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A PROOF OF RADO'S THEOREM
ON REMOVABLE SINGULARITIES OF ANALYTIC FUNCTIONS

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Abstract. In this self-contained paper, a lucid proof of Rado's theorem on removable singularities of analytic functions is given.

In a 1924 paper, T. Rado proved a Lemma which is essentially equivalent to the Theorem below concerning a rather significant case of the removable singularities of analytic functions. Some 27 years later, in 1951, Rado's result was further modified by H. Behnke and K. Stein. In 1952, H. Cartan reformulated the result in its present form giving a proof based on the properties of subharmonic functions. In 1956, E. Heinz improved the proof based on the representation of harmonic functions by means of the Poisson's integral. Further generalizations were given by I. Glicksberg in 1964 and E. L. Stout in 1968 (see references in [2] and [4]).

In the 1967 paper of R. Kaufman [3], a proof of Rado's theorem was given in an extremely terse form. The present self-contained paper is written based on the central lines of Kaufman's proof. Certainly, the important theorem of Rado and an elegant and easy to follow proof of it should be of interest to anyone concerned with the properties of analytic functions

In what follows, unless otherwise indicated, all functions are complex valued and of a complex variable.

Lemma 1. *Let f be a continuous function on the closed unit disk $\bar{D}$ and analytic at every point of the open unit disk D at which f is not zero. Then the maximum modulus of f is attained at a point on the boundary ∂D of D.*

Proof. Let A be the set of all the points of D where f is not zero, i.e.,

$$A = \{z \mid z \in D \text{ and } f(z) \neq 0\} \tag{1}$$

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Let C be the set of all complex numbers. Clearly, A is the preimage of the open set $C - \{0\}$ under the continuous function f from D into C. Thus, A is an open subset of D. Consequently, f is continuous on $\bar{A}$ and analytic in the open set A. Therefore [1] the maximum modulus of f is attained at a point m on the boundary ∂A of A. However, $m \in (\partial D \cap \partial A)$ since f is continuous in D and $f(z) = 0$ for every $z \in (D - A)$. Thus, $m \in \partial D$, as desired.

Remark. The open set A given by (1) is defined in connection with the function f. Nevertheless, as Lemma 2 shows, any function g which is defined only on $\bar{A}$ and which is continuous on $\bar{A}$ and analytic in A shares with f the property of attaining its maximum modulus on ∂D. This is indeed a curious fact. We observe that, in part, this is due to the fact (as the proof of Lemma 1 indicates) that:

$$(\partial D \cap \partial A) \neq \emptyset \text{ whenevever } A \neq \emptyset. \tag{2}$$

Lemma 2. *Let f, D and $A \neq \emptyset$ be as in Lemma 1 and* (1). *Let g be a function continuous on $\bar{A}$ and analytic in A. Then the maximum modulus of g is attained at a point on the boundary ∂D of D.*

Proof. Clearly, the function gf is continuous on $\bar{A}$ and analytic in (the open set) A. Thus, its maximum modulus $\max|gf|$ is attained on ∂A. However, since $f(z) = 0$ for every $z \in (D - A)$, we see from (2) that $\max|gf|$ is attained on ∂D. A similar argument shows that for every positive integer n, $\max|g^n f|$ is attained on ∂D. Consequently, for $c \in A$, we have:

$$|g(c)|^n |f(c)| \leq (\max|g|)^n (\max|f|)$$

or

$$|g(c)|\,|f(c)|^{1/n} \leq \max|g|\,(\max|f|)^{1/n}$$

which by letting $n \to \infty$ implies $|g(c)| \leq \max|g|$ for every $c \in A$, as desired.

Lemma 3. *Let f, D and $A \neq \emptyset$ be as in Lemma 1 and* (1). *Then A is dense in D.*

Proof. We must show that $D \subset \bar{A}$. Let us assume the contrary. Thus, $D - \bar{A}$ and A are nonempty disjoint open subsets of D implying that there exists a boundary point b of A such that $b \in D$. But then it can be readily verified that there exist $m \in (D - \bar{A})$ and $a \in A$ such that

$$|a - m| < 1 - |m| \tag{3}$$

We observe that $1 - |m|$ is the shortest distance between m and any point on ∂D. Next, let us consider the function g given by $g(z) = 1/(z - m)$. Clearly, g is continuous on $\bar{A}$ and analytic in A. Therefore, by Lemma 2, $\max|1/(z - m)|$ is attained on ∂D. But then, because of our above observation, $\max|1/(z - m)| \leq$ $\leq 1/(1 - |m|)$. However, by (3), for $z = a$ we have $1/|a - m| > 1/(1 - |m|)$. This contradicts our assumption and Lemma 3 is proved.

