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UNYFYING PRINCIPLES IN ANALYSIS AND A COMPACTNESS CRITERION

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Various theorems in elementary analysis may be proved using a unified approach. There are several equivalent and sometime also non equivalent forms of such an attitude. Since all of them are of a considerable didactic value it may be useful for the teacher to have in a sense relatively complete information in this direction. So we start with some references. Here they are: [1], [4—8]. We do not present the application of these principles to prove such fundamental theorems of elementary analysis as are the theorems on continuous functions on compact interval because it is clearly done in the union of the referred papers.

As to our knowledge such principles have not been applied to such kind of proofs as that of Arzela-Ascoli theorem. To present such an application is, beside of the above information, the main aim of this note.

To propagate the advantage of such an application we prove a more general theorem than Arzela-Ascoli, namely a theorem of Bolzano Weierstrass type for some sets of regulated functions. The result itself is not new. A simple but not quite elementary proof has been given by Hildebrandt [3]. Further generalisation was presented in the paper [2]. We consider the real functions on $\langle a, b \rangle$ only, with the emphasis on the didactic attitude involving one of the mentioned unifying principles. We hope that the proof is suitable for undergraduates.

Regulated function in this paper means a function f: $\langle a, b \rangle \to R$ such, that the unilateral limits $f(x + 0)$ $(f(x - 0))$ exist for all $x \in \langle a, b)$ $(x \in (a, b\rangle)$. Denote Q the set of all regulated functions on $\langle a, b \rangle$.

Theorem 1. Necessary and sufficient conditions that for a sequence $\{f_n\}$ of regulated functions on $\langle a, b \rangle$ there exist a subsequence $\{g_n \subset \{f_n\}$ which uniformly converges in Q are

(i) The set $\{f_n\}$ is uniformly bounded

(ii) For every $x_0 \in \langle a, b \rangle$ with $a < x_0 \leqq b$ $(a \leqq x_0 < b)$

$$\lim_{x\to x_0-0} f_n(x) = f_n(x_0-0)\ \Big(\lim_{x\to x_0+0} f_n(x) = f_n(x_0+0)\Big)$$

uniformly with respect to n.

Corollary (Arzela-Ascoli theorem). Necessary and sufficient conditions that for a subset F of the class C of continuous functions there exists a sequence $\{f_n\} \subset F$ which uniformly converges in C are

a) F is uniformly bounded

b) F is equicontinuous at every $x \in \langle a, b\rangle$.

From the set of unifying principles that of Patric Shanahan [6] will be used. To formulate it note that a family I of subintervals of $\langle a, b\rangle$ is said to be local if for each $x \in \langle a, b\rangle$ there exists $\langle c, d\rangle \in I$ containing x as an interior point in the relative topology. It is said additive if whenever $\langle c_1, d_1\rangle$, $\langle c_2, d_2\rangle \in I$, $\langle c_1, d_1\rangle \cap \langle c_2, d_2\rangle = \emptyset$, then $\langle c_1, d_1\rangle \cup \langle c_2, d_2\rangle \in I$. The Shanahan's principle states.

Lemma 1. If I is local and additive then $\langle a, b\rangle \in I$. Note that the simple proof given in [6] uses only the principle of nested intervals.

To make the proof of the Theorem 1 transparent we prove the following

Lemma 2. Let (i) and (ii) of the Theorem 1 be satisfied. Then given a sequence $\{f_n\} \subset Q$ a point $x_0 \in \langle a, b)$ and $\varepsilon > 0$, there exists a subsequence $\{g_n\} \subset \{f_n\}$, a positive integer n_0 and a positive number δ such that

$$|g_m(x) - g_n(x)| < \varepsilon \text{ for all } m, n \geqq n_0 \text{ and all } x \in \langle x_0, x_0 + \delta\rangle$$

Proof. Choose a subsequence $\{g_n\} \subset \{f_n\}$ such that $\{g_n(x_0)\}$ and $\{g_n(x_0 + 0)\}$ are convergent. Let n_0 be such that for $m, n \geqq n_0$

$$|g_n(x_0) - g_m(x_0)| < \frac{\varepsilon}{3}, \qquad |g_n(x_0 + 0) - g_m(x_0 + 0)| < \frac{\varepsilon}{3}$$

