UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LVIII—LIX

# QUASICONTINUOUS SELECTIONS FOR ONE-TO-FINITE MULTIFUNCTIONS

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**Key words**. Multifunction, selection, quasicontinuous.

**Abstract**. The aim of this paper is to show that we can guarantee the existence of a quasicontinuous selection for one-to-finite quasicontinuous multifunctions defined on spaces where an one-to-finite continuous multifunction need not have a continuous selection.

## 1. Introduction

Many authors have studied the problem of the existence of continuous selections for continuous multifunctions. This research was started by Michael [5] who proved continuous selection theorems for l.s.c. multifunctions with closed convex values in Banach spaces.

In general, if a continuous multifunction $F: X \to Y$ has nonconvex compact values, or even finite ones, then $F$ need not have a continuous selection. In [1] Carbone gives an example of a continuous multifunction $F$ from a circle $C$ onto the boundary of Möbius band such that, for each $x$ in $C$, the set $F(x)$ has exactly two points and $F$ has no continuous selection.

The reason for proving quasicontinuous selection theorems when we cannot prove the continuous ones is a relatively good connection between the continuity and quasicontinuity in spite of the generality of the latter. More information about quasicontinuity can be found in the survey paper by Neubrunn [7].

First quasicontinuous selection theorems were proved in 1987 by Matejdes ([4]). The results and the ideas of the present paper are different.

## 2. Preliminaries

In what follows, $X$ and $Y$ will be topological spaces.

A subset $B$ of $X$ is said to be semiopen if there exists an open set $G$ such that $G \subset B \subset \mathrm{cl}\, G$ or, which is equivalent, $B \subset \mathrm{cl}\,(\mathrm{int}\, B)$ ([3])
Int $B$ and cl $B$ denote the interior and the closure of the set $B$ respectively.

We will use the following properties of semiopen sets ([3], [6]):

1. The union of any family of semiopen sets is semiopen.
2. The intersection of an open set and a semiopen one is semiopen.
3. A set $S$ is semiopen if and only if $S = (\mathrm{int}\, S) \cup B$ where $B$ is a subset of the boundary of $\mathrm{int}\, S$.

Levine in [3] defined a function $f\colon X \to Y$ to be semi-continuous if $f^{-}(G)$ is semiopen for any open subset $G$ of $Y$. Kempisty in [2] introduced the notion of quasicontinuous function. A function $f\colon X \to Y$ is said to be quasicontinuous at $x \in X$ if, for any open set $V$ such that $f(x) \in V$ and any open $U$ such that $x \in U$, there exists a nonempty open set $W \subset U$ such that $f(W) \subset V$. If $f$ is quasicontinuous at each point $x \in X$ it is said to be quasicontinuous in $X$. A.Neubrunnová in [8] proved that a function $f\colon X \to Y$ is semi-continuous if and only if it is quasicontinuous.

We say that a set $S$ is a semineighbourhood of a point $x$ if there exists a semiopen set $A$ such that $x \in A \subset S$ holds. It is easy to see (using (2) and (3)) that a function $f\colon X \to Y$ is quasicontinuous if and only if for each $x$ in $X$ and for each open $G \subset Y$ such that $f(x) \in G$ the set $f^{-}(G)$ is a semineighbourhood of $x$.

Now we give several definitions of continuity of multifunctions. If $F\colon X \to Y$ is a multifunction then for subsets $A$, $V_1$, $V_2$, $\dots$, $V_n$ of $Y$ we denote

$$F^{-}(A) = \{x\colon F(x) \cap A \neq \emptyset\},\ F^{+}(A) = \{x\colon F(x) \subset A\} \text{ and}$$

$$F^{-}(V_1, \dots, V_n) = \{x\colon F(x) \cap V_i \neq \emptyset\ i = 1, 2, \dots, n\}.$$

A multifunction $F\colon X \to Y$ is said to be upper (lower) semicontinuous at a point $z \in X$ if for any open $V$ such that $F(z) \subset V\,(F(z) \cap V \neq \emptyset)$ there exists an open set $U$ containing $z$ and such that $F(t) \subset V\,(F(t) \cap V \neq \emptyset)$ for any $t \in U$. $F$ is said to be upper (lower) semi-continuous if it is upper (lower) semi-continuous at any $y \in X$.

