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# POLARITIES ON ALGEBRAICAL STRUCTURES

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## 1. Introduction

A fundamental construction of polarity on an algebraic structure has been described in G. Birkhoff [2]. This polarity has many important properties on lattice ordered groups published for example in books by P. Conrad [3] and V. M. Kopytov [4]. Many authors generalized the classical polarity on lattice ordered groups to other algebraical structures. We mention F. Šik [8], J. Rachůnek [6] and B. Šmarda [9], [10].

**Definition** ([2], V. §7) Let $\varrho$ be a binary relation on a non-empty set $M$. If $X \subseteq M$ then the set $X' = \{m \in M : m\varrho x \text{ for each } x \in X\}$ is called a polar of $X$ in $\varrho$.

By induction we can define $X^n = [X^{n-1})'$, for every natural number $n$.

General properties of sets of all polars which form Boolean algebras are investigated in this paper. Some results show that a natural restriction are polars for symmetric and antireflexive relations (called polarities). Connections between polarities and equivalences are described in the last part of the paper.

## 2. Basic properties of polars

Some basic properties of polars and Galois connections are described in [2], V. §7,8. We can show also the following basic properties:

**2.1.** If $M_o = \{m \in M : m\varrho m\}$ then

a) $\varrho$ is reflexive iff $M_o = M$,

b) $\varrho$ is antireflexive iff $M_o = M'$,

c) $\varrho$ is areflexive iff $M_o = \emptyset$.

**Remark**. Recall that a relation $\varrho$ on $M$ is antireflexive (areflexive, resp.) when $\varrho$ has the following property:

If $x\varrho x$ for some $m \in M$ then $x\varrho y$ for each $y \in M$, (for no $x \in M$ is $x\varrho x$, resp.).

**2.2.** The following assertions are equivalent:

a) $\varrho$ is symmetric,

b) $X \to X''$ is a closure operator on $M$,

c) $X \subseteq X''$ for every $X \subseteq M$.

**2.3.** The following assertiosn are equivalent:

a) $\varrho$ is an equivalence,

b) $a \in a'' = a'$ for each $a \in M$,

c) $a \in a'$ and $A \subseteq A' = A''$ or $A' = \emptyset$ for each $a \in M$, $A \subseteq M$.

**Proof** follows from 2.2 and the fact that $\varrho$ is transitive iff $\emptyset \neq A \subseteq B' \Rightarrow$ $\Rightarrow A' \subseteq B'$ for each $A, B \subseteq M$.

**2.4.** $\varrho$ is a tolerance iff $a \in A' \Rightarrow a \in a' \supseteq A$ for each $A \subseteq M$.

**2.5.** $\varrho$ is a partial order iff for every non-empty sets $A, B \subseteq M$ the following holds:

a) $A \subseteq A' \Leftrightarrow \operatorname{card} A = 1$,

b) $A \neq B \wedge A \subseteq B' \Rightarrow B' \nsubseteq A' \subseteq B'$.

**Proof** follows from the mentioned characterization of transitivity and the following fact: $\varrho$ is antisymmetric iff $A \subseteq B'$, $B \subseteq A' \Rightarrow A = B = \{m\}$, for a suitable $m \in M_o$, where $A, B \subseteq M$, $A \neq \emptyset \neq B$.

**2.6.** 1. $\varrho$ is antireflexive iff $A \cap A' \subseteq M'$, for every $A \subseteq M$.

2. If $\varrho$ is symmetric then the following holds: $\varrho$ is antireflexive iff $A'$, $A''$ are complementary polars, for every $A \subseteq M$.

**Proof** follows from 2.2.

We can investigate other easy results between relations and corresponding polars.

## 3. The set of polars

**3.1. Lemma** (A generalization of [1], Th. 1) If $(A,.)$ is a grupoid, $\bar{\ }$ is a closure operator on $A$ and $X * Y = \overline{X.Y}$, $\vee Y_i = \overline{\cup Y_i}$, for each $X, Y, Y_i \subseteq A$ then the following assertions are equivalent:

1. $X.Y \subseteq X * Y$,
2. $X * Y = X * \overline{Y}$,
3. $X.(\vee Y_i) \subseteq \vee (X.Y_i)$,
4. $X * (\vee Y_i) = \vee (X * Y_i)$.

