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AN APPLICATION OF CHANGE OF INDEPENDENT VARIABLE IN THE OSCILLATION THEORY OF DIFFERENTIAL EQUATIONS WITH UNBOUNDED DELAYS

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1. Introduction

We are interested in oscillation theorems for functional differential equations of the form

$$x'(t) + \sum_{i=1}^{n} p_i(t)\, x(t - \tau_i(t)) = 0, \tag{1.1}$$

where $n \geqq 1$, p_i, τ_i: $(t_0, \infty) \to (0, \infty)$ are continuous functions and $\lim_{t\to\infty} (t - \tau_i(t)) = \infty$, $i = 1, \dots, n$.

The special case of (1.1) in which the p_i' s and τ_i's are constants has been studied by a number of authors. For instance, Tramov [17] has proved the following

Theorem A. Suppose that $p_i(t) = p_i$ and $\tau_i(t) = \tau_i$, $i = 1, \dots, n$, on $[t_0, \infty)$. Then all solutions of (1.1) are oscillatory if and only if

$$-\lambda + \sum_{i=1}^{n} p_i e^{\lambda\tau_i} > 0 \tag{1.2}$$

for all $\lambda > 0$.

For a different proof of the above theorem see also [9].

Another necessary and sufficient condition for oscillation of all solutions of (1.1) in the case of constant parameters appears in [10].

Theorem B. Suppose that $p_i(t) = p_i$ and $\tau_i(t) = \tau_i$, $i = 1, \dots, n$, on $[t_0, \infty)$. Then all solutions of (1.1) are oscillatory if and only if there exist positive constants N_i, $i = 1, \dots, n$, such that

$$\sum_{i=1}^{n} \mathrm{N}_i = 1 \tag{1.3}$$

and

$$\sum_{i=1}^{n} \frac{N_i}{\tau_i}\left(1 - \ln \frac{N_i}{p_i \tau_i}\right) > 0. \tag{1.4}$$

We note that in [10] only the sufficiency part of Theorem B has been proved. However, it is not difficult to see, with the aid of Proposition 1 from [1], that it is also a necessary condition for oscillation of all solutions of (1.1).

In [6], Hunt and Yorke studied also the case where $p_i(t)$ and $\tau_i(t)$ are not necessarily constants. In particular, they proved the following result.

Theorem C. Suppose that

$$0 < \tau_i(t) \leqq \tau_0, \qquad i = 1, \ldots, n,\ t \geqq t_0, \tag{1.5}$$

for some constant τ_0 and

$$\liminf_{t \to \infty} \sum_{i=1}^{n} p_i(t)\, \tau_i(t) > \frac{1}{e}. \tag{1.6}$$

Then all solutions of (1.1) are oscillatory.

However, the criterion given by Theorem C does not work for such important equations with non-constant $p_i(t)$ and $\tau_i(t)$ as the following Euler-like equation

$$x'(t) + \sum_{i=1}^{n} \frac{p_i}{t} x(\sigma_i t) = 0, \tag{1.7}$$

where $p_i > 0$ and $0 < \sigma_i < 1$, $i = 1, \ldots, n$, are constants. This is due to the fact that the delays $(1 - \sigma_i)t$, $i = 1, \ldots, n$, are unbounded in this case, i.e. the condition (1.5) is not satisfied. Since equations with deviating arguments of the form σt, $0 < \sigma < 1$, play an important role in various applications (see, for example, Kato and McLeod [7], Ladde, Lakshmikantham and Zhang [10], Ockendon and Tayler [13], Staikos and Tsamatos [14], Tomaras [16] and the references cited therein), it would be of practical interest to know effective oscillation criteria also for the equations of the form (1.1) where the deviations of arguments are not necessarily bounded.

The purpose of this note is twofold. First we show, by means of an "oscillation invariant" transformation of the independent variable, that the result of Hunt and Yorke can be extended also to differential equations with unbounded delays.

Secondly, we enlarge considerably the class of delay differential equations for which the oscillation situation can be completely characterized. The class includes the equation

$$x'(t) + \sum_{i=1}^{n} p_i t^{k-1} x((t^k - \tau_i)^{1/k}) = 0, \tag{1.8}$$

where k is a positive integer and $p_i > 0$ and $\tau_i > 0$, $i = 1, \ldots, n$, are real constants, as well as equations of the forms (1.7) and

$$x'(t) + \sum_{i=1}^{n} \frac{p_i}{t \ln t} x(t^{\sigma_i}) = 0, \tag{1.9}$$

where $p_i > 0$ and $0 < \sigma_i < 1$, $i = 1, \ldots, n$, are given constants.

