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A REMARK ON ONE TYPE OF POINCARÉ'S INEQUALITY

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1 Introduction

It is well known that the weak solutions to the equation

$$\sum_{i,j=1}^{n} -\frac{\partial}{\partial x_i}\left[a_{ij}\frac{\partial w}{\partial x_j}\right] = 0 \qquad x\in\Omega, \tag{1.1}$$

where $\Omega\subset \mathbf{R}^n$ is a bounded domain and $a_{ij}\in L^{\infty}(\Omega)$, $i, j = 1, \dots, n$, such that

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \geq \gamma|\xi|^2, \qquad \gamma>0, \quad \xi\in \mathbf{R}^n \tag{1.2}$$

are Hölder continuous, i. e. $w\in C^{0,\mu}(\Omega)$, $\mu\in(0, 1]$ [2, Theorem 5.4.32]. Now a question arises what can be said about the value of Hölder exponent μ.

In this note we shall deal with a special case of Poincaré's inequality on the cube and then, as a consequence, we may obtain some information about the exponent μ.

Adopting the notation of [2] we can state our result.

2 Poincaré's inequality

The aim of this part is to give a direct proof of the following assertion:

Theorem 2.1. Let $Q_R(x^0)\subset \mathbf{R}^n$, $Q_R(x^0) = \{x\in \mathbf{R}^n: |x_i - x_i^0| < R,\ i = 1, \dots, n\}$

$$x^0 = (x_1^0, \dots, x_n^0)$$

and

$$N = \{u\in W^{1,2}(Q_R(x^0)): u = 0 \text{ on } S \text{ with } |S|\geq c_1|Q_R(x^0)|,\ c_1>0\}$$

(S depends on u but not on c_1).

Then for $u\in N$

$$\int_{Q_R(x^0)} |u|^2\,\mathrm{d}x \leq c_2\int_{Q_R(x^0)} |\nabla u|^2\,\mathrm{d}x, \qquad c_2 = 2n\left(1+\frac{1}{c_1^2}\right)R^2 \tag{2.1}$$

This assertion is well known [2, Lemma 5.4.26] but its proof is not direct. A construction of that proof is analogous to the proof of Theorem 7.1 in [1]. There is a disadvantage of this type of proof that we do not see an exact dependence of the constant c_2 on $Q_R(x^0)$ and c_1.

The result of the following Lemma (proved in [1]) will be used later:

Lemma 2.1: Let $u \in W^{1,2}(Q_R(x^0))$. Then

$$\int_{Q_R(x^0)} |u|^2 \, \mathrm{d}x \le 2R^2 n \int_{Q_R(x^0)} |\nabla u|^2 \, \mathrm{d}x + \frac{1}{|Q_R(x^0)|} \left| \int_{Q_R(x^0)} u(x) \, \mathrm{d}x \right|^2. \tag{2.2}$$

The proof of Theorem 2.1:

Let indicate $$Q_R(x^0) = Q.$$

If we put $$\bar{u} = \frac{1}{|Q|} \int_Q u \, \mathrm{d}x, \quad \text{for} \quad u \in \boldsymbol{N},$$

then it is easy to see that

$$\bar{u} = \frac{1}{|S|} \int_{Q \setminus S} (u - \bar{u}) \, \mathrm{d}x$$

and thus

$$\left| \int_Q u \, \mathrm{d}x \right| = ||Q|\bar{u}| \le \frac{|Q|}{|S|} \int_{Q \setminus S} |u - \bar{u}| \, \mathrm{d}x \tag{2.3}$$

Next (2.3) yields by Hölder's inequality

$$\left| \int_Q u \, \mathrm{d}x \right|^2 \le \frac{|Q|}{c_1^2} \int_Q |u - \bar{u}|^2 \, \mathrm{d}x \tag{2.4}$$

To estimate the right-hand side of (2.4), we use Lemma 2.1 for $(u - \bar{u})$ and obtain

$$\frac{1}{|Q|} \left| \int_Q u \, \mathrm{d}x \right|^2 \le \frac{1}{c_1^2} 2nR^2 \int_Q |\nabla u|^2 \, \mathrm{d}x \tag{2.5}$$

which together with (2.2) immediately gives the assertion of our theorem.

3 Regularity of solutions to (1.1)

Results which will be stated in this part are analogous to those in [2].