**Proof.** $3 \Rightarrow 4$: $X * (\vee Y_i) = \overline{X.\vee Y_i} \subseteq \overline{\vee (X.Y_i)} \subseteq \overline{\cup(\overline{X.Y_i})} = \vee (X * Y_i)$ and $\vee (X * Y_i) = \overline{\cup \overline{X.Y_i}} \subseteq \overline{X.\overline{\cup Y_i}} = X * \vee Y_i$.

$4 \Rightarrow 2$: We choose $Y_i = Y$.

$2 \Rightarrow 1$: $X.\overline{Y} \subseteq X * \overline{Y} = X * Y$.

$1 \Rightarrow 3$: $X.\vee Y_i = X.\overline{\cup Y_i} \subseteq X * (\cup Y_i) = \overline{X.(\cup Y_i)} = \overline{\cup X.Y_i} = \vee (X.Y_i)$.

**Remark**. If we change factors in the operations $*$ and . then we receive a similar lemma. For example 3. $(\vee Y_i).X \subseteq \vee (Y_i.X)$.

**3.2. Corollary.** Let $(A, .)$ be a semigroup and $\bar{\ }$ be a closure operator on $A$. Then the set $\sigma(A)$ of all closed sets in $A$ with regard to $\bar{\ }$ is a quantale, where $X * Y = \overline{X . Y}$, $\vee\, Y_i = \overline{\cup\, Y_i}$ for every $X, Y, Y_i \in \sigma(A)$ iff $X * \overline{Y} = X * Y = \overline{X} * Y$ for every $X, Y \subseteq A$.

**Proof.** Let us recall (see [5]) that $\sigma(A)$ is a quantale when the operation $*$ is associative and $X * \vee\, Y_i = \vee\, (X * Y_i)$, $\vee\, Y_i * X = \vee\, (Y_i * X)$ hold for every $X$, $Y_i \in \sigma(A)$. From Lemma 3.1 and the Remark after 3.1 it follows that we must prove only the associativity of $*$: But $X * (Y * Z) = X * (\overline{Y . Z}) = X * (Y . Z) = \overline{X . (Y . Z)} = \overline{(X . Y) . Z} = (\overline{X . Y}) * Z = (X * Y) * Z$, for every $X, Y, Z \in \sigma(A)$.

**3.3. Theorem.** If $\varrho$ is a symmetric relation on a semigroup $(A, .)$ then the following assertions are equivalent:

1. The set $\varrho(A)$ of all polars on $A$ in $\varrho$ is a complete Boolean algebra with regard to the operations $X * Y = (X . Y)''$, $\vee\, X_i = (\cup\, X_i)''$ for each $X, Y, X_i \in \varrho(A)$.

2. $\varrho$ is an antireflexive relation with the following properties:

a) $a \varrho b \Rightarrow a . b \varrho m$ for every $m \in A$,

b) $X * Y = Y * X$, $X * X = X$ holds for every $X, Y \in \varrho(A)$ and $X * Y = X * Y'' = X'' * Y$ holds for every $X, Y \subseteq A$.

**Proof.** $1 \Rightarrow 2$: a) We have $A = x' \vee x'' = (x' \cup x'')'' = (x' \cap x'')'$ for each $x \in A$ and thus $x' \cap x'' \subseteq A'$. It means that $x \varrho m$ for each $m \in A$ and $\varrho$ is antireflexive. Further, $a \varrho b \Rightarrow a \in b' \Rightarrow a . b \in a . a' \subseteq a'' * a' = A' \Rightarrow a . b \varrho m$ for $a, b \in A$ and each $m \in A$.

b) It follows immediately from 3.1 and the Remark after 3.1.

$2 \Rightarrow 1$: Propositions 2.2, 3.2 imply that $\varrho(A)$ is an idempotent quantale and Proposition 2.6 implies that $X' \vee X'' = A$ holds for each $X \subseteq A$. We have $x \varrho y$ for every $x \in X'$, $y \in X''$ and thus $x . y \varrho m$ holds for every $m \in A$. From this $X' . X'' \subseteq A'$, i.e., $X' * X'' = (X . X'')'' = A'$. Finally, $X'$, $X''$ are complementary polars.