By (ii) there exists $\delta > 0$ such that for each $x \in (x_0, x_0 + \delta\rangle$ and $n = 1, 2, \dots$

$$|g_n(x) - g_n(x_0 + 0)| < \frac{\varepsilon}{3}$$

So for all $m, n \geqq n_0$ and all $x \in (x_0, x_0 + \delta\rangle$ we have

$$|g_m(x) - g_n(x)| \leqq |g_m(x) - g_m(x_0 + 0)| + |g_m(x_0 + 0) - g_n(x_0 + 0)| + \\ + |g_n(x_0 + 0) - g_n(x)| < \varepsilon$$

Since $|g_m(x_0) - g_n(x_0)| < \varepsilon$ for all $m, n \geqq n_0$ is true, we have $|g_m(x) - g_n(x)| < \varepsilon$ for each $x \in \langle x_0, x_0 + \delta\rangle$ and all $m, n \geqq n_0$.

Remark 1. The proof of an analogical "left hand side" lemma for $x_0 \in (a, b\rangle$ is the same.

Proof of Theorem 1. Necessity is the easier part and it is proved in the standard way. We give it for the sake of completness. The condition (i) is evident. Suppose that (ii) is not true.

Then either $\lim_{x \to x_0 - 0} f_n(x) = f_n(x_0 - 0)$ or $\lim_{x \to x_0 + 0} f_n(x_0 + 0)$ does not hold uniformly with respect to n. Let us restrict to the second case. The first is analogical. So we have a point $x_0 \in \langle a, b)$ a sequence $\{x_n\}$ where $x_n > x_0$ for $n = 1, 2, \ldots$ and a subsequence $\{g_n\} \subset \{f_n\}$ such that

$$|g_n(x_n) - g_n(x_0 + 0)| > \varepsilon \text{ for } n = 1, 2, \ldots \tag{1}$$

But $\{g_n\}$ by the assumption contains a uniformly convergent subsequence. Without changing the notation $\{g_n\}$ is supposed to be such subsequence and it may be supposed chosen in such a way that $\{g_n(x_0 + 0)\}$ is Cauchy.
We have for $m, n = 1, 2, \ldots$

$$\begin{aligned} |g_n(x_n) - g_n(x_0 + 0)| &= |g_n(x_n) - g_m(x_n)| + |g_m(x_n) - g_m(x_0 + 0)| + \\ &+ |g_m(x_0 + 0) - g_n(x_0 + 0)| \end{aligned} \tag{2}$$

The right hand side of (2) is for suitable chosen m, n less than ε and it is a contradiction with (1). Such choice is evidently possible. First we have n_0 such that $|g_m(x_0 + 0) - g_n(x_0 + 0)| < \frac{\varepsilon}{3}$ and $|g_m(x) - g_n(x)| < \frac{\varepsilon}{3}$ for all $m, n \geqq n_0$ and all $x \in \langle a, b\rangle$. Having this we choose $m \geqq n_0$ fixed. Because g_m are regulated and $x_n \to x_0 + 0$, we have $|g_m(x_n) - g_m(x_0 + 0)| < \frac{\varepsilon}{3}$ if $n \geqq n_0$ is sufficiently large.

Sufficiency. (Here we use the unifying principle (Lemma 1)). Denote I the set of all subintervals $\langle c, d\rangle \subset \langle a, b\rangle$ for which the following is true: there exist some positive integer N and a subsequence $\{h_n\} \subset \{f_n\}$ such that for all $m, n \geqq N$ and all $x \in \langle c, d\rangle$ we have $|h_m(x) - h_n(x)| < \varepsilon$. It follows from Lemma 2 that I is local. A simple consideration shows that I is additive. Hence by Lemma 1 $\langle y, b\rangle \in I$ i.e. there exists a positive integer n_0 such that $|f_m(x) - f_n(x)| < \varepsilon$ for every $m, n \geqq n_0$ and every $x \in \langle a, b\rangle$. We use now the preceding considerations in the following way. For $\varepsilon = 1$ we obtain a subsequence $\{f_{n,1}\} \subset \{f_n\}$ with the property that there is n_1 such that for $m, n \geqq n_1$ we have

$$|f_{m,1}(x) - f_{n,1}(x)| < 1 \text{ for every } x \in \langle a, b\rangle.$$

Repeating the procedure we obtain for $\varepsilon = \frac{1}{2}$ a subsequence $\{f_{n,2}\} \subset \{f_{n,1}\}$ a positive integer n_2 such that for $m, n \geqq n_2$ and every $x \in \langle a, b\rangle$ $|f_{m,2}(x) - f_{n,2}(x)| < \frac{1}{2}$. Continuing in this way we obtain by induction for $k = 2, 3, \ldots$ and $\varepsilon = \frac{1}{k}$ a sequence $\{f_{n,k}\} \subset \{f_{n,k-1}\}$ and a positive integer k such that for all $m, n \geqq n_k$ we have $|f_{n,k}(x) - f_{m,k}(x)| < \frac{1}{k}$ for all $x \in \langle a, b\rangle$. Putting $g_n = f_{n,n}$ for $n = 1, 2, \ldots$ we have a subsequence of $\{f_n\}$ which is evidently uniformly Cauchy.

