \cap a^\perp & \text{if } 0, 1 \notin E, \end{cases}$$

for any $E \in B(R)$.

Remark. If, in the definition of x, $B(R)$ is replaced by any Boolean $\sigma -$ algebra $\mathscr{A}$, then x is called an $\mathscr{A}$-observable of (Ω, M). A non-void subset A of M is said to be a Boolean σ-algebra of (Ω, M) if

(i) there are the minimal and maximal elements 0_A, $1_A \in A$ such that $a \cap a^\perp = 0_A$, $a \cup a^\perp = 1_A$ for any $a \in A$;

(ii) with respect to 0_A, 1_A, $\perp$, $\cap$, and $\cup$, A is a Boolean σ-algebra (in the sense of Sikorski [11]).

From the definition, it is immediately clear that if x is an observable, then $R(x) = \{x(E): E \in B(R)\}$ is a Boolean σ-algebra of (Ω, M) with the minimal and maximal elements $x(\emptyset)$ and $x(R)$, respectively.

For any $a \in M$, we put

$$M_a = \{b \in M: b \cup b^{\perp} = a \cup a^{\perp}\}, \tag{2.1}$$

then either $M_a \cap M_b$ is the empty set or $M_a = M_b$. In other words, $\{M_a: a \in M\}$ is a partition of M.

Define

$$\Omega_a = \{\omega \in \Omega: a(\omega) \neq 1/2\}, \tag{2.2}$$

and, for any $b \in M_a$, put

$$\Omega_a(b) = \{\omega \in \Omega_a: b(\omega) = (a \cup a^{\perp})(\omega)\}, \tag{2.3}$$

moreover, let

$$Q_a: = \{\Omega_a(b): b \in M_a\}. \tag{2.4}$$

We recall that a $q - \sigma$-algebra is a nonempty system Q of subsets of a given set $X \neq \emptyset$ such that (i) if $A \in Q$, then $X - A \in Q$; (ii) $\bigcup_{i=1}^{\infty} A_i \in Q$ whenever $\{A_i\}$ is a sequence of mutually disjoint subsets of Q.

Theorem 2.1. (i) Q_a is a $q - \sigma$-algebra, for any $a \in M$.

(ii) The mapping $\Omega_a(.)$: $M_a \to Q_a$ defined via (2.3) is an ortho σ-isomorphism, i.e. it is bijective and preserves the maximal elements, complements, and joins of any sequences of mutually orthogonal elements.

Proof. It follows from the following simple properties:

(i) $\Omega_a(b) = \Omega_a(c)$ iff $b = c$, $\quad b, c \in M_a$;

(ii) $\Omega_a(b) \subseteq \Omega_a(c)$ iff $b \leq c$, $\quad b, c \in M_a$;

(iii) $\Omega_a(b^{\perp}) = \Omega_a - \Omega_a(b)$, $\quad b \in M_a$;

(iv) $\Omega_a(b) \cap \Omega_a(c) = \emptyset$ iff $b \perp_F c$, $b, c \in M_a$;

(v) $\Omega_a\left(\bigcup_{i=1}^{\infty} b_i\right) = \bigcup_{i=1}^{\infty} \Omega_a(b_i)$, $\{b_i\} \subset M_a$, $b_i \perp_F b_j$ for $i \neq j$.

Q.E.D.

An F-state of (Ω, M) is a mapping m: $M \to [0, 1]$ such that (i) $m(a \cup a^{\perp}) = 1$ for any $a \in M$; (ii) $m\left(\bigcup_{i=1}^{\infty} a_i\right) = \sum_{i=1}^{\infty} m(a_i)$ if $a_i \perp_F a_j$ for $i \neq j$, $\{a_i\} \subset M$.

Theorem 2.2. Let m be an F-state on M. Then the mapping $\mu_a\colon Q_a \to [0, 1]$ defined via

$$\mu_a(\Omega_a(b)) = m(b), \qquad b \in M_a,$$

is a probability measure on Q_a..

Proof is evident from Theorem 2.1.