Now, $X'' = \vee\, (x'' : x \in X'')$ and $x' \vee X'' = A$ holds evidently for each $X \subseteq A$ and each $x \in X''$. It means that $A$ is a regular quantale and Theorem 2.5 from [5] implies that $A$ is a locale. Finally, $A$ is a distributive complementare complete lattice, i.e., $A$ is a complete Boolean algebra.

**3.4. Corollary.** Let $\varrho$ be a symmetric relation on a semigroup $(A, .)$ and the set $\varrho(A)$ of all polars on $A$ in $\varrho$ be a complete Boolean algebra with regard to the operations $*$, $\vee$. Then the following holds:

1. $X * Y = X \cap Y$, for each $X, Y \in \varrho(A)$.
2. Polars from $\varrho(A)$ are ideals of $A$.
3. $x \varrho y . z \Rightarrow x \varrho z . y$, for each $x, y, z \in A$.

**Proof.** 1. $\varrho(A)$ is an idempotent quantale and [7, Prop. 1] implies that $\varrho(A)$ is a locale.

2. $A . X \subseteq A * X = A \cap X = X$ and $X . A \subseteq X$ similarly, for each $X \in \varrho(A)$.

3. We have $x\varrho y.z \Rightarrow x\in(y.z)' \subseteq (y''.z'')' = (y''*z'')' = (z''*y'')' = (z''.y'')' \Rightarrow$ $\Rightarrow x\varrho z.y$. The second implication follows similarly.

**3.5. Theorem.** If $\varrho$ is a symmetric relation on a set $M$ then the following assertions are equivalent:

1. The set $\varrho(M)$ of all polars on $M$ in $\varrho$ is a complete Boolean algebra with regard to the operations $X \wedge Y = X \cap Y$, $X \vee Y = (X \cup Y)''$, for $X, Y \in \varrho(M)$, where $X'$ is the complement of $X$.

2. $\varrho$ is an antireflexive relation and $(X'' \cap Y) \cup (X \cap Y'') \subseteq (X \cap Y)''$ holds for every $X, Y \subseteq M$.

3. $x\varrho y \Leftrightarrow x'' \cap y'' = M'$ for every $x, y \in M$.

**Proof.** $1 \Rightarrow 2$: Similarly as in the proof of 3.3.

$2 \Rightarrow 1$: Follows from 2.6,2 and 3.1.

$1 \Rightarrow 3$: Theorems 3, 4 from [8] imply that $\varrho(M)$ is a Boolean algebra iff $\varrho$ is antireflexive and $x$ non $\varrho y \Rightarrow$ there exists $z \in M$ such that $z$ non $\varrho z$, $z \in x'' \cap y''$. If $x\varrho y$ then $x \in y'$ and $x'' \cap y'' \subseteq x'' \cap x' \subseteq M_o = M'$ follow, i.e., $x'' \cap y'' = M'$. On the contrary, $x'' \cap y'' = M'$ implies $x\varrho y$ immediately.

$3 \Rightarrow 1$: We have $x\varrho x \Rightarrow x'' = M' \Rightarrow x \in M'$ and thus $\varrho$ is antireflexive. Now, $x$ non $\varrho y \Rightarrow x'' \cap y'' \neq M'$ and then $z \notin M'$ exists, i.e., $z$ non $\varrho z$ and $z \in x'' \cap y''$. Finally, $\varrho(M)$ is a complete Boolean algebra.

## 4. Polarities

From the previous results we can recognize the motivation for the following definition.

**Definition**. A symmetric antireflexive relation on a set $M$ is called a polarity on $M$.

**4.1.** ([9], Prop. 2.8) If $S$ is a system of all subsets in $M$ which has a given property $(K)$, $S$ is closed with respect to intersections, $\cup S = M$ and $P = \cap\{\langle m\rangle : m \in M\}$, where $\langle m\rangle$ is the smallest set from $S$ containing $m$, then the following holds: $\varrho$ is the greatest polarity on $M$ such that all polars on $M$ in $\varrho$ form a complete Boolean algebra and $M' = P$ iff $\varrho$ has the following property: $a\varrho b \Leftrightarrow$ $\Leftrightarrow \langle a\rangle \cap \langle b\rangle = \{O\}$.