By [7], a multifunction $F\colon X \to Y$ is said to be upper quasicontinuous — briefly u-quasicontinuous (lower quasicontinuous — briefly l-quasicontinuous) at a point $x \in X$ if for any open set $V$ containing $F(x)$ (for any point $z$ from $F(x)$ and for any neighbourhood $V$ containing $z$) and any neighbourhood $U$ of $x$, there exists a nonempty open set $W \subset U$ such that $F(t) \subset V\,(F(t) \cap V \neq \emptyset)$ for any $t \in W$. If $F$ is u-quasicontinuous (l-quasicontinuous) at any $z$ in $X$, then it is said to be u-quasicontinuous (l-quasicontinuous). A multifunction $F\colon X \to Y$ is u-quasicontinuous (l-quasicontinuous) if and only if for any open set $G$ the set $F^+(G)$ $(F^-(G))$ is semiopen (see [7]).

To obtain a quasicontinuous selection theorem we will need the following notion of generalized continuity for multifunctions: A multifunction $F\colon X \to Y$ is said to be strongly lower quasicontinuous at a point $z$ in $X$ if for any finite collection $\{V_1, V_2, \ldots, V_n\}$ of open subsets of $Y$ such that $F(z) \cap V_i \neq \emptyset$ $i = 1, 2, \ldots, n$ the set $F^-(V_1, V_2, \ldots, V_n)$ is a semineighbourhood of $z$. If $F$ is strongly lower quasicontinuous at any $x$ in $X$ than it is said to be strongly lower quasicontinuous.

We shall denote by u.s.c., l.s.c., u.q.c., l.q.c. and s.l.q.c. the upper semicontinuity, the lower semicontinuity, the upper quasicontinuity, the lower quasicontinuity and the strong lower quasicontinuity. In this paper a multifunction $F$ is said to be continuous (quasicontinuous) if it is u.s.c. and l.s.c. (u. q.c. and s.l.q.c.).

It is easy to see that a multifunction $F\colon X \to Y$ is s.l.q.c. if and only if the set $F^-(V_1, V_2, \ldots, V_n)$ is semiopen for every finite collection $\{V_1, V_2, \ldots, V_n\}$ of open subsets of $Y$. We can see that every l.s.c. multifunction is s.l.q.c.. The contrary is not true. For example every quasicontinuous function is s.l.q.c. but it need not be l.s.c..

In what follows card$A$ will mean the number of elements of a finite set $A$. Let $F\colon X \to Y$ be a multifunction. Let $V$ be an open subset of $X$. The symbol $S(F, V)$ will denote the set of all quasicontinuous functions $g\colon V \to Y$ such, that for each $v$ in $V$ $g(v)$ is an element of $F(v)$, i.e. the set of all quasicontinuous selections of $F$ on $V$.

## 3. Quasicontinuous selection theorems

**Lemma 1.** Let $X$ be a topological space. Let $Y$ be a Hausdorff topological space. Let $n$ be a positive integer. Let $F\colon X \to Y$ be a s.l.q.c. multifunction with exactly $n$ values for each $x$ in $X$. Then for every nonempty open subset $U$ of $X$ there exists a nonempty open subset $W$ of $U$ such, that the set $S(F, W)$ is nonempty.

**Proof.** Let $U$ be a nonempty open subset of $X$. Let us take an arbitrary point $t$ from $U$. Let $F(t) = \{a_1, \ldots, a_n\}$. There exist nonempty open sets $V_1, \ldots, V_n$ which are pairwise disjoint and such that $a_i \in V_i$ $i = 1, 2, \ldots, n$. Since $F$ is s.l.q.c. there exists a semineighbourhood $S$ of the point $t$ such that

(i) For each $z$ in $S$ $F(z) \cap V_i \neq \emptyset$ for $i = 1, 2, \ldots, n$.