Lemma 2.3. Let $\{a_n\} \subset M_a$. Then $\bigcap_{n=1}^{\infty} a_n \in M_a \left(\bigcup_{n=1}^{\infty} a_n \in M_a\right)$ iff $\bigcap_{n=1}^{\infty} \Omega_a(a_n) \in Q_a \cdot \left(\bigcup_{n=1}^{\infty} \Omega_a(a_n) \in Q_a\right)$, and in this case $\Omega_a\left(\bigcap_{n=1}^{\infty} a_n\right) = \bigcap_{n=1}^{\infty} \Omega_a(a_n) \left(\Omega_a \bigcup_{n=1}^{\infty} a_n\right) = \bigcup_{n=1}^{\infty} \Omega_a(a_n)\Big)$.

Proof. is straightforward. Q.E.D.

Theorem 2.4. Let x be an observable of (Ω, M), then there is a unique functiom $\varphi\colon \Omega_{x(R)} \to R$ such that φ is $Q_{x(R)}$-measurable, and

$$\Omega_{x(R)}(x(E)) = \varphi^{-1}(E), \qquad E \in B(R). \tag{2.5}$$

Conversely, for any Q_a-measurable mapping $\varphi\colon \Omega_a \to R$, where $a \in M$, there is a unique observable x of (Ω, M) with $x(R) = a \cup a^\perp$ such that (2.5) holds.

Proof. Let x be an observable. From Theorem 2.1 we conclude that $\Omega_{x(R)}(R(x)) = \{\Omega_{x(R)}(x(E))\colon E \in B(R)\}$ is a Boolean σ-algebra of subsets of $\Omega_{x(R)}$. Due to Theorem 1.4 by Varadarajan [12], there exists a unique mapping φ: $\Omega_{x(R)} \to R$ with (2.5).

The converse follows from Theorem 2.1. Q.E.D.

3. Compatibility

A nonempty set A of M is said to be f-compatible if, for all $a_1, a_2, \dots, a_{n+1} \in A$, we have (i) $b_1 := a_1 \cap \dots \cap a_n \cap a_{n+1} \in M$, $b_2 := a_1 \cap \dots \cap a_n \cap a_{n+1}^\perp \in M$; (ii) $b_1 \cup b_2 = a_1 \cap \dots \cap a_n$. The subset A is strongly f-compatible if $A \cup A^\perp$ is f-compatible, where $A^\perp = \{a^\perp\colon a \in A\}$.

We recall that if $\{a, b\}$ is strongly f-compatible, then $a \cup a^\perp = b \cup b^\perp$. Indeed, we have $a = a \cap b \cup a \cap b^\perp$, and $a^\perp = a^\perp \cap b \cup a^\perp \cap b^\perp$ which entail $a \cup a^\perp \leq b \cup b^\perp$. In the same way we have $b \cup b^\perp \leq a \cup a^\perp$. In some particular cases, this condition is also sufficient in order that A be strongly f-compatible. Moreover, if $1_a = a_1 \cup a_1^\perp = a_2 \cup a_2^\perp = \dots$, then $1_a = \left(\bigcup_i a_i\right) \cup \left(\bigcup_i a_i\right)^\perp = \left(\bigcap_i a_i\right) \cup \left(\bigcap_i a_i\right)^\perp$.

Lemma 3.1. Let $a_1 \leq a_2 \leq \ldots \leq a_n$, $a_i \in M$ for $i = 1, \ldots, n$. Then $\{a_1 \ldots, a_n\}$ is strongly f-compatible iff

$$a_1 \cup a_1^\perp = a_2 \cup a_2^\perp = \ldots = a_n \cup a_n^\perp . \tag{3.1}$$