Lemma 3.1: Let $n \ge 3$ and $v \in W^{1,2}(Q_{2R}(x^0))$ be a weak non-negative sub-solution to the equation (1.1) where $\|a_{ij}\|_{L^r(Q_{2R}(x^0))} \le \alpha$, $i, j = 1, \dots, n$ and let the condition (1.2) be satisfied in $\boldsymbol{R}^n$. Then

$$\sup_{x \in Q_R(x^0)} v(x) \le c_3 (2R)^{-\frac{n}{2}} \left(\int_{Q_{2R}(x^0)} v^2 \, dx \right)^{\frac{1}{2}} \tag{3.1}$$

where

$$c_3 = 2^{\frac{n(n-2)}{4}} 2^s \left\{ \left(\frac{2n-2}{n-2} \right)^2 (n+1) 2^5 \left(1 + 2^2 n^3 \left(\frac{\alpha}{\gamma} \right)^2 \right) \right\}^{\frac{n}{4}}$$

$$s = \sum_{i=0}^{\infty} \frac{i}{k^i}, \qquad k = \frac{n}{n-2}$$

(s may be estimated by integral criterion).

The definition of subsolution is in [2].

Lemma 3.1 is proved in [2, Theorem 5.4.9] except possibly for calculation of c_3. This calculation follows directly from construction of the proof except possibly for fact that we put

$$\Phi_l = \left(\int_{Q_l} h^{2k^l} \, dx \right)^{\frac{1}{2k^l}} \quad \text{in} \quad (5.4.15).$$

Lemma 3.2: (of the Harnack type)

Let $n \ge 3$ and $u \in W^{1,2}(Q_{4R}(x^0))$, $u \ge 0$, be a weak solution to (1.1).

Let $\|a_{ij}\|_{L^\infty(Q_{4R}(x^0))} \le \alpha$, $i, j = 1, \dots, n$ and (1.2) holds.

Let $S = \{x \in Q_{4R}(x^0) : u(x) \ge 1\}$, $|S| \ge c_4 |Q_{4R}(x^0)|$. Then

$$u(x) \ge c_5(\alpha, \gamma, c_4) > 0 \quad \text{in} \quad Q_R(x^0) \tag{3.2}$$

where

$$c_5 = e^{-A},$$

$$A = c_3 2^{\frac{3}{2}} n^2 . r \left(1 + \frac{4}{c_4^2} \right)^{\frac{1}{2}} \cdot \left(\frac{4}{4-r} \right) \left(\frac{4+r}{2} \right)^{\frac{n}{2}} \left(\frac{\alpha}{\gamma} \right),$$

$$r = 4 \left(1 - \frac{c_4}{2} \right)^{\frac{1}{3}}.$$

This Lemma is in [2, Theorem 5.4.20] but it does not determine the concrete constant c_5. We obtain this constant by using Theorem 2.1 in (5.4.25), [2].

Lemma 3.3: [2, Theorem 5.4.32]

Let $u \in W^{1,2}(\Omega)$, $\Omega \subset \boldsymbol{R}^n$, $n \ge 3$, be a bounded domain and u be a solution to (1.1) with $\|a_{ij}\|_{L^\infty(\Omega)} \le \alpha$ and let (1.2) hold. Then $u \in C^{0,\mu}(\Omega)$, $\mu = -\dfrac{\ln(1 - c_5)}{\ln 4}$ $\left(c_5 \text{ — constant from (3.2) with } c_4 = \frac{1}{2} \right)$.

The proof of Lemma 3.3 follows immediately from Lemma 3.2. Thus Theorem 2.1 enables us to determine the constant μ (which is determined by parameters α, γ of equation (1.1)).

On the other hand it is possible to find optimal constants α and γ (to determine a class of equations) such that μ be the greatest.

REFERENCES

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SÚHRN

POZNÁMKA K JEDNÉMU TYPU POINCARÉHO NEROVNOSTI

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V práci je uvedený priamy dôkaz (s presným určením konštanty) jedného typu Poincarého nerovnosti, keď oblasť Q je n-rozmerná kocka.

РЕЗЮМЕ

ЗАМЕЧАНИЕ ОБ ОДНОМ ТИПЕ НЕРАВЕНСТВА ПУАНКАРЕ

Е. ВИСЗУС, Братислава

В работе предложено прямое доказательство (с точной константой) одного типа неравенства Пуанкаре в случае, когда область Q есть n-мерный куб.