Let us denote $W = \operatorname{int}(S \cap U)$. $W$ is a nonempty open subset of $U$. Let us define a function $g\colon W \to Y$ as follows:

for each $w$ in $W$ $g(w) = F(w) \cap V_1$

By (i), for every $s$ from $W$, card $F(w) \cap V_1 = 1$ hods. To prove that $S(F, W)$ is nonempty it suffices to show that $g$ is quasicontinuous.

Let $G$ be an open subset of $Y$. Then $g^{-}(G) = g^{-}(G \cap V_1) = \{x \in W$: for $i = 1, 2, \ldots, n$ $F(x) \cap V_i \neq \emptyset$ and $F(x) \cap (G \cap V_1) \neq \emptyset\} = W \cap F^{-}(G \cap V_1, V_2, \ldots, V_n)$. Since $F$ is s.l.q.c., the set $g^{-}(G)$ is semiopen. Therefore g is an element of S(F, W).

**Lemma 2.** Let $X$ be a topological space. Let $Y$ be a Hausdorff topological space. Let $F$: $X \to Y$ be a multifunction such that, for every nonempty open subset $W$ of $X$, there exists a nonempty open subset $D$ of $W$ such that $S(F, D) \neq \emptyset$. Then there exists an open subset $U$ of $X$ dense in $X$ and such that $S(F, U)$ is nonempty.

**Proof.** Let $Z = \{(g, D)$: $D$ is a nonempty open subset of $X$ and $g \in S(F, D)\}$. We define a partial order $\leq$ on $Z$ as follows:

$(h, A) \leq (g, B)$ if and only if $A \subseteq B$ and for each $a \in A$ $g(a) = h(a)$ holds.

Let $S \subset Z$ be a linearly ordered subset of $Z$. We will show that $S$ has an upper bound in $Z$. Let us define an ordered pair $(p, P) \in Z$ as follows:

$P = \bigcup_{(g,\, D) \in S} D$ and $p$: $P \to Y$ is a function such, that for every $x \in P p(x) = g(x)$ for each $g$ such, that there exists $(g, D) \in S$ and $x \in D$. It is easy to see that $p$ is a function and that $p$ is a selection for $F$ on the set $P$.

Let $U$ be an open subset of $Y$. From the definition of $p$ it follows that

$p^{-}(U) = \bigcup_{(g,\, D) \in S} g^{-}(U)$ so $p^{-}(U)$ is a semiopen set.

Therefore $p$ is quasicontinuous and $(p, P)$ is an element of $Z$.

Moreover we can see that for every $(h, V) \in S$ $(p, P) \geq (h, V)$ holds, so $S$ has an upper bound. By Zorn's lemma the set $Z$ has at least one maximal element. Let us denote one of these maximal elements by $(m, M)$. We will prove that the open set $M$ is dense in $X$.

Suppose, to the contrary, $M$ is not dense in $X$. Then the set $W = X - \operatorname{cl} M$ would be a nonempty open subset of $X$. Under this assumption there exists a nonempty open subset $G$ of $W$ and a function $h$: $G \to Y$ such that $(h, G) \in S(F, G)$.

Let us denote $C = M \cup G$. Let us define a function $c$: $C \subset Y$ as follows:

$$
\begin{aligned}
c(x) &= m(x) \quad && \text{for each } x \text{ in } M \\
c(x) &= h(x) \quad && \text{for each } x \text{ in } G.
\end{aligned}
$$

It is easy to see that $c \in S(F, C)$, since if $U \subset Y$ is open then $c^{-}(U) = m^{-}(U) \cup h^{-}(U)$. Therefore $(c, C) \in Z$ and we see that $(c, C) > (m, M)$ holds. This is a contradiction. So $M$ must be dense in $X$. Moreover $m$ is an element of $S(F, M)$. This completes the proof.

**Lemma 3.** Let $X$ and $Y$ be topological spaces. Let $F$: $X \to Y$ be an u.q.c. multifunction with compact values. Let there exist a dense open subset $U$ of $X$ and a function $g$: $U \subset Y$ such that $g \in S(F, U)$. Then $F$ has a quasicontinuous selection on $X$.