Proof. One direction is now clear. For the second one, suppose (3.1) holds. First of all we consider that, for any i, j, $1 \leq i, j \leq n$ and any $u, v \in \{0, 1\}$ we have $a_i^u \cap a_j^v \in M$, where $a_i^0 = a_i^\perp$, $a_i^1 = a_i$. Moreover, for any $i_1, \ldots, i_{k+1}$, $1 \leq i_1, \ldots, i_{k+1} \leq n$ and any $j_1, \ldots, j_{k+1} \in \{0, 1\}$,

$$b_1 = a_{i_1}^{j_1} \cap \ldots \cap a_{i_{k+1}}^{j_{k+1}} = \begin{cases} a_{i_1} \in M \text{ if } j_1 = \ldots j_{k+1} = 1 \\ a_{i_{k+1}}^\perp \in M \text{ if } j_1 = \ldots = j_{k+1} = 0 \\ a_i \cap a_j^\perp \in M \text{ otherwise,} \end{cases}$$

where $i = \min\{i_u : j_u = 0,\ 1 \leq u \leq k+1\}$, $j = \max\{i_v : j_v = 1,\ 1 \leq v \leq k+1\}$. Hence

$$a_{i_1}^{j_1} \cap \ldots \cap a_{i_{k+1}}^{j_{k+1}} \cup a_{i_1}^{j_1} \cap \ldots \cap (a_{i_{k+1}}^{j_{k+1}})^\perp = a_{i_1}^{j_1} \cap \ldots \cap a_{j_k}^{j_k} \in M.$$

Q.E.D.

Theorem 3.2. Let A be a nonvoid subset of M. The following two statements are equivalent:

(i) A is strongly f-compatible.

(ii) There is a Boolean σ-algebra of (Ω, M) containing A.

Proof. The second statement evidently entails the first one. Let now (i) hold. Since $a \cup a^\perp = b \cup b^\perp$ for all $a, b \in A$, we have $A \subseteq M_a$ for each $a \in A$. Let $\mathscr{A} = \Omega_a(A) = \{\Omega_a(b) : b \in A\}$. Then $\mathscr{A}$ is a nonempty subset of the q-σ-algebra Q_a. Let $\{a_1, \ldots, a_n\}$ be an arbitrary finite subset of A. The strong f-compatibility of A and Lemma 2.3 imply that $\{\Omega_a(a_1^{j_1} \cap \ldots \cap a_n^{j_n}) : j_i, \ldots, j_n \in \{0, 1\}\}$ is an orthogonal covering of the set $\{\Omega_a(a_1), \ldots, \Omega_a(a_n)\}$. Due to Theorem 3 by Brabec [1], there exists a σ-complete Boolean subalgebra $\mathscr{R}$ of Q_a containing $\mathscr{A}$. Therefore $\Omega_a^{-1}(\mathscr{R})$ is a Boolean σ-algebra of (Ω, M) containing A.

Q.E.D.

Lemma 3.3. Let for a finite set $\{a_1, \ldots, a_n\}$ of M the condition (3.1) hold. Then $\{a_1, \ldots, a_n\}$ is strongly f-compatible iff $\bigcap_{i \in D} a_i \in M$ for any nonempty subset D of $\{1, \ldots, n\}$.

Proof. The proof follows from Lemma 2.3 and the observation that $\left\{\bigcap_{i \in D} \Omega_a(a_i) : D \subseteq \{1, \ldots n\}\right\}$, where $a = a_1$, generates an orthogonal covering of $\{\Omega_a(a_1), \ldots, \Omega_s(a_n)\}$. The rest follows from the criterion of Brabec [1] and Theorem 3.2. Q.E.D.

4. On joint observables

In the present section, we shall consider the relation between the joint σ-observable and the f-compatibility of a system of observables.

We say that a system $\{\mathscr{A}_t: t \in T\}$ of Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$ is independent (σ-independent) if, for any finite (countable) subset $\alpha \subseteq T$

$$\bigcap_{t < \alpha} A_t \neq 0$$

for any $0 \neq A_t \in \mathscr{A}_t$ and any $t \in \alpha$.

Let $\mathscr{D} = \mathscr{D}(T) = \left\{\bigwedge_{t \in \alpha} A_t\colon A_t \in \mathscr{A}_t, t \in \alpha, \emptyset \neq \alpha \subseteq T, \bar{\alpha} < \infty\right\}$, $\mathscr{R} = \mathscr{R}(T)$ be the minimal subalgebra of $\mathscr{A}$ containing $\mathscr{D}$, and let $\mathscr{A}(T)$ be the minimal sub-σ-algebra of $\mathscr{A}$ containing $\mathscr{R}$.

