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SOME TYPES OF CONVERGENCE OF SEQUENCES OF REAL VALUED FUNCTIONS

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1. Introduction

We shall deal with some types of convergence of sequences of real valued functions. We recall some of them (see [2], [3], [6]).

Definition 1.1. Let X be a non-empty set, $f, f_n\colon X \to \mathrm{R}$, $n = 1, 2, \ldots$ being real valued functions defined on X.

a) We say that the sequence $\{f_n\}_{n=1}^{\infty}$ quasinormally converges to f on X (= equally in [3]) if there exists a sequence $\{\varepsilon_n\}_{n=1}^{\infty}$ of positive reals, $\lim\limits_{n\to\infty} \varepsilon_n = 0$ such that

$$(\forall x \in X)(\exists n_x)(\forall n \geq n_x)\,|f_n(x) - f(x)| \leq \varepsilon_n.$$

b) We say that the sequence $\{f_n\}_{n=1}^{\infty}$ quasiuniformly converges to f on X if $f_n \to f$ on X (i.e. pointwise) and the following holds

$$(\forall \varepsilon > 0)(\forall m)(\exists p)(\forall x \in X) \min\{|f_m(x) - f(x)|, \ldots |f_{m+p}(x) - f(x)|\} < \varepsilon.$$

c) We say that the sequence $\{f_n\}_{n=1}^{\infty}$ discretly converges to f on X if for every $x \in X$ there is an integer n_x such that $f_n(x) = f(x)$ for every $n \geq n_x$..

We shall investigate the relationship between quasinormal and quasiuniform convergence and the metrizability problem of all convergences from Definition 1.1.

2. Relationship between quasinormal and quasiuniform convergence

There is the natural question whether the quasiuniform convergence implies the quasinormal one and conversely. The following simple examples show that in general the answer is negative.

Example 2.1. It is known that there exists a sequence of continuous functions pointwise converging to a continuous function but not quasinormally (see [2], [3]). E.g. for $X = \mathrm{R}$ such a sequence can be constructed as follows (see [2]). Let $\{r_1, r_2, \ldots, r_k, \ldots\}$ be a one-to-one enumeration of the set Q of rationals. Set

$$f_n(r_k) = \frac{1}{2^k} \text{ for } k \leq n,$$

$$f_n(x) = 0 \text{ for } x \notin \bigcup_{k=1}^{n} \left(r_k - \frac{1}{2^n}, r_k + \frac{1}{2^n}\right)$$

are extend f_n piecewise linearly to a continuous function on R.

One can easily see that $f_n \to f$ pointwise, where

$$f(r_k) = \frac{1}{2^k},\ k = 1, 2, \ldots$$

$$f(x) = 0, \text{ for } x \notin \mathrm{Q}.$$

The sequence $\{f_k\}_{k=1}^{\infty}$ does not quasinormally converge (see [2]). We set

$$g_{2n} = f_n,\ n = 1, 2, \ldots$$

$$g_{2n+1} = f + \frac{1}{2n+1},\ n = 0, 1, 2, \ldots$$

Evidently the sequence $\{g_n\}_{n=1}^{\infty}$ pointwise converges to f and $\{g_{2n+1}\}_{n=0}^{\infty}$ uniformly converges to f. Thus, $\{g_n\}_{n=1}^{\infty}$ quasiuniformly converges to f. Since every subsequence of a quasinormally convergent sequence quasinormally converges, we obtain that $\{g_n\}_{n=0}^{\infty}$ does not quasinormally converge to f.

Example 2.2. Let X be the closed unit interval $\langle 0, 1\rangle$, $f_n(x) = x^n$ for $n = 1, 2, \ldots$. The sequence $\{f_n\}_{n=1}^{\infty}$ quasinormally converges to the function f, where

$$f(x) = 0 \text{ for } x \in \langle 0, 1)$$

$$f(1) = 1.$$

Evidently, it does not quasiuniformly converge.