As it is customary, we restict our attention to those solutions $x(t)$ of Eq. (1.1) which exist on some half-line $[t_x, \infty)$, $t_x \geqq t_0$, and satisfy $\sup\{|x(t)|: t \geqq T\} > 0$ for any $T \geqq t_x$ (the so called proper solutions). Such a solution is said to be oscillatory if it has arbitrarily large zeros and it is said to be nonoscillatory otherwise.

2. Main results

The key tool in establishing our results is a transformation of the original equation (1.1) into an equation with bounded delays.

Theorem 2.1. Assume that for some continuous functions $\bar{\tau}_i$, $i = 1, \ldots, n$, defined on $[t_0, \infty)$ and such that

$$0 < \bar{\tau}_i(t) \leqq \tau_0, \qquad i = 1, \ldots, n,\ t \geqq t_0, \tag{2.1}$$

for some constant τ_0, there exists a function $\varphi \in C_1([t_0, \infty))$ with the properties $\lim_{t\to\infty} \varphi(t) = \infty$ and $\varphi'(t) > 0$ on $[t_0, \infty)$, which is the simultaneous solution of the functional equations

$$\varphi(t) - \varphi(t - \tau_i(t)) = \bar{\tau}_i(t), \qquad i = 1, \ldots,, n, \tag{2.2}$$

for all large t. If, moreover,

$$\liminf_{s\to\infty} \sum_{i=1}^{n} \frac{p_i(\varphi^{-1}(s))}{\varphi'(\varphi^{-1}(s))} \bar{\tau}_i(\varphi^{-1}(s)) > \frac{1}{\mathrm{e}}, \tag{2.3}$$

then all proper solutions of (1.1) are oscillatory.

Proof. Assume that there exist functions $\bar{\tau}_i \in C([t_0, \infty))$, $i = 1, \ldots, n$, and $\varphi \in C_1([t_0, \infty))$ such that $\varphi'(t) > 0$ on $[t_0, \infty)$, $\lim_{t\to\infty} \varphi(t) = \infty$ and (2.1) and (2.2) hold. Then the change of variables

$$s = \varphi(t), \qquad y(s) = x(t), \tag{2.4}$$

in (1.1) leads to the equation of the form

$$y'(s) + \sum_{i=1}^{n} \frac{p_i(\varphi^{-1}(s))}{\varphi'(\varphi^{-1}(s))} y(s - \bar{\tau}_i(\varphi^{-1}(s))) = 0. \tag{2.5}$$

Obviously, $x(t)$ is a proper solution of Eq. (1.1) if and only if $y(s)$ is a proper solution of the transformed equation (2.5) and, moreover, $x(t)$ is oscillatory if and only if $y(s)$ is oscillatory. The assertion of Theorem 2.1 now immediately follows from Theorem C applied to the equation (2.5).

Remark 2.1. Although the assumption of the existence of a solution φ to (2.2) with desired properties seems to be rather restrictive (for related questions see Heard [5], Kuczma [8] and Neuman [11, 12]), we shall show that it is satisfied for a large variety of equations. As an immediate consequence of Theorem 2.1 we have, for example, the following result.

Corollary 2.1. Suppose that there exists a constant σ_0, $0 < \sigma_0 < 1$, such that

$$0 < \tau_i(t) \leqq \sigma_0 t, \qquad i = 1, \ldots, n, \tag{2.6}$$

for all large t, If, moreover,

$$\liminf_{t \to \infty} t \sum_{i=1}^{n} p_i(t) \ln \frac{t}{t - \tau_i(t)} > \frac{1}{e}, \tag{2.7}$$

then all proper solutions of (1.1) are oscillatory.

In fact, the functions $\varphi(t) = \ln t$ and $\bar{\tau}_i(t) = \ln \dfrac{t}{t - \tau_i(t)}$ satisfy the conditions (2.1) and (2.2) of Theorem 2.1 with $\tau_0 = \ln(1/(1 - \sigma_0))$, and (2.3) becomes (2.7).