Let $\{\mathscr{A}_t: t \in T\}$ be a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$. We say that a system $\{x_t: t \in T\}$, where x_t is an $\mathscr{A}_t - \sigma$-observable of a fuzzy quantum poset (Ω, M), has a joint σ-observable if there is an $\mathscr{A}(T) - \sigma$-observable x of (Ω, M) such that

$$x\left(\bigwedge_{t \in \alpha} A_t\right) = \bigcap_{t \in \alpha} x_t(A_t) \tag{4.1}$$

for any $A_t \in \mathscr{A}_t$ and any finite nonvoid subset $\alpha \subseteq T$, supposing that the fuzzy set intersection on the right-hand side of (4.1) exists in M. It is clear that if the joint σ-observable exists for $\{x_t: t \in T\}$, then it is unique.

We say that $\{x_t: t \in T\}$ is a system of f-compatible observables if $\bigcup_{t \in T} R(x_t)$ is an f-compatible set in M. It is clear that for $\bigcup_{t \in T} R(x_t)$ the f-compatibility and the strong f-compatibility are equivalent.

Theorem 4.1. Let $\{\mathscr{A}_t: t \in T\}$ be a system of σ-independent Boolean sub-σ-algebras of a Boolean σ-algebra $\mathscr{A}$. For any $t \in T$, let x_t be an $\mathscr{A}_t$-σ-observable of a fuzzy quantum poset (Ω, M). Then $\{x_t: t \in T\}$ has a joint σ-observable iff $\{x_t: t \in T\}$ are f-compatible.

Proof. Let x be a joint σ-observable of $\{x_t: t \in T\}$, and denote by $R(x)$ the range of x. Conversely, let $\{x_t; t \in T\}$ be f-compatible. Due to (4.1), $R(x_t) \subseteq R(x)$ for any $t \in T$, so that $\{x_t: t \in T\}$ is a system of f-compatible observables. Due to Theorem 3.2, there exists the minimal Boolean σ-algebra A of (Ω, M) containing $\bigcup_{t \in T} R(x_t)$. We can prove that every Boolean σ-algebra of (Ω, M) is σ-distributive. Since A is a Boolean σ-algebra, the mapping x defined via (4.1) is defined well on $\mathscr{D}(T)$. According to Sikorski [11, Theorem 37.1], this mapping may be uniquely extended to an $\mathscr{A}(T)$-σ-observable of (Ω, M), that is, to a joint σ-observable of the system $\{x_t: t \in T\}$.

Q.E.D.

If, in particular, in Theorem 4.1, $\mathscr{A}_t = B(R)$ for any $t \in T$, we may build up a calculus for f-compatible observables, since the following theorem holds:

Theorem 4.2. Let $\{x_n\}$ be a sequence of observables of (Ω, M). Then the following criteria are equivalent:

(i) $\{x_n\}$ are f-compatible.

(ii) There eists a joint observable for $\{x_n\}$.

(iii) There exists a sequence $\{f_n\}$ of real-valued, Borel measurable functions and an observable x of (Ω, M) such that

$$x_n = f_n(x), \qquad n \geq 1, \tag{4.2}$$

where $f_n(x)$: $E \mapsto x(f_n^{-1}(E)), \qquad E \in B(R)$.

Proof. Theorem 4.2 is a consequence of Theorem 4.1 and Theorem 6.9 by Varadajan [12] Q.E.D.

Therefore, for f-compatible observables x and y we may define, for example, sum and product as follows: $x + y = (f + g)(z)$, $x . y = (f . g)(z)$, etc.

5. On representation of observables

The following result gives a characterization of observables through a special class of fuzzy sets.

