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REMARK ON PROPERTIES OF REAL VALUED NON NEGATIVE FUNCTIONS ON SEMIRINGS

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Some authors construct a notion of an integral coming out from more general functions than a measure, at the same time these functions are defined on various classes of sets. A survey of such approaches can be found in [2]. While the mutual connection of such properties as monotony (m), additivity (ad), finite additivity (fad), subadditivity (sa), finite subadditivity (fsa), valuation (v) and superadditivity ($\mu(A \cup B) \geqq \mu(A) + \mu(B)$, $A \cap B = \emptyset$; (sp)) on a ring is trivial, the situation on a semiring is a little different. In this paper we consider a semiring as a non empty class of sets $\mathscr{P}$ closed under the formation of intersections, which satisfies the condition

(a) if A, $B \in \mathscr{P}$ and $A \subset B$ then there are sets $C_i \in \mathscr{P}$, $i = 0, 1, 2, \ldots, n$ such that $A = C_0 \subset C_1 \subset \ldots \subset C_n = B$ and $D_i = C_i - C_{i-1} \in \mathscr{P}$, $i = 1, 2, \ldots, n$ (see [1]).

By μ we shall always denote a real non negative function defined on a semiring (although in Theorem 1 and Proposition 4 non-negativity is not essential).

The following theorem is a basic assertion concerning systemization of the above mentioned properties on a semiring.

1. Theorem. If a function μ is additive on a semiring $\mathscr{P}$, then μ is finitely additive.

Proof. Let some additive function μ on a semiring $\mathscr{P}$ be not finitely additive. Let m be the smallest positive integer such that there are sets A, $A_i \in \mathscr{P}$, $i = 1, 2, \ldots, m$, A_i are pairwise disjoint and $A = \bigcup_{i=1}^{m} A_i$ that $\mu(A) \neq \sum_{i=1}^{m} \mu(A_i)$. Evidently $m > 2$. There are sets $C_0, C_1, \ldots, C_r \in \mathscr{P}$ such that $A_1 = C_0 \subset C_1 \subset \ldots \subset C_r = A$, $D_j = C_j - C_{j-1} \in \mathscr{P}, j = 1, 2, \ldots, r$. Since $A_1 \cup D_1 \cup D_2 \cup \ldots \cup D_j = C_j \in \mathscr{P}, j = 1, 2, \ldots, r$, we obtain $\mu(A) = \mu(A_1) + \sum_{j=1}^{r} \mu(D_j)$. Analogously $\bigcup_{k=1}^{j} (A_i \cap D_k) =$

$= A_i \cap C_j \in \mathscr{P}$, $i = 2, 3, \ldots, m, j = 1, 2, \ldots, r$ and $\mu(A_i) = \sum_{j=i}^{r} \mu(A_i \cap D_j)$. Since for every $j = 1, 2, \ldots, r$ $D_j = \bigcup_{i=2}^{m} (A_i \cap D_j)$, it follows from the definition of the number m that

$$\mu(D_j) = \sum_{i=2}^{m} \mu(A_i \cap D_j) \text{ for every } j = 1, 2, \ldots, r.$$

Utilizing the previous knowledge we obtain $\sum_{i=1}^{m} \mu(A_i) = \mu(A_1) + \sum_{i=2}^{m} \sum_{j=1}^{r} \mu(A_i \cap$ $\cap D_j) = \mu(A_1) + \sum_{j=1}^{r} \sum_{i=2}^{m} \mu(A_i \cap D_j) = \mu(A_1) + \sum_{j=1}^{r} \mu(D_j) = \mu(A)$ and this is the contradiction.

2. Remark. If we replace the condition (a) from the definition of the semiring $\mathscr{P}$ by the condition

(a)′ if $A, B \in \mathscr{P}$, then there are pairwise disjoint sets $E_i \in \mathscr{P}$, $i = 1, 2, \ldots, n$ such that $B - A = \bigcup_{i=1}^{n} E_i$ (see [2]), then an additive function μ on thus defined semiring is not necessarily finitely additive. (Here, in fact, the definition of a semiring is weaker, as the sets $C_i = A \cup \left(\bigcup_{k=1}^{i} E_k \right)$ need not belong to $\mathscr{P}$.)

Let $V = \{$ad, fad, sa, fsa, v, sp, m$\}$.