Let us now consider a more general delay differential equation than (1.1), namely,

$$x'(t) + p(t) f(x(t - \tau_1(t)), \ldots, x(t - \tau_n(t))) = 0, \tag{2.8}$$

where p, τ_i: $[t_0, \infty) \to (0, \infty)$ are continuous functions, $\lim_{t \to \infty} (t - \tau_i(t)) = \infty$, $i = 1, \ldots, n$, $f(y_1, \ldots, y_n)$ is continuous on R^n, increasing in each y_i, $i = 1, \ldots, n$, and satisfies

$$f(\alpha y_1, \ldots, \alpha y_n) = \alpha f(y_1, \ldots, y_n)$$

for all $\alpha \in R$ (cf. Fukagai and Kusano [2]).

The following theorem can be regarded as an extension of Theorem A stated in the introduction.

Theorem 2.2. Let for some positive constants $\bar{\tau}_i$, $i = 1, \ldots, n$, the functional equations

$$\varphi(t) - \varphi(t - \tau_i(t)) = \bar{\tau}_i, \qquad i = 1, \ldots, n, \tag{2.9}$$

have a simultaneous solution $\varphi \in C_1([t_0, \infty))$ such that $\varphi'(t) > 0$ on $[t_0, \infty)$ and

$\lim_{t \to \infty} \varphi(t) = \infty$. Assume, moreover, that there is a constant $c > 0$ such that

$$p(t) = c\varphi'(t) \tag{2.10}$$

for all large t. Then all proper solutions of Eq. (2.8) are oscillatory if and only if

$$-\lambda + cf(e^{\lambda\bar{\tau}_1}, \ldots, e^{\lambda\bar{\tau}_n}) > 0 \tag{2.11}$$

for all $\lambda > 0$.

Proof. As in the proof of Theorem 2.1, the change of variables $s = \varphi(t)$, $y(s) = x(t)$ in (2.8) leads to the equation

$$y'(s) + cf(y(s - \bar{\tau}_1), \ldots, y(s - \bar{\tau}_n)) = 0 \tag{2.12}$$

with constant parameters and a direct application of Theorem 6 from [2] to (2.12) proves our claim.

Example 2.1. Consider the delay differential equation

$$x'(t) + t^{k-1}f(x((t^k - \tau_1)^{1/k}), \ldots, x((t^k - \tau_n)^{1/k})) = 0, \tag{2.13}$$

where k is a positive integer, $\tau_i > 0$, $i = 1, \ldots, n$, are given constants and f is as before.

The increasing continuously differentiable function $\varphi(t) = t^k$, $t \geqq t_0 \geqq 0$, satisfies (2.9) with $\bar{\tau}_i = \tau_i$, $i = 1, \ldots, n$, and $p(t) = k^{-1}\varphi'(t)$. Thus, by Theorem 2.2, all proper solutions of (2.13) are oscillatory if and only if

$$-\lambda + k^{-1}f(e^{\lambda\tau_1}, \ldots, e^{\lambda\tau_n}) > 0 \tag{2.14}$$

for all $\lambda > 0$.

Example 2.2. Consider the equations

$$x'(t) + t^{-1}f(x(\sigma_1 t), \ldots, x(\sigma_n t)) = 0 \tag{2.15}$$

and

$$x'(t) + (t \ln t)^{-1}f(x(t^{\sigma_1}), \ldots, x(t^{\sigma_n})) = 0, \tag{2.16}$$

where $0 < \sigma_i < 1$, $i = 1, \ldots, n$, are constants and f is as before.

Then (2.9) are satisfied for $\bar{\tau}_i = \ln(1/\sigma_i)$, $i = 1, \ldots, n$, and $\varphi(t) = \ln t$ (resp. $\varphi(t) = \ln(\ln t)$). According to Theorem 2.2, all proper solutions of (2.15) (or (2.16)) are oscillatory if and only if

$$-\lambda + f(\sigma_1^{-\lambda}, \ldots, \sigma_n^{-\lambda}) > 0 \tag{2.17}$$

for all $\lambda > 0$.

Motivated by the above examples (2.15) and (2.16) it is easy to indicate an infinite sequence of "equivalent" equations which can be transformed into an equation with constant coefficients and constant delays. In fact, let $\tau_i > 0$, $i = 1, \ldots, n$, be real constants. Define

$$\tau_0^i(t) = t - \tau_i, \qquad \tau_m^i(t) = \exp[\tau_{m-1}^i(\ln t)], \qquad m = 1, 2, \ldots$$

$$p_0(t) = 1, \qquad p_m(t) = \frac{1}{t} p_{m-1}(\ln t), \qquad m = 1, 2, \ldots$$

and consider the equations

$$x'(t) + p_m(t) f(x(\tau_m^1(t)), \ldots, x(\tau_m^n(t))) = 0, \tag{2.18$_m$}$$

$m = 0, 1, \ldots$. Then for every $m = 0, 1, \ldots$ all proper solutions of (2.18$_m$) are oscilatory if and only if

$$-\lambda + f(e^{\lambda\tau_1}, \ldots, e^{\lambda\tau_n}) > 0 \tag{2.19}$$

for all $\lambda > 0$.