Theorem 5.1. Let x be an observable of a fuzzy quantum poset (Ω, M) and let Q be any countable, dense subset in R. Denote, for any $r \in Q$, $B_x(r) = x((-\infty, r))$. The system $\{B_x(r): r \in Q\}$ fulfils the following conditions:

$$\begin{aligned}
&\text{(i)} B_x(s) \leq B_x(t) \text{ if } s < t, \qquad s, t \in Q;\\
&\text{(ii)} \bigcup_{r \in Q} B_x(r) = a;\\
&\text{(iii)} \bigcap_{r \in Q} B_x(r) = a^{\perp};\\
&\text{(iv)} \bigcup_{s < r} B_x(s) = B_x(r), \qquad r \in Q;\\
&\text{(v)} \ B_x(r) \cup B_x(r)^{\perp} = a, \qquad r \in Q,
\end{aligned} \tag{5.1}$$

where $a = x(R)$ and $a^{\perp} = x(\emptyset)$.

Conversely, if a system $\{B(r)\colon r\in Q\}$ of fuzzy sets of M fulfils the conditions (i) – (v) for some $a\in M$, then there is a unique observable x of (Ω, M) such that $B_x(r) = B(r)$ for any $r\in Q$ and $x(R) = a$.

Proof. Because $R(x) = \{x(E)\colon E\in B(R)\}$ is a Boolean σ-algebra, any countable union of fuzzy sets of $R(x)$ also belongs to $R(x)$. Thereby, the first statement of the theorem can be proved easily.

Conversely, suppose that for (Ω, M) and for some $a\in M$, a system $\{B(r)\colon r\in Q\}\subset M$ satisfying the conditions (i) – (v) be given. Due to Lemma 3.1 and Theorem 3.2, there exists the minimal Boolean σ-algebra $\mathscr{A}$ of M containing all $B(r)$'s. So, the remaining part can be proved similarly to the proof of Theorem 2.3 in [7]. Q.E.D.

Now we present the main result of this section. For that, we need the following: Let (Ω, M) be a fuzzy quantum poset, and, according to [9], we introduce

$$\begin{aligned} K(M) = \{A \subseteq \Omega\colon &\text{ there is an } a\in M \text{ such that} \\ &\{a > 1/2\} \subseteq A \subseteq \{a \geq 1/2\}\}. \end{aligned} \tag{5.2}$$

Then by [9], $K(M)$ is a $q-\sigma$-algebra.

Theorem 5.2. Let x be an observable of a fuzzy quantum poset (Ω, M). Then there is a $K(M)$- measurable function $f\colon \Omega\to R$ such that

$$\{x(E) > 1/2\} \subseteq f^{-1}(E) \subseteq \{x(E) \geq 1/2\} \tag{5.3}$$

for any $E\in B(R)$. If g is any $K(M)$-measurable, real-valued function on Ω with (5.3), then $\{\omega\in\Omega\colon f(\omega)\neq g(\omega)\}\in K(M)$ and

$$\{\omega\in\Omega\colon f(\omega)\neq g(\omega)\} \subseteq \{x(\emptyset) = 1/2\}. \tag{5.4}$$

Conversely, let $f\colon \Omega\to R$ be any $K(M)$-measurable function. Then there is an observable x of (Ω, M) with (5.3). If y is any observable of (Ω, M) with (5.3), then $x(E)\perp_F y(E)^\perp$ for any $E\in B(R)$.

Proof. Let $\bar{M} = M/I_0$ be a quotient poset which has been defined in [6]. Then by [6], $\bar{M}$ is an orthomodular σ-orthoposet, and $\varphi\colon M\to\bar{M}$ defined via $\varphi(a) = \bar{a}$ is a canonical ortho-σ-homomorphism from M onto $\bar{M}$. On the other hand, according to the Loomis-Sikorski analogue theorem for (Ω, M) [6], there is a mapping h from the q-σ-algebra $K(M)$ onto $\bar{M}$ such that $h(A) = \bar{a}$ if $\{a > 1/2\}\subseteq A\subseteq\{a\geq 1/2\}$.