**Proof**. Let us denote $Z = X - U$. We will show that the following assertion is true.

(a) For each $z$ in $Z$ there exists an element $y_z \in F(z)$ such that for every open neighbourhood $V$ of $y_z$ and for every open neighbourhood $G$ of $z$ there exists a nonempty open subset $W \subset G \cap U$ such that $g(W) \subset V$.

Suppose that, contrary to what we wish to prove, there exists $z \in Z$ such that for every $y \in F(z)$ there exist open sets $V_y \subset Y$ and $U_y \subset X$ and a dense subset $H_y$ of $U_y \cap U$ such that

(i) $y \in V_y$, $z \in U_y$ and $g(H_y) \cap V_y = \emptyset$.

Since $g$ is quasicontinuous on $U$ and $H_y$ is dense in $U \cap U_y$ we have $g^{-}(V_y) \cap \cap (U_y \cap U) = \emptyset$. So

(ii) $g(U_y \cap U) \cap V_y = \emptyset$.

The system of sets $V_y$ mentioned above forms a cover of $F(z)$. There can be selected a finite cover $\{V_1, \dots, V_m\}$ of $F(z)$ from this cover. Let $U_1, \dots, U_m$ be the corresponding open neighbourhoods of $z$ such that

(iii) $g(U_i \cap U) \cap V_i = \emptyset$ for $i = 1, \dots, m$.

Let us denote

$$\text{(iv)} \quad S = U \cap U_1 \cap \dots \cap U_m \cap F^{+}\left(\bigcup_{i=1}^{m} V_i\right)$$

$S$ is a nonempty semiopen set, since the set $U$ is dense and open and the rest of the intersection is a nonempty semiopen set containing $z$.

By (iv) we obtain $g(S) \subset \bigcup_{i=1}^{m} V_i$ but by (iii) for every $t$ in $S$ and for every $i = 1, \dots, m$ $g(t) \cap V_i = \emptyset$ holds. This is a contradiction. Therefore the assertion (a) is true.

Let us define a function $h$: $X \to Y$ as follows;

$h(z) = g(z)$ for any $z \in U$

$h(z) = y_z$ for every $z \in X - U$ where $y_z$ is an element of $F(z)$ mentioned in (a).

The function $h$ is quasicontinuous at each $z \in U$ since if $V \subset Y$ is open and $h(z) \in V$ then $z \in g^{-}(V) \subset h^{-}(V)$. The set $g^{-}(V)$ is semiopen therefore $h^{-}(V)$ is a semineighbourhood of $z$. Using (a) we can see that $h$ is quasicontinuous also in each point $z$ of $X - U$. Therefore $h \in S(F, X)$. This completes the proof.

Before using Lemmas 1, 2, 3 as a proof of Theorem 1 we need the following

**Lemma 4.** Let $X$, $Y$ and $F$ be as in Lemma 1. Then $F$ is u.q.c..

**Proof.** Let $V \subset Y$ be an open set. We shall prove that for every $x \in F^+(V)$ the set $F^+(V)$ is a semineighbourhood of $x$, which implies that $F^+(V)$ is semiopen.

So let $x \in F^+(V)$. Let $F(x) = \{y_1, \ldots, y_h\}$. Since $Y$ is Hausdorff, there exist open disjoint sets $V_1, \ldots, V_n$ such that for $i = 1, \ldots, n$ $y_i \in V_i$ and $V_i \subset V$. Let us denote $S = F^-(V_1, \ldots, V_n)$. We can see that $x \in S$ and $S$ is semiopen. But $S \subset F^+(V)$ holds since the sets $V_i$ are disjoint and for every $s \in S$ card $F(s) = n$. Therefore the set $F^+(V)$ is a semineighbourhood of $X$. Q.E.D.

Using Lemmas 1, 2, 3, 4 we can establish the following

**Theorem 1.** Let $X$ be a topological space. Let $Y$ be a Hausdorff topological space. Let $F: X \to Y$ be a s.q.l.c. multifunction with exactly $n$ values for each $x$ in $X$. Then $F$ has a quasicontinuous selection $f: X \to Y$.