Lemma 4. *Let f be a continuous function on the boundary ∂D of the unit disk D. Then for every $\varepsilon > 0$ there exists a polynomial P such that*

$$|\operatorname{Re}(f(z) - P(z))| < \varepsilon \text{ for every } z \in \partial D \tag{4}$$

Proof. Let us consider the continuous real valued function $\operatorname{Re}(f)$ on ∂D. Let us parametrize ∂D by the arc length θ and let $\varepsilon > 0$ be given. Then it is always possible to determine a truncation $a_0 + a_1 \cos\theta + b_1 \sin\theta + \dots + a_n \cos n\theta + b_n \sin n\theta$ of the Fourier series of a polygonal line which suitably approximates the graph of $\operatorname{Re}(f)$ on ∂D, and, such that

$$|\operatorname{Re} f(z) - (a_0 + a_1 \cos\theta + b_1 \sin\theta + \dots + a_n \cos n\theta + b_n \sin n\theta)| < \varepsilon \tag{5}$$
$$\text{for every } z \in \partial D \text{ i.e., } z = e^{i\theta}$$

Clearly $\sum_{k=0}^{n} (a_k \cos k\theta + b_k \sin k\theta)$ appearing in (5) is (say, with $b_0 = 0$) the real part of the polynomial $P(z)$ given by

$$P(z) = a_0 + (a_1 - ib_1)z + (a_2 - ib_2)z^2 + \dots + (a_n - ib_n)z^n$$

which, in view of (5), establishes (4).

In (5), letting ε run through $1/k$ with $k = 1, 2, 3, \dots$, in view of (4) we obtain:

Lemma 5. *Let f be a continuous function on the boundary ∂D of the unit disk D. Then there exists a sequence (P_k) of polynomials such that*

$$|\operatorname{Re}(f(z) - P_k(z))| < 1/k \text{ for every } z \in \partial D \text{ and } k = 1, 2, \dots \tag{6}$$

For the proof of the Theorem below, we need also to observe that:

$$|\operatorname{Re}(z)| < 1/k \text{ iff } |e^z| < e^{1/k} \text{ and } |e^{-z}| < e^{1/k} \text{ for } k = 1, 2, \dots \tag{7}$$

Finally, we prove:

Theorem (Rado). *Let f be a continuous function on the closed unit disk $\bar{D}$ and analytic at every point of the open unit disk D at which f is not zero. Then f is analytic in D.*

Proof. From (6) and (7) it follows that there exists a sequence of polynomials P_k such that for $k = 1, 2, 3, \dots$

$$|e^{f(z) - P_k(z)}| < e^{1/k} \text{ and } |e^{P_k(z) - f(z)}| < e^{1/k} \text{ for } z \in \partial D \tag{8}$$

Without loss of generality we assume that f is not the zero function, i.e., A (as given by (1)) is nonempty. By the hypothesis of the Theorem, for every k, we see that $e^{f(z) - P_k(z)}$ as well as $e^{P_k(z) - f(z)}$ is a continuous function on $\bar{A}$ and analytic in A. Hence, by Lemma 2, the maximum modulus of either of these functions (with $z \in \bar{A}$) is attained on ∂óéšD. Thus, the inequalities in (8) are valid not only for $z \in \partial D$ but also for $z \in \bar{A}$. From Lemma 3, however, it follows that A is dense in D and therefore (because of the continuity of the functions involved) the inequalities in (8) are valid also for $z \in \bar{D}$. From this and (7), we conclude that the inequalities in (6) are valid also for $z \in \bar{D}$, i.e.,

$$|\operatorname{Re}(f(z) - P_k(z))| < 1/k \text{ for every } z \in \bar{D} \text{ and } k = 1, 2, \ldots \tag{9}$$

Since $P_k(z)$ is a polynomial for every k, (9) implies that the sequence $(\operatorname{Re} P_k(z))$ of harmonic functions converges uniformly to $\operatorname{Re} f(z)$ on $\bar{D}$. Consequently, $U = \operatorname{Re} f$ is a harmonic function in D with continuous partial derivatives U_x and U_y in D. Since $\operatorname{Im} f = \operatorname{Re}(-if)$, the same reasoning shows that $V = \operatorname{Im} f$ is a harmonic function in D with continuous partial derivatives V_x and V_y in D. However, f is analytic in A (as given (by (1)) and hence the Cauchy-Riemann equations $U_x = V_y$ and $U_y = -V_x$ are valid in A. But then since A is dense in D and, as just noted, these partial derivatives exist and are continuous throughout D, the above Cauchy-Riemann equations are valid also in D. Thus, f is analytic in D, as desired.

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SÚHRN

DÔKAZ RADÓOVEJ VETY O ODSTRÁNITEĽNÝCH SINGULARITÁCH ANALYTICKÝCH FUNKCIÍ

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V článku je podaný jasný a pomerne jednoduchý dôkaz Radóovej vety o odstrániteľných singularitách analytických funkcií. Založený je na myšlienkach predchádzajúceho dôkazu R. Kaufmana.

РЕЗЮМЕ

ДОКАЗАТЕЛЬСТВО ТЕОРЕМЫ РАДО О УСТРАНИМЫХ ОСОБЕННОСТЯХ АНАЛИТИЧЕСКИХ ФУНКЦИЙ

Александр Абиан, США

В статье просто и ясно доказана теорема Радо о устранимых особенностях аналитических функций. Доказательство основано на идеях предидущего доказательства принадлежающего Р. Кауфману.