UNIVERSITAS COMENIANA
ACTA MATHEMATICA UNIVERSITATIS COMENIANAE
LVIII—LIX

A REFINEMENT OF SIERPIŃSKI'S INEQUALITIES

HORST ALZER, FRG

Abstract. We prove: If A_n, G_n and H_n denote the unweighted arithmetic, geometric and harmonic means of the positive real numbers $x_1, \ldots, x_n$ $(n \geq 3)$, then

$$\frac{1}{n}(G_n)^{n-1}(G_n - H_n) \leq (A_n)^{n-1}H_n - (G_n)^n$$

and

$$\frac{1}{n}(H_n)^{n-1}(A_n - G_n) \leq (G_n)^n - A_n(H_n)^{n-1},$$

with equality holding if and only if $x_1 = \ldots = x_n$.

In this note we designate by A_n, G_n and H_n the unweighted arithmetic, geometric and harmonic means of the positive real numbers $x_1, \ldots, x_n$, i.e.

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i,\ G_n = \prod_{i=1}^{n} x_i^{1/n} \text{ and } H_n = n \Big/ \sum_{i=1}^{n} 1/x_i.$$

For the classical double inequality

$$H_n \leq G_n \leq A_n \tag{1}$$

one can find many proofs in literature. We refer to the recently published book "Means and Their Inequalities" by Bullen/Mitrinović/Vasić [4] which contains more than fifty approaches to (1). Besides different proofs for (1) a lot of interesting generalizations, sharpenings and variants have been published [4]. A very remarkable refinement of (1) was discovered by W. Sierpiński [7] in 1909. He proved the inequalities

$$A_n(H_n)^{n-1} \leq (G_n)^n < (A_n)^{n-1}H_n, \tag{2}$$

where the sign of equality holds for $n \geq 3$ if and only if $x_1 = \ldots = x_n$. If $n = 1, 2$, then (2) reduces to identities. In the last years Sierpiński's result has evoked new interest among several mathematicians and diverse proofs as well as extensions and sharpenings of (2) were published [1—3, 5, 6, 8]. The aim of this paper is to prove a new refinement of Sierpiński's inequalities. We establish the following.

Theorem. For all positive real numbers $x_1, \ldots, x_n$ $(n \geq 3)$ we have

$$\frac{1}{n}(G_n)^{n-1}(G_n - H_n) \leq (A_n)^{n-1}H_n - (G_n)^n \tag{3}$$

and

$$\frac{1}{n}(H_n)^{n-1}(A_n - G_n) \leq (G_n)^n - A_n(H_n)^{n-1}. \tag{4}$$

The sign of equality holds in (3) as well as in (4) if and only if $x_1 = \ldots = x_n$.

Proof. If we replace in (3) the numbers $x_1, \ldots, x_n$ by $1/x_1, \ldots, 1/x_n$, then we obtain (4). Hence it suffices to establish inequality (3). If $x_1 = \ldots = x_n$, then the sign of equality holds in (3). Without loss of generality we may assume

$$0 < x_1 \leq x_2 \leq \ldots \leq x_n, \qquad x_1 < x_n,$$

and we shall establish (3) with "$<$" instead of "$\leq$". We use induction on n.

To prove (3) for $n = 3$ we investigate the function

$$f\colon M = \{(a, b) \in R^2 \mid 0 \leq a \leq b \leq 1\} \to R,$$

$$f(a, b) = (a - b)^2 - 2(a + b) + 1 + 3(ab)^{2/3}.$$

We denote the absolute minimum of f by (x, y). If (x, y) is an interior point of M, then

$$\nabla f(x, y) = 0$$

and we obtain the equations

$$\frac{1}{2}f_a(x, y) = x - y - 1 + (xy)^{-1/3}y = 0, \tag{5}$$

$$\frac{1}{2}f_b(x, y) = y - x - 1 + (xy)^{-1/3}x = 0,$$

which lead to

$$(y - x)[(xy)^{-1/3} - 2] = 0.$$

Since $x \neq y$ we get $(xy)^{-1/3} = 2$ and equation (5) yields $x + y = 1$, which implies $x = 1/2 - 1/\sqrt{8}$ and $y = 1/2 + 1/\sqrt{8}$; and because of $f(x, y) = 1/4 > 0 = f(0, 1)$ f attains its minimum at a boundary point. If $x = 0$, then

$$f(0, y) = (y - 1)^2 \geq 0.$$

If $x = y$, then

$$f(x, x) = 3x^{4/3} - 4x + 1$$

is strictly decreasing on $[0, 1]$ and we get

$$f(x, x) \geq f(1, 1) = 0.$$

Finally, if $y = 1$, then

$$f(x, 1) = x^2 - 4x + 3x^{2/3}$$

is concave on $[0, (1/3)^{3/4}]$ and convex on $[(1/3)^{3/4}, 1]$. Because of

$$f(1, 1) = 0 \text{ and } \left.\frac{\mathrm{d}f(x, 1)}{\mathrm{d}x}\right|_{x=1} = 0$$

we conclude

$$f(x, 1) \geq 0 \text{ on } [0, 1].$$

Hence we have proved:

$$\text{If } 0 < a \leq b \leq 1,\ a < 1, \text{ then } f(a, b) > 0. \tag{6}$$

Setting $a = x_1/x_3$ and $b = x_2/x_3$, implication (6) leads to (3) with $n = 3$ and with "<" instead of "≤".

Next we assume that (3) is valid for $n - 1$ with $n \geq 4$. We define

$$p\colon [A_{n-1}, \infty) \to R,$$

$$p(x) = n\left[\frac{n-1}{n} A_{n-1} + \frac{x}{n}\right]^{n-1} + (G_{n-1})^{(n-1)^2/n} x^{(n-1)/n}$$

$$- \frac{n^2 - 1}{n} \frac{(G_{n-1})^{n-1}}{H_{n-1}} x - \frac{n+1}{n} (G_{n-1})^{n-1}$$

and we will show:

$$p(x_n) > 0 \tag{7}$$

which is equivalent to (3) with "<" instead of "≤".

