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TYPICAL CONTINUOUS FUNCTION HAS THE SET OF CHAIN RECURRENT POINTS OF ZERO LEBESGUE MEASURE

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We shall consider $C(I, I)$ the set of all continuous functions $f\colon I \to I$, where I is a real compact interval, and the usual norm $\|f\| = \max_{x \in I} |f(x)|$. For any $n = 1, 2, \ldots f^n$ denotes the n-th iterate of f, $Per(f)$ denotes the set of periodic points of f and $\Omega(f)$ is the set of all non-wandering points of f; here $x \in \Omega(f)$ means that for every neighborhood $U(x)$ of x, there exists $n \geq 1$ such that $f^n(U(x)) \cap U(x) \neq \emptyset$.

Definition. (*i*) For $\varepsilon > 0$ let $\{Q^j_\varepsilon(x, f)\}_{j=0}^\infty$ be a sequence defined in the following way:

$$Q^0_\varepsilon(x, f) = U_\varepsilon(x) \text{ is the } \varepsilon\text{-neighborhood of } x,$$

$$Q^{j+1}_\varepsilon(x, f) = U_\varepsilon(f(Q^j_\varepsilon(x, f)).$$

Let $Q_\varepsilon(x, f) = \bigcap_{i=0}^{\infty} \bigcup_{j \geq i} \overline{Q^j_\varepsilon(x, f)}$ and $Q(x, f) = \bigcap_{\varepsilon \geq 0} Q_\varepsilon(x, f)$. A point x is called a chain recurrent point of f, if $x \in Q(x, f)$. It is possible to give another equivalent definition:

(*ii*) Point $x \in I$ is called a chain recurrent point of f, if for any $\varepsilon > 0$ there exists a sequence of points (chain $\{x_k\}_{k=0}^n$ such that $x_0 = x = x_n$ and

$$|f(x_i) - x_{i+1}| < \varepsilon \qquad i = 0, 1, \ldots n - 1.$$

The set of all chain recurrent points of f is denoted by $B(f)$. It is easy to see that $Per(f) \subset \Omega(f) \subset B(f)$. For more details see e.g. [3] or [5]. The set $B(f)$ has special importance in general analysis of non-wandering points. Making decomposition of non-wandering or chain recurrent points by using the relation $x \sim y$ if and only if $Q(x, f) = Q(y, f)$, we get classes of equivalences, which are stable in some sense (more details in [5]).

In [1] it is proved, that the set of functions $f \in C(I, I)$ with the property that

$\Omega(f)$ is a nowhere dense, closed subset of a null set is residual in $C(I, I)$. Now we shall generalize it to the set $B(f)$. The proof is based on two lemmas.

Lemma 1. Let $f \in C(I, I)$. Then for every $\varepsilon > 0$ and $\beta > 0$ there exists $g \in C(I, I)$ and a neighborhood $O_1(g)$ of the function g, such that $\|f - g\| < \varepsilon$ and for every $g^* \in O_1(g)$ $\mu(B(g^*)) < \beta$.

Proof. Let $f \in C(I, I)$, $\varepsilon > 0$ and $\beta > 0$ be given. Since I is a real compact interval, there exists δ such that $0 < \delta < \frac{\varepsilon}{8}$ and for all $x, y \in I$

$$|x - y| < \delta \text{ implies } |f(x) - f(y)| < \frac{\varepsilon}{8} \tag{1}$$

Take a positive integer m, large enough, and points $a_0 < b_0 < a_1 < b_1 < \ldots < < a_m < b_m$ where $I = [a_0, b_m]$, $\sum_{i=0}^{m} \mu([a_i, b_i]) < \beta$ and

$$|a_{i+1} - a_i| < \delta \text{ for } i = 0, \ldots m - 1, \text{ and } |b_m - a_m| < \frac{\delta}{2} \tag{2}$$

Denote $[a_i, b_i] = I_i$; $[b_i, a_{i+1}] = K_i$ and let $c_i = (b_i + a_{i+1})/2$ denote the middle of K_i for any $i \in \{0, \ldots m - 1\}$. It is easy to see that for every i there exists $n(i)$ such that

$$|f(c_i) - c_{n(i)}| < \delta < \frac{\varepsilon}{8} \tag{3}$$

This follows from (2). Now define a function $g \in C(I, I)$ in this way: $g(x) = c_{n(i)}$ if $x \in K_i$ and let g be a linear function on I_i for every i, and let $f(a_0) = g(a_0)$ and $f(a_m) = g(a_m)$. So if for some i we have $x \in K_i$, then by (1) and (3) we get

$$|f(x) - g(x)| \leq |f(x) - f(c_i)| + |f(c_i) - c_{n(i)}| < \frac{\varepsilon}{4} \tag{4}$$

and if $x \in I_i = [a_i, b_i]$ then by (1) and (4),

$$|f(x) - g(x)| \leq |f(x) - f(a_i)| + |f(a_i) - g(a_i)| + |g(a_i) - g(x)| < \\ < \frac{\varepsilon}{8} + \frac{\varepsilon}{4} + |g(a_i) - g(x)| \tag{5}$$