3. Topology on C(0, 1) and convergences of continuous functions

An L-space is a set X endowed with the following structure. To some sequences of points of X a point, called the limit of this sequence, is assigned in such a way that

1. if $x_n = x$ for all $n = 1, 2, \dots$ then $\lim\limits_{n \to \infty} x_n = x$,

2. if $\lim\limits_{n \to \infty} x_n = x$ and $\{n_k\}_{k=1}^{\infty}$ is a strictly increasing sequence of integers then $\lim\limits_{k \to \infty} x_{n_k} = x$ (see [5], p. 84).

If Y is a non-empty set then the set $X = {}^{Y}\mathrm{R}$ of all real valued functions defined on Y with the pointwise convergence is an example of an L-space. Similarly, X endowed with the quasinormal or uniform convergence is an L-space, but the quasiuniform convergence does not make of X an L-space.

If Y is a topological space then the set $C(Y)$ of all real valued continuous functions endowed with the pointwise or the quasinormal or the uniform convergence is also an L-space. J. Ewert remarked that $C(Y)$ endowed with the quasiuniform convergence is not an L-spacse in general, i.e. for an arbitrary Y.

It is an L-space if Y is compact. Given an L-space X one can naturally ask whether there is a topology $\mathcal{O}$ on X which generates the L-structure, i.e. $\lim\limits_{n \to \infty} x_n = x$ if and only if $\{x_n\}_{n=0}^{\infty}$ converges to x in the sense of the topology $\mathcal{O}$. In general, such a topology need not be determined uniquely.

If X is the set of all real valued function defined on $\langle 0, 1 \rangle$, i.e. $X = {}^{\langle 0, 1 \rangle}\mathrm{R}$, then the Tychonoff product topology on X generates the pointwise convergence on X. Similarly this topology relativized to $C(0, 1)$ generates the pointwise convergence on $C(0, 1)$. Can we find some nice topology on $C(0, 1)$ generating this convergence? M. K. Fort [4] has proved that such a topology cannot be metrizable. We prove a little stronger result for all considered types of convergences.

A. V. Archangelskij (see [1]) considers five properties of L-spaces. The weakest one of them reads as follows $\langle 4 \rangle$: If for every n the sequence $\{x_{n,k}\}_{k=1}^{\infty}$ is one-to-one (i.e. $x_{n,k} \neq x_{n,l}$ for $k \neq l$) and $\lim\limits_{k \to \infty} x_{n,k} = x$ then there exist sequences $n_1 < n_2 < \dots < n_j \dots, k_1, k_2, \dots, k_j, \dots$ such that

$$\lim_{j \to \infty} x_{n_j, k_j} = x.$$

In [1] A. V. Archangelskij has proved the following

Theorem A. If the convergence of an L-space X is generated by a topology satisfying the first axiom of countability, the X has then property $\langle 4 \rangle$.

The purpose of this paper is to prove

Theorem 3.1. None of the pointwise, discrete quasinormal quasiuniform convergences on $C(0, 1)$ has the property $\langle 4 \rangle$.

Corollary. The L-space $C(0, 1)$ with the discrete, quasinormal and quasiuniform convergence is not generated by a first-countable topology, hence, it is not metrizable.

Proof of the theorem. We shall slightly modify and simplify the argument given in [4].

For given natural numbers n, k, we define a function $f_{n,k}\colon \langle 0, 1\rangle \to \mathsf{R}$ as follows:

$$f_{n,k}(x) = 0 \text{ for } x \in \left\langle \frac{i}{2^n}, \frac{i}{2^n} + \frac{1}{(k+3)\,2^n} \right\rangle \cup \left\langle \frac{i}{2^n} + \frac{1}{k\,.\,2^n}, \frac{i+1}{2^n} \right\rangle,$$

$$i = 0, 1, \dots, 2^n - 1,$$

$$f_{n,k}(x) = 1 \text{ for } x \in \left\langle \frac{i}{2^n} + \frac{1}{(k+2)\,2^n}, \frac{i}{2^n} + \frac{1}{(k+1)\,2^n} \right\rangle$$

$$i = 0, 1, \dots, 2^n - 1$$

and we extend $f_{n,k}$ in piecewise linear way to a continuous function.