Let x be an observable of (Ω, M). Let $r_1, r_2, \ldots$ be any distinct enumeration of the rational numbers in R. We shall construct sets $A_1, A_2, \ldots$ in $K(M)$ such that

(i) $h(A_i) = \varphi(x(-\infty, r_i))), \qquad i\geq 1,$

(ii) $A_i\subseteq A_j$ whenever $r_i < r_j$.

Indeed, it is enough to choose $A_i = \{x((-\infty, r_i)) > 1/2\}$. Since $\{A_i\}$ is a sequence of monotonne subsets of $K(M)$, there exists $\bigcap_i A_i$, $\bigcup_i A_i$ in $K(M)$.

An $h\left(\bigcap_j A_j\right) = \bigwedge_j \varphi(x((-\infty, r_j))) = \varphi\left(x\left(\bigcap_j (-\infty, r_j)\right)\right) = \bar{0}$, we may, by replacing A_k by $A_k - \bigcap_j A_j$, if necessary, assume that $\bigcap_j A_j = \emptyset$. Further $h\left(\bigcup_j A_j\right) = \varphi\left(x\left(\bigcap_j (-\infty, r_j)\right)\right) = \bar{1}$. Therefore, $h(N) = \bar{0}$, where $N = \Omega - \bigcup_j A_j$. If we define a mapping $f\colon \Omega \to R$ via

$$f(\omega) = \begin{cases} 0 & \text{if } \omega \in N \\ \inf\{r_j\colon \omega \in A_j\} & \text{if } \omega \in N, \end{cases}$$

then f is a well-defined, $K(M)$-measurable function such that $h(f^{-1}(E) = \varphi(x(E))$, $E \in B(R)$. In other words, f satisfies (5.3).

Conversely, suppose that f is any $K(M)$-measurable, real-valued function on Ω, and let $Q = \{r_1, r_2, \ldots\}$. For any integer $n \geq 1$, we find a fuzzy set $a_n \in M$ such that $\{a_n > 1/2\} \subseteq f^{-1}((-\infty, r_n)) \subseteq \{a_n \geq 1/2\}$. Put $1_K = \bigcap_n (a_n \cup a_n^{\perp})$, $0_K = 1_K^{\perp}$. For the system $\{b_n\colon n \geq 1\} \subset M$ defined via $b_n = a_n \cap 1_K \cup 0_K$, we have

(i) $\{b_n > 1/2\} \subseteq \{a_n > 1/2\} \subseteq f^{-1}((-\infty, r_n)) \subseteq \{a_n \geq 1/2\} \subseteq \{b_n \geq 1/2\}$;

(ii) $b_n \cup b_n^{\perp} = 1_K$ for any $n \geq 1$;

(iii) $b_n \leq b_m$ whenever $r_n < r_m$.

Due to Lemma 3.1, the system $\{b_1, b_2, \ldots\}$ is strongly f-compatible, and, thereby it is contained in some Boolean σ-algebra of (Ω, M).

If we put $B(r_n) = b_n$ for any $n \geq 1$, then the system $\{B(r)\colon r \in Q\}$ fulfils (i)$-$(v) of Theorem 5.1, consequently, there is a unique observable x such that $B(r) = x((-\infty, r))$ for each $r \in Q$. Hence, $\{x((-\infty, r)) > 1/2\} \subseteq f^{-1}((-\infty, r)) \subseteq \{x((-\infty, r)) \geq 1/2\}$ for any rational r, which entails the validity of (5.3) for any Borel set E in R.

Suppose now g is any $K(M)$-measurable, real-valued function on Ω for which (5.3) holds. Then we have

$$\{f < g\} = \bigcup_{r \in Q} \{f < r < g\} = \bigcup_{r \in Q} \{f < r\} \cap \{g > r\}$$

and

$$\emptyset = \{x(\emptyset) > 1/2\} \subseteq \{f < g\} = \bigcup_{r \in Q} \{f < r\} \cap \{g > r\} \subseteq \bigcup_{r \in Q} \{x((-\infty, r\} \geq 1/2\} \cap$$

$$\cap \{x([r, \infty)) \geq 1/2\} = \{x(\emptyset) \geq 1/2\} = \{x(\emptyset) = 1/2\}$$

which proves $\{f \neq g\} \in K(M)$ and (5.4).