Because of linearity of g, we have by (1) and (4)

$$|g(a_i) - g(x)| \leq |g(a_i) - g(b_i)| \leq |g(a_i) - f(a_i)| + |f(a_i) - f(b_i)| +$$
$$+ |f(b_i) - g(b_i)| < \frac{5}{8} \varepsilon$$

This along with (5) implies $|f(x) - g(x)| < \varepsilon$.
This proves that $\| f - g \| < \varepsilon$.

Now we are going to prove that $\mu(B(g^*)) < \beta$ for any $g^* \in C(I, I)$ sufficiently near to g.
Take $\alpha > 0$ such that $M_i = (c_i - \alpha, c_i + \alpha) \subset K_i$ for all i and

$$\mu\left(\bigcup_{i=0}^{m} I_i\right) + \mu\left(\bigcup_{i=0}^{m-1} M_i\right) < \beta \tag{6}$$

It is obvious that $g(K_i) \subset M_{n(i)}$ for any i and because of continuity of g there exists such $\lambda > 0$ that $U_\lambda(g(U_\lambda(K_i))) \subset M_{n(i)}$. The same is true for any $g^* \in C(I, I)$ sufficiently near to g. Now for every such g^*, if $x \in K_i \backslash M_i$ then $x \notin Q_\lambda(x, g^*)$ and also $x \notin Q(x, g^*)$; this implies $x \notin B(g^*)$. Consequently $(K_i \backslash M_i) \cap B(g^*) = \emptyset$ for every i and by (6), $\mu(B(g^*)) < \beta$.

Lemma 2. Let $\beta > 0$. Then exists a set $A_\beta \subset C(I, I)$ with the property that $C(I, I) \backslash A_\beta$ is nowhere dense in $C(I, I)$ and $\mu(B(f)) < \beta$ for every $f \in A_\beta$.

Proof. $A_\beta := \{f \in C(I, I);\ \mu(B(f)) < \beta\}$ is the set we are looking for. In Lemma 1 we have proved that A_β contains a dense open set, hence $C(I, I) \backslash A_\beta$ is nowhere dense in $C(I, I)$ and this proves our lemma.

Theorem. The set $A := \{f \in C(I, I);\ \mu(B(f)) = 0\}$ is residual in $C(I, I)$.

Proof. It is easy to see that $A = \bigcap_{n=1}^{\infty} A_{1/n}$ and $A_{1/n}$ is residual in $C(I, I)$ according Lemma 2.

Remarks. As a consequence of this theorem we get that, for a residual subset of continuous functions on the closed interval, there is no absolutely continuous (relative to Lebesgue measure) invariant measure. The proof is based on the fact that the support of an invariant measure is a subset of the set of non-wandering points of a given function. For necessary notions and results see e.g. [6, 7].

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РЕЗЮМЕ

ТИПИЧНОЕ НЕПРЕРЫВНОЕ ОТОБРАЖЕНИЕ ИМЕЕТ МНОЖЕСТВО ПОЧТИ НЕБЛУЖДАЮЩИХ ТОЧЕК НУЛЕВОЙ МЕРЫ ЛЕБЕГА

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Известно, что типичное непрерывное отображение имеет множество периодических точек нулевой меры Лебега. Агронски, Бракнер и Лацкович это самое доказали о множестве неблуждающих точек. В статьи доказано, что множество слабо неблуждающих точек обладает тем же свойством и это представляет последнее обобщение значительное в теории динамических систем.

SÚHRN

TYPICKÁ SPOJITÁ FUNKCIA MÁ MNOŽINU SKORO NEBLÚDIVÝCH BODOV NULOVEJ MIERY

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Je známe, že typická spojitá funkcia na uzavretom intervale, má množinu periodických bodov nulovej Lebesguovej miery. Ten istý výsledok pre množinu neblúdivých bodov dokázali Agronsky, Bruckner a Laczkovich. V článku je tento výsledok zovšeobecnený na množinu skoro neblúdivých bodov, pričom táto množina je maximálnou zovšeobecňujúcou spomedzi tých, ktoré sú dôležité pri klasifikácii dynamických systémov.