One can easily see that for given n the sequence $\{f_{n,k}\}_{k=1}^{\infty}$ discretly converges to f, where $f(x) = 0$ for any $x \in \langle 0, 1\rangle$. Hence, it converges to f also pointwise, quasinormally and by some well-known facts ([6], p. 143) also quasiuniformly.

Let $n_1 < n_2 < \dots$ and $k_1, k_2, \dots$ be natural numbers. We show that there exists an $x_0 \in \langle 0, 1\rangle$ such that

$$\lim_{i\to\infty} f_{n_i, k_i}(x_0) \neq 0.$$

Thus the sequence $\{f_{n_i, k_i}\}_{i=1}^{\infty}$ does not converge pointwise to zero and hence neither quasinormally, nor discretly.

To find such an $x_0 \in \langle 0, 1\rangle$, let I_1 be a closed interval such that $f_{n_1, k_1}(x) = 1$ for $x \in I_1$ (e.g, we can set)

$I_1 = \left\langle \frac{1}{(k_1+2)\,2^{n_1}}, \frac{1}{(k_1+1)\,2^{n_1}} \right\rangle$. Let $m_1 = 1$. If m_2 is sufficiently large then

$$\left\langle \frac{i_2}{2^{n_{m_2}}}, \frac{i_2+1}{2^{n_{m_2}}} \right\rangle \subseteq I_1$$

for some i_2. Set

$$I_2 = \left\langle \frac{i_2}{2^{n_{m_2}}} + \frac{1}{(k_{m_2}+2)\,2^{n_{m_2}}}, \frac{i_2}{2^{n_{m_2}}} + \frac{1}{(k_{m_2}+1)\,2^{n_{m_2}}} \right\rangle.$$

Evidently $f_{n_{m_2}, k_{m2}}(x) = 1$ for $x \in I_2$.

Going on by induction we construct a decreasing sequence $I_1 \supseteq I_2 \supseteq \dots$ of closed intervals and an increasing sequence $\{m_j\}_{j=1}^{\infty}$ of integers such that

$$f_{n_{m_j}, k_{m_j}}(x) = 1 \text{ for } x \in I_j, j = 1, 2, \dots$$

By the Cantor theorem (see [5], p. 5) there is an $x_0 \in \bigcap_{j=1}^{\infty} I_j$, for which $\lim_{j \to \infty} f_{n_{mj}, k_{mj}}(x_0) = 1$.

q.e.d.

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РЕЗЮМЕ

О НЕКОТОРЫХ ВИДАХ СХОДИМОСТИ ПОСЛЕДОВАТЕЛЬНОСТЕЙ ДЕЙСТВИТЕЛЬНЫХ ФУНКЦИЙ

Зузана Буковска, Кошице — Тибор Шалат, Братислава

В работе исследованы отношения между следующими видами сходимости последовательностей действительных функций: квазинормальна, квазиравномерная и дискретная сходимость. На предвыдующие виды сходимости расширенный знакомый результат М. К. Форта (см. [4]) о неметризуемости точечной сходимости в классе $C(0, 1)$.

SÚHRN

O NIEKTORÝCH TYPOCH KONVERGENCIE POSTUPNOSTÍ REÁLNYCH FUNKCIÍ

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V práci sa študujú vzájomné vzťahy medzi týmito typmi konvergencie postupností reálnych funkcií: kvázinormálna, kvázirovnomerná a diskrétna konvergencia. Na predošlé typy konvergencie je rozšírený známy výsledok M. K. Forta (pozri [4]) o nemetrizovateľnosti bodovej konvergencie v triede $C(0, 1)$.