We prove:

$$p''(x) > 0 \text{ for } x \geq A_{n-1}, \tag{8}$$

$$p'(A_{n-1}) \geq 0, \tag{9}$$

and

$$p(A_{n-1}) \geq 0. \tag{10}$$

Because of $x_n > A_{n-1}$, inequalities (8), (9) and (10) imply (7). In what follows we use the following abbreviations:

$a = A_{n-1}$, $g = G_{n-1}$ and $h = H_{n-1}$.

A simple calculation yields

$$\frac{n^2}{n-1} x^{1+1/n} p''(x) = n(n-2) \left[\frac{n-1}{n} a + \frac{x}{n}\right]^{n-3} x^{1+1/n} - g^{(n-1)^2/n}$$

and from the arithmetic mean-geometric mean inequality we conclude

$$\frac{n^2}{n-1} x^{1+1/n} p''(x) \geq n(n-2)\, a^{(n-1)^2/n} - g^{(n-1)^2/n} > 0.$$

This proves inequality (8).

From the induction hypothesis:

$$-g^{n-1}h^{-1} \geq -\frac{n-1}{n} a^{n-2} - \frac{1}{n} g^{n-2} \tag{11}$$

we obtain

$$p'(a) = (n-1)\, a^{n-2} + \frac{n-1}{n} g^{(n-1)^2/n} a^{-1/n} - \frac{n^2-1}{n} g^{n-1} h^{-1}$$

$$\geq \frac{n-1}{n^2} a^{-1/n} [a^{(n-1)^2/n} + n g^{(n-1)^2/n} - (n+1)\, g^{n-2} a^{1/n}].$$

If we denote the term in square brackets on the right-hand side by $q(a)$, then we have

$$n a^{1-1/n} q'(a) = (n-1)^2\, a^{n-2} - (n+1)\, g^{n-2} > 0$$

and

$$q(a) \geq q(g) = 0$$

which implies (9).

Finally we verify inequality (10).

From (11) we obtain

$$p(a) = n a^{n-1} + g^{(n-1)^2/n} a^{(n-1)/n} - \frac{n^2-1}{n} g^{n-1} h^{-1} a - \frac{n+1}{n} g^{n-1}$$

$$\geq \frac{n^2+n-1}{n^2} a^{n-1} + a^{(n-1)/n} g^{(n-1)^2/n} - \frac{n+1}{n} g^{n-1} - \frac{n^2-1}{n^2} \text{a}\, g^{n-2}. \tag{12}$$

We designate the right-hand side of (12) by φ(a).

From

$$\frac{n^2}{n-1} a^{1+1/n} \varphi''(a) = (n^2 + n - 1)(n-2) a^{(n-1)^2/n} - g^{(n-1)^2/n} > 0,$$

$$\varphi'(g) = \frac{(n-1)(n^2+n-2)}{n^2} g^{n-2} > 0,$$

and

$$\varphi(g) = 0$$

we conclude $\varphi(a) \geq 0$ and hence $p(a) \geq 0$.
This completes the proof of the Theorem.

Remark. If $n = 2$, then in (3) and (4) the inequalities must be reversed.

REFERENCES

1. Alzer, H.: Über die Ungleichung zwischen dem geometrischen und dem arithmetischen Mittel. Quaest. Math. 10 (1987), 351—356.
2. Alzer, H., Ando, T. and Nakamura, Y.: The inequalities of W. Sierpiński and Ky Fan. J. Math. Anal. Appl. (to appear).
3. Bauer, H.: A class of means and related inequalities. Manuscripta Math. 55 (1986), 199—212.
4. Bullen, P. S., Mitrinović, D. S. and Vasić, P. M.: Means and Their Inequalities. D. Reidel Publ. Co., Dordrecht 1988.
5. Mitrinović, D. S. and Vasić, P. M.: On a theorem of W. Sierpiński concerning means. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 544—576 (1976), 113—114.
6. Pečarić, J. E. and Wang, C.-L.: An extension of a Sierpiński inequality. Congr. Numer. 61 (1988), 35—38.
7. Sierpiński, W.: Sur une inegalité pour la moyenne arithmétique, géométrique et harmonique. (Polish). Warsch. Sitzungsber. 2 (1909), 354—357.
8. Wang, C.-L.: An extension ot two sequences of inequalities of Mitrinović and Vasić. Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. Fiz. 634—677 (1979), 94—96.

Author's address:

Horst Alzer
Morsbacher Str. 10
5220 Waldbröl
Germany

Received: 10. 10. 1989

РЕЗЮМЕ

УТОЧНЕНИЕ НЕРАВЕНСТВ ШЕРПИНЬСКОГО

Горст Алгер, ФРГ

В работе доказываются неравенства (3) и (4) (см. Теорему) для A_n, G_n и H_n, где A_n, G_n и H_n означают арифметическое, геометрическое и гармоническое невзведшенное среднее из действительных чисел $x_1, \ldots, x_n$ ($n \geq 3$).

SÚHRN

ZJEMNENIE SIERPIŃSKÉHO NEROVNOSTÍ

Horst Alzer, SRN

V práci sa dokazujú nerovnosti (3) a (4) (viď Teorému) pre A_n, G_n a H_n, kde A_n, G_n a H_n znamenajú po rade aritmetickú, geometrickú a harmonickú nevyváženú strednú hodnotu z reálnych čísel $x_1, \ldots, x_n$ pre $n \geq 3$.