3. Corollary. If a function μ is additive on a semiring $\mathscr{P}$, then μ has every property from the set V.

Proof. Let $\mathscr{S}(\mathscr{P})$ be the ring generated by the class $\mathscr{P}$. Let us put $\bar{\mu}(E) = \sum_{i=1}^{n} \mu(E_i)$ for every $E \in \mathscr{S}(\mathscr{P})$. Then the function $\bar{\mu}$, which is an additive extension of the function μ, has all the properties from the set V.

4. Proposition. Let a function μ be a valuation on a semiring $\mathscr{P}$ and $\mu(\emptyset) = c$. Let $A, A_i \in \mathscr{P}$, $i = 1, 2, \ldots, n$, A_i be pairwise disjoint and $A = \bigcup_{i=1}^{n} A_i$. Then it holds

$$\text{(b)} \quad \mu(A) + (n-1)\, c = \sum_{i=1}^{n} \mu(A_i).$$

Proof. If for every $i = 2, 3, \ldots, n$ $\bigcup_{k=1}^{i} A_k \in \mathscr{P}$, then the equation (b) is true. Let μ be a valuation on $\mathscr{P}$ and let the equation (b) be not true for every $A \in \mathscr{P}$. Let us consider a positive integer m and sets $C_i(D_i)$ analogously as in the proof

of Theorem. Then for every $i = 2, 3, \dots, m$ $\mu(A_i) = \sum_{j=1}^{r} \mu(A_i \cap D_j) - (r-1)c$ and for every $j = 1, 2, \dots, r$ $\mu(D_j) = \sum_{i=2}^{m} \mu(A_i \cap D_j) - (m-2)c$.
Using the previous relations we obtain

$$\sum_{i=1}^{m} \mu(A_i) = \mu(A_1) + \sum_{i=2}^{m} \left(\left(\sum_{j=1}^{r} \mu(A_i \cap D_j)\right) - (r-1)c\right) = \mu(A_1) +$$

$$+ \sum_{j=1}^{r} \sum_{i=2}^{m} \mu(A_i \cap D_j) - (m-1)(r-1)c = \mu(A_1) + \sum_{j=1}^{r} (\mu(D_j) + (m-2)c) -$$

$$-(m-1)(r-1)c = \mu(A) + rc + (m-r-1)c = \mu(A) + (m-1)c$$

and this is the contradiction.

5. Corollary. If a function μ is a valuation on a semiring $\mathscr{P}$, then μ is finitely subadditive.

6. Proposition. If a function μ is superadditive on a semiring $\mathscr{P}$, then μ is monotone.

Proof. Let A, B, C_i, D_i be sets in (a). Then $\mu(B) \geqq \mu(D_n) + \mu(C_{n-1}) \geqq$ $\geqq \mu(D_n) + \mu(D_{n-1}) + \mu(C_{n-2}) \geqq \dots \geqq \mu(A) + \sum_{j=1}^{n} \mu(D_n)$.

A subadditive function μ on a semiring $\mathscr{P}$ is not necessarily finitely subadditive.

7. Example. Let $\mathscr{P} = \{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c, d\}\}$. Let us define a function μ on the semiring $\mathscr{P}$ as follows $\mu(\emptyset) = \mu(\{a\}) = \mu(\{a, b\}) = \mu(\{a, c\}) = \mu(\{a, d\}) = 0$ and $\mu(A) = 1$ for every other set A in $\mathscr{P}$. Then the function μ is subadditive but it is not finitely subadditive.

We can represent the systemization of the properties from the set V of a real valued non negative function defined on a semiring by the following diagram

m $\leftarrow$ sp $\leftarrow$ (sp and sa) $\leftrightarrows$ ad $\leftrightarrows$ fad $\leftrightarrows$ (sp and v) $\leftrightarrows$ (v and $\mu(\emptyset) = 0$) $\rightarrow$ v $\rightarrow$ fsa $\rightarrow$ sa.

A property x implies a property y if it is possible to pass by arrows from x to y.

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РЕЗЮМЕ

ЗАМЕТКА О СВОЙСТВАХ ДЕЙСТВИТЕЛЬНЫХ НЕОТРИЦАТЕЛЬНЫХ ФУНКЦИЙ НА ПОЛУКОЛЬЦАХ

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В статье систематизированы взаимоотношения свойств действительных неотрицательных функций определенных на полукольцах.

SÚHRN

POZNÁMKA O VLASTNOSTIACH REÁLNYCH NEZÁPORNÝCH FUNKCIÍ NA POLOOKRUHOCH

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V práci sú systemizované vlastnosti nezáporných reálnych funkcií definovaných na polookruhoch.