**4.2.** If $\sigma$ is a polarity on a set $M$ then

a) $a\sigma b \Rightarrow a'' \cap b'' = M'$, for $a, b \in M$,

b) $\sigma(M)$ is a complete Boolean algebra iff we have: $a\sigma b \Leftrightarrow a'' \cap b'' = M'$, for every $a, b \in M$.

**Proof.** a) We have $a\sigma b \Rightarrow a \in b'$ and if $x \in a'' \cap b''$ then $x \in a'' \subseteq b' \Rightarrow x\sigma x \Rightarrow$ $\Rightarrow x \in M'$.

b) see Theorem 3.5, $1 \Leftrightarrow 3$.

**Definition**. Let $\sigma$ be a polarity on a set $M$. Then the relation $\varrho(\sigma)$ on $M$ such that

$$a\varrho(\sigma)\, b \Leftrightarrow a'' = b'' \text{ in } \sigma, \text{ for } a,\, b \in M,$$

is an equivalence, which we shall call an equivalence induced by the polarity $\sigma$.

The corresponding decomposition $M/\varrho(\sigma)$ is a partially ordered set ($\alpha \leq \beta$, $\alpha$, $\beta \in M/\varrho(\sigma) \Leftrightarrow a'' \subseteq b''$ for every $a \in \alpha$, $b \in \beta$) with the smallest element $O_{\varrho(\sigma)} = M'$.

**Definition**. Let $\varrho$ be an equivalence on a set $M$ and the corresponding decomposition $M/\varrho$ be a partially ordered set with the smallest element $O_\varrho$. Then the relation $\sigma(\varrho)$ on $M$ defined by

$$a\sigma(\varrho)\, b \Leftrightarrow A_a \cap A_b = O_\varrho, \text{ for } a,\, b \in M,$$

where $A_a = \{\gamma \in M/\varrho\colon \gamma \leq \alpha \in M/\varrho,\, a \in \alpha\}$ and $A_b$ similarly, is a polarity which we shall call a polarity induced by the equivalence $\varrho$ belonging to the given partial order on $M/\varrho$.

**4.3.** If $\varrho$ is an equivalence on M then we have:

1. $\varrho(\sigma(\varrho)) = \varrho$.
2. $\sigma(\varrho(M))$ is a complete Boolean algebra.

**Proof**. 1. We have $a\varrho b \Leftrightarrow A_a = A_b \Leftrightarrow \{x\sigma(\varrho)\, a \Leftrightarrow x\sigma(\varrho)\, b$ for $x \in M\} \Leftrightarrow a'' = b'' \Leftrightarrow a\varrho(\sigma(\varrho))\, b$, for arbitrary partial order of classes of $M/\varrho$ with the smallest element and for every $a$, $b \in M$.

2. Proposition 4.2 yields that it suffices to prove that $a'' \cap b'' = M' \Rightarrow a\sigma(\varrho)\, b$ for every $a$, $b \in M$. If $\alpha$, $\beta \in \mathrm{M}/\varrho$, $\alpha \leq \beta$ then $A_a \subseteq A_b$ for every $a \in \alpha$, $b \in \beta$ and thus $A_z \cap A_b = O_\varrho \Rightarrow A_z \cap A_a = O_\varrho$, i.e., $z\sigma(\varrho)\, b \Rightarrow z\sigma(\varrho)\, a$ for $z \in M$. It means that $a'' \subseteq b''$. Now, if $a'' \cap b'' = M'$ and $\gamma \in A_a \cap A_b$, $c \in \gamma \in M/\varrho$ then $\alpha \geq \gamma$, $\beta \leq \gamma$ hold for $\alpha$, $\beta \in M/\varrho$ such that $\alpha = \{x \in M\colon x'' = a''\}$, $\beta = \{x \in M\colon x'' = b''\}$. Finally, we have $a'' \cap b'' \supseteq c''$, i.e., $c'' = M' = O_\varrho$ and thus $A_a \cap A_b = O_\varrho$ and $a\sigma(\varrho)\, b$.

**4.4. Theorem.** If $\varrho$ is a map which maps a polarity $\sigma$ on $M$ onto the equivalence $\varrho$ induced by $\sigma$, then the following assertions are equivalent:

1. $\sigma(M)$ is a complete Boolean algebra for each polarity $\sigma$ on $M$.
2. $\sigma = \sigma(\varrho(\sigma))$ for each polarity $\sigma$.
3. $\varrho$ is an injection.