We note that the equation (2.8) may not be linear as the following example shows.

Example 2.3. Consider the nonlinear delay differential equations

$$x'(t) + \frac{p}{t} \prod_{i=1}^{n} |x(\sigma_i t)|^{\alpha_i} \operatorname{sgn} x(\sigma_i t) = 0 \tag{2.20}$$

and

$$x'(t) + \frac{p}{t \ln t} \prod_{i=1}^{n} |x(t^{\sigma_i})|^{\alpha_i} \operatorname{sgn} x(t^{\sigma_i}) = 0, \tag{2.21}$$

where $p > 0$, $0 < \sigma_i < 1$ and $\alpha_i \geqq 0$, $i = 1, \ldots, n$, are constants with

$$\sum_{i=1}^{n} \alpha_i = 1.$$

According to Theorem 2.2, all proper solutions of (2.20) (or (2.21)) are oscilatory if and only if

$$-\lambda + p\left[\prod_{i=1}^{n} (1/\sigma_i)^{\alpha_i}\right]^{\lambda} > 0 \tag{2.22}$$

for all $\lambda > 0$. It is easy to check that the condition (2.22) is satisfied if and only if

$$p \sum_{i=1}^{n} \alpha_i \ln(1/\sigma_i) > \frac{1}{e}. \tag{2.23}$$

Remark 2.2. All the results presented in this paper have their analogues for advanced equations of the form

$$x'(t) - \sum_{i=1}^{n} p_i(t) x(t + \tau_i(t)) = 0 \qquad (2.24)$$

and

$$x'(t) - p(t) f(x(t + \tau_1(t)), \dots, x(t + \tau_n(t))) = 0, \qquad (2.25)$$

where $n \geqq 1$, p, p_i, τ_i: $[t_0, \infty) \to (0, \infty)$, $i = 1, \dots, n$, are continuous functions and f satisfies the same conditions as before.

Finally we remark that all the results remain valid if the equations (1.1), (2.8), (2.24) and (2.25) are replaced by the inequalities

$$\left\{x'(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i(t))\right\} \operatorname{sgn} x(t) \leqq 0, \qquad (2.26)$$

$$\{x'(t) + p(t) f(x(t - \tau_1(t)), \dots, x(t - \tau_n(t)))\} \operatorname{sgn} x(t) \leqq 0, \qquad (2.27)$$

$$\left\{x'(t) - \sum_{i=1}^{n} p_i(t) x(t + \tau_i(t))\right\} \operatorname{sgn} x(t) \geqq 0 \qquad (2.28)$$

and

$$\{x'(t) - p(t) f(x(t + \tau_1(t)), \dots, x(t + \tau_n(t)))\} \operatorname{sgn} x(t) \geqq 0, \qquad (2.29)$$

respectively.

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SÚHRN

POUŽITIE ZMENY NEZÁVISLEJ PREMENNEJ V TEÓRII OSCILÁCIE DIFERENCIÁLNYCH ROVNÍC S NEOHRANIČENÝMI ONESKORENIAMI

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V práci sú odvodené postačujúce podmienky oscilácie všetkých regulárnych riešení diferenciálnych rovníc prvého rádu s neohraničenými oneskoreniami. Pre určité triedy rovníc, ktoré môžu byť transformované na rovnice s konštantnými koeficientami a konštantnými oneskoreniami, sú odvodené podmienky aj nutnými podmienkami.

РЕЗЮМЕ

ПРИМЕНЕНИЕ ПРЕОБРАЗОВАНИЯ НЕЗАВИСИМОЙ ПЕРЕМЕННОЙ В ТЕОРИИ КОЛЕБЛЕМОСТИ ДИФФЕРЕНЦИАЛЬНЫХ УРАВНЕНИЙ С НЕОГРАНИЧЕННЫМИ ЗАПАЗДЫВАНИЯМИ

Ярослав Ярош, Братислава

В работе приведены достаточные условия колеблемости всех правильных решений дифференциальных уравнений первого порядка с неограниченными запаздываниями. Для некоторых классов уравнений, которые могут быть преобразованы в уравнения с постоянными коэффициентами и постоянными запаздываниями, приведенные условия являются и необходимыми.