Let now y be any observable of (Ω, M) with (5.3). Then

$$\{y(E)^{\perp} > 1/2\} \subseteq f^{-1}(E)^c \subseteq \{y(E)^{\perp} \geq 1/2\} \text{ and } \{x(E) \cap y(E^c) > 1/2\} =$$
$$= \{x(E) > 1/2\} \cap \{y(E^c) > 1/2\} \subseteq f^{-1}(E) \cap f^{-1}(E^c) = \emptyset$$

which yields $x(E) \perp_F y(E)^{\perp}$ for any $E \in B(R)$. Q.E.D.

If an observable x of (Ω, M) and a $K(M)$-measurable, real-valued function f on Ω satisfy (5.3), we shall write $x \sim f$.

Proposition 5.3. Let $x \sim f_i$, for $i = 1, \ldots, n$. Then $f_1, \ldots, f_n$ are f-compatible in $K(M)$.

Proof. According to Brabec [1], it is necessary to show that, for any $E_1, \ldots, E_n \in B(R)$, $\bigcap_{i=1}^{n} f_i^{-1}(E_i) \in K(M)$. But this follows from the following observation

$$\left\{x\left(\bigcap_{i=1}^{n} E_i\right) > 1/2\right\} = \bigcap_{i=1}^{n} \{x(E_i) > 1/2\} \subseteq \bigcap_{i=1}^{n} f_i^{-1}(E_i) \subseteq$$
$$\subseteq \bigcap_{i=1}^{n} \{x(E_i) \geq 1/2\} = \left\{x\left(\bigcap_{i=1}^{n} E_i\right) \geq 1/2\right\}. \qquad \text{Q.E.D.}$$

Theorem 5.4. Let x be an observable of (Ω, M) and $\varphi: \Omega_{x(R)} \to R$ be a unique $Q_{x(R)}$-measurable function corresponding to x via (2.5). Let f be any $K(M)$-measurable, real-valued function. Then $x \sim f$ iff

$$f(x) = \begin{cases} \varphi(\omega) & \text{if } \omega \in \Omega_{x(R)} \\ \varphi_0(\omega) & \text{if } \omega \in \Omega - \Omega_{x(R)} \end{cases} \tag{5.5}$$

where φ_0 is any mapping from $\Omega - \Omega_{x(R)}$ into R.

Proof. Let $x \sim f$, put $\varphi_0 = f | \Omega - \Omega_{x(R)}$, and define φ via (2.5). Then $\{x(E) > 1/2\} = \varphi^{-1}(E) \subseteq f^{-1}(E) \subseteq \varphi^{-1}(E) \cup \{x(E) = 1/2\}$ which gives $\varphi^{-1}(E) \subseteq f^{-1}(E) \cap \Omega_{x(R)} \subseteq \varphi^{-1}(E)$, so that (5.5) holds.

Conversely, suppose that f has the form (5.5). Then $f^{-1}(E) = \varphi^{-1}(E) \cup \varphi_0^{-1}(E)$ for any $E \in B(R)$ and

$$\{x(E) > 1/2\} = \varphi^{-1}(E) \subseteq \varphi^{-1}(E) \cup \varphi_0^{-1}(E) \subseteq \{x(E) \geq 1/2\}. \qquad \text{Q.E.D.}$$

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SÚHRN

POZOROVATEĽNÉ VO FUZZY KVANTOVÝCH POSETOCH

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Práca pojednáva o reprezentáciách pozorovateľných fuzzy kvantových priestorov pomocou bodových funkcií na základnom priestore a tiež študuje problém kompatibility.

РЕЗЮМЕ

НАБЛЮДАЕМЫЕ В НЕЧЕТКИХ КВАНТОВЫХ ЧАСТИЧНО УПОРЯДОЧЕННЫХ ПРОСТРАНСТВАХ

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В работе исследуются предсравления наблюдаемых в нечетких квантовых частично упорядоченных пространствах с помощью точечных функций в основном пространстве, и также изучается проблема компатибильности.