**Proof**. $1 \Rightarrow 2$: If $\bar{\sigma} = \sigma(\varrho(\sigma))$ then $a\sigma b \Rightarrow a'' \cap b'' = M'$ (by 4.2) $\Rightarrow x'' \subseteq M'$, for every $x \in \varkappa \in A_a \cap A_b$, i.e., $x$ satisfying $x'' \subseteq a'' \cap b'' \Rightarrow A_a \cap A_b = M' \Rightarrow a\bar{\sigma}b$.

Otherwise, $a\bar{\sigma}b \Rightarrow A_a \cap A_b = M'$ holds because $M' = O_{\varrho(\sigma)} \Rightarrow c \in \gamma \in M/\varrho$ for $c \in a'' \cap b''$, where $\gamma \leq \alpha$, $\gamma \leq \beta \Rightarrow c'' = M' \Rightarrow a'' \cap b'' = M' \Rightarrow a\sigma b$ holds (see again 4.2).

$2 \Rightarrow 3$: If $\sigma_1$, $\sigma_2$ are polarities on $M$ such that $\varrho(\sigma_1) = \varrho(\sigma_2)$ then $a''_{\sigma_1} = a''_{\sigma_2}$

for every $a \in M$. Therefore decompositions $M/\varrho(\sigma_1)$, $M/\varrho(\sigma_2)$ are equal including partial orders defined by $\varrho(\sigma_1)$ and $\varrho(\sigma_2)$. Then $\sigma_1 = \sigma(\varrho(\sigma_1)) = \sigma(\varrho(\sigma_2)) = \sigma_2$ and $\varrho$ is an injection.

$3 \Rightarrow 1$: If $a'' \cap b'' = M'$ for $a, b \in M$ then $c'' \subseteq a'' \cap b''$ for each $\gamma \in A_a \cap A_b$ and each $c \in \gamma$, i.e., $c \in M'$ and $A_a \cap A_b = M' = O_{\varrho(\sigma)}$. From this $a\bar{\sigma}b$. Proposition 4.2 implies that $a\bar{\sigma}b \Leftrightarrow a'' \cap b'' = M'$, $\bar{\sigma}(M)$ is a complete Boolean algebra and Proposition 4.3 implies that $\varrho(\bar{\sigma}) = \varrho(\sigma(\varrho(\sigma))) = \varrho(\sigma)$. The fact that $\varrho$ is an injective map implies $\bar{\sigma} = \sigma$ and thus $\sigma(M)$ is also a complete Boolean algebra.

**4.5. Corollary.** Let $E$ be the set of all equivalences on a set $M$ and $P$ be the set of all polarities on $M$ such that $\sigma(M)$ is a complete Boolean algebra. Then the mapping $\varrho\colon P \to E$ such that it maps $\sigma \in P$ onto the equivalence $\varrho(\sigma)$ induced on $M$ by $\sigma$ is a bijection and the mapping $\sigma\colon E \to P$ which maps $\varrho \in E$ onto $\sigma(\varrho)$ is the inverse mapping to $\varrho$.

**Proof** follows from 4.3 and 4.4.

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# РЕЗЮМЕ

## ПОЛЯРНОСТИ НА АЛГЕБРАИЧЕСКИХ СТРУКТУРАХ

Богумил Шмарда, Брно

В теории решеточно упорядоченных групп хорошо известны понятия поляра и полярность. В этой статье изучаются эти понятия в обшчности на любом множестве. Именно, свойства множества всех поляр которые являются булевыми алгебрами для симметрических и антирефлексивных отношений. В конце описана связь между полярностями и эквивалентностями.

# SÚHRN

## POLARITY NA ALGEBRAICKÝCH STRUKTURÁCH

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V teorii svazově uspořádaných grup jsou dobře známy pojmy polára a polarita. V tomto článku se vyšetřují tyto pojmy obecně na libovolné množině. Zejména vlastnosti množiny všech polár, které jsou Booleovými algebrami pro symetrické a antireflexivní relace. Závěrem je popsán vztah mezi polaritami a ekvivalencemi.