Now we will use the previous results to obtain a selection theorem for one-to-finite quasicontinuous multifunctions.

**Theorem 2.** Let $X$ be a Baire topological space. Let $Y$ be a Hausdorff topological space. Let $F: X \to Y$ be a quasicontinuous multifunction such that, for every $x$ in $X$, the set $F(x)$ is finite. Then $F$ has a quasicontinuous selection $f: X \to Y$.

**Proof.** Let us define for every positive integer $i$ sets $B_i = \{x \in X$: there exists an open neighbourhood $P$ of $x$ such that for every $t$ in $P$ card $F(t) = i\}$ and $L_i = \{x \in X: \text{card}\, F(x) \leq i\}$. Let us denote $U = \bigcup_{i=1}^{\infty} B_i$. Note that $B_i$ are open and pairwise disjoint.

First let us show that $X - U$ is nowhere dense. Suppose, to the contrary, that the set $Z = X - U$ is not nowhere dense. Denote $G = \text{int}\, Z$ and $Z_i = Z \cap L_i$ for $i = 1, 2, \ldots$

The open set $G$ is not of the first category, so the set $Z$ is not of the first category.

$Z = \bigcup_{i=1}^{\infty} Z_i$ holds. Let $n$ be the first integer such that $\text{int}(\text{cl}\, Z_n) \neq \emptyset$. Let us denote $H = \text{int}(\text{cl}\, Z_n) - \text{cl}\, Z_{n-1}$. Then $H$ is a nonempty open subset of $G$. Note that the set $Z_n$ is dense in $H$.

Since $H \cap B_n = \emptyset$ and $H \cap Z_{n-1} = \emptyset$, there exists a point $h \in H$ such that card $F(h) = k$ and $k > n$.

Let $F(h) = \{y_1, \ldots, y_k\}$. There exist open disjoint subsets $V_1, \ldots, V_k$ of $Y$ such that $y_i \in V_i$ $i = 1, \ldots, k$. Let us denote $W = F^-(V_1, \ldots, V_k) \cap H$. $W$ is a nonempty semiopen subset of $H$ and we can see that, for every $t$ from $W$, card $F(t) \geq k > n$.

But this implies $W \cap Z_n = \emptyset$. This is a contradiction with the density of $Z_n$ in $H$.

We have just proved that $Z$ is nowhere dense. Thus, $U$ is dense in $X$. To complete this proof it suffices, by Lemma 3, to show that $S(F, U)$ is nonempty.

By Theorem 1 for every $i \in N$ such that $B_i \neq \emptyset$ there exists a function $g_i \in S(F, B_i)$. Let us define a function $g\colon U \to Y$ as follows:

$g(t) = g_i(t)$ if and only if $t \in B_i$. The function $g$ is a selection of $F$ on $U$. Let $V$ be an open subset of $Y$, then $g^-(V) = \bigcup_{i=1}^{\infty} g^-(V) \cap B_i = \bigcup_{i=1}^{\infty} g_i^-(V) \cap B_i$, which is a semiopen set since $B_i$ are open. Thus, $g$ is quasicontinuous and $g \in S(F, U)$. Q.E.D.

*Acknowledgment*. The author wishes to thank the referee for his valuable comments.

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Received: 23. 11. 1989

## SÚHRN

### KVÁZISPOJITÉ SELEKTORY PRE KONEČNOHODNOTOVÉ MULTIFUNKCIE

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V článku sa dokazujú vety o existencii kvázispojitých selektorov pre kvázispojité multifunkcie s konečnými hodnotami v priestoroch, v ktorých spojité multifunkcie nemusia mať spojité selektory.

## РЕЗЮМЕ

## КВАЗИНЕПРЕРЫВНЫЕ СЕЧЕНИЯ ДЛЯ МНОГОЗНАЧНЫХ ОТОБРАЖЕНИЙ С КОНЕЧНЫМИ ЗНАЧЕНИЯМИ

И. Купка, Братислава

В статье доказаны теоремы об существований квазинепрерывных сечений многозначных отображений с конечными значениями в пространствах, в которых непрерывные многозначные отображения не дольжны иметь непрерывные сечения.