Thus it is uniformly convergent. Moreover it can be immediately seen using (ii) that the limit function belongs to Q.

Remark 2. To prove Theorem 1 also the principle formulated by Leinfelder [4] or that one introduced in [1] may be used essentially in the same way.

No doubt that the application of unifying principles to convergence of functional sequences is broader. As an easy application we mention the proof of Dini's theorem. For this purpose choose its following simple version.

Theorem 2 (Dini). Let $\{f_n\}$ be a nonincreasing sequence of continuous functions on $\langle a, b\rangle$ pointwise converging to 0. Then $\{f_n\}$ converges uniformly.

Proof. Denote by I the set of all closed subintervals $\langle c, d\rangle$ of $\langle y, b\rangle$ having the following property: To arbitrary $\varepsilon > 0$ there exists some positive integer n_0 such that for all $n \geqq n_0$ and all $x \in \langle c, d\rangle\ |f_n(x)| = f_n(x) < \varepsilon$. It follows immediately from the convergence $\{f_n(y)\}$ where $y \in \langle a, b\rangle$ that for $\varepsilon > 0$ there is n_0 such that $|f_n(y)| = f_n(y) < \frac{\varepsilon}{2}$ for all $n \geqq n_0$ and using continuity of f_{n_0} at y we obtain $f_{n_0}(x) < \varepsilon$ for all x belonging to a small interval $\langle c, d\rangle$ containing y. So for all $n \geqq n_0$ and all $x \in \langle c, d\rangle\ |f_n(x)| = f_n(x) \leqq f_{n_0}(x) < \varepsilon$ showing that I is local. The fact that I is additive follows immediately. So by Lemma 1 $\langle a, b\rangle \in I$ and the uniform convergence of $\{f_n\}$ is proved.

REFERENCES

1. Botsko, M. W.: A unified treatment of various theorems in elementary analysis, Amer. Math. Monthly, 94 (1987) 450—452.
2. Dyer, J. A. and Johns, W. B.: Some properties of uniformly quasi-continuous functions, Amer. Math. Monthly, (1969) 489—494.
3. Hildebrandt, T. H.: Compactness in the space of quasincontinuous functions, Amer. Math. Monthly, 73 (1966) 144—145.
4. Leinfelder, H.: A unifying principle in real analysis, Real Analysis Exchange, 8 (1982—83) 511—518.
5. Neubrunnová, A.: On unifying principle in real analysis, Acta math. Univ. Comen., 48—49 (1986) 123—126.
6. Shanahan, P.: A unified proof of several basic theorems of real analysis, Amer. Math. Monthly, 79 (1972) 895—898.
7. Šalát, T.: Unifying principles for proving fundamental theorems in real analysis, Real Analysis Exchange, 10 (1984—85) 343—348.
8. Thomson, B. S.: On full covering properties, Real Analysis Exchange, 6 (1980—81) 77—93.

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SÚHRN

JEDNOTIACE PRINCÍPY V ANALÝZE A KRITÉRIUM KOMPAKTNOSTI

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Jednotiace princípy dôkazov mnohých základných viet matematickej analýzy sa často formulujú a dokazujú v matematickej literatúre. Ukazujú sa ako veľmi vhodný metodický nástroj, sú však zaujímavé aj samy o sebe ako matematické tvrdenia. Najbežnejšia je ich aplikácia na dôkazy základných viet o vlastnostiach spojitých funkcií. Cieľom tejto práce je prezentovať aplikáciu jedného takého princípu na dôkaz istého zovšeobecnenia Arzala—Ascoliho vety.

РЕЗЮМЕ

ОБЪЕДИНЯЮЩИЕ ПРИНЦИПЫ МАТЕМАТИЧЕСКОГО АНАЛИЗА И ОДИН КРИТЕРИЙ КОМПАКТНОСТИ

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Показано, что применением одной общей точки зрения на доказательства основных теорем математического анализа можно получить простое доказательство одного обощения теоремы Арцела—Асколи.