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# ON A PROBLEM FROM EXTREMAL GRAPH THEORY

ROMAN NEDELA, Banská Bystrica

## 0 Introduction

We say that a system $F_1, \ldots, F_k$ of factors of a graph $G$ presents an *edge decomposition* of $G$ if every edge of $G$ belongs to exactly one of the factors $F_1, \ldots, F_k$. Let $f(k)$ be the smallest natural number such that the complete graph $K_{f(k)}$ is decomposable into $k$ factors with diameter 2.

Decompositions of complete graphs into factors with a given diameter were introduced by Bosák, Rosa and Znám in [4]. In [2] Bosák showed that $6k - 52 \leq f(k) \leq 6k$ for $k \geq 2$. Later B. Bollobás proved $f(k) \geq 6k - 9$ for $k \geq 6$ in [1]. Š. Znám proved $f(k) = 6k$ for sufficiently large $k$ in [7]. This has been improved in [5] by showing that $f(k) = 6k$ for $k \geq 153$. For small $k$ the unique value exactly known is $f(2) = 5$. From [4], [2], [1] we have the following bounds: $12 \leq f(3) \leq 13$, $15 \leq f(4) \leq 24$, $20 \leq f(5) \leq 30$, $27 \leq f(6) \leq$ $\leq 36$.

In the present paper we prove $f(4) \geq 17, f(5) \geq 22, f(6) \geq 28$. The problem, what are the exact values $f(3), f(4), f(5), f(6)$ remains open.

## 1 Some general results

We summarise some general results here. All graphs considered in our paper are undirected, without loops and multiple edges. We will use usual notations: $d(x)$ — the degree of vertex $x$ in a given graph, $\Gamma(x)$ — the neighbourhood of a vertex $x$ $(x \notin \Gamma(x))$, $e(G)$ — the number of edges of a graph $G$.

The basic result of [4] reads as follows:

**Theorem 1.** If the complete graph $K_n$ is decomposable into $k$ factors with diameters $d_1, d_2, \ldots, d_k$ then for $N > n$ the complete graph $K_N$ is also decomposable into $k$ factors with diameters $d_1, d_2, \ldots, d_k$.

Denote by $S(u)$ the sum of degrees of the vertices adjacent to a vertex $u$ in a given graph $G$. It is generally known that, if $G$ of order $n$ has diameter 2, then $S(u) \geq n - 1$ for every vertex $u$ of $G$. However the following stronger result holds.

**Proposition 1.** Let $F$ be a graph of order $n$ and of diameter 2. Let $u$ be its vertex of degree $m$. Let $p_i$ $(i = 1, \ldots, m)$ be the numbers of the vertices which are not adjacent to $u$ and have at least $i$ common neighbours with $u$. Finally, let $r$ be the number of edges between vertices adjacent to $u$. Then

$$S(u) = n - 1 + p_2 + \ldots + p_m + 2r \tag{1.1}$$

**Proof.** Let $c_i$ $(i = 1, 2, \ldots, m)$ be the numbers of the vertices which are not adjacent to $u$ and have exactly $i$ common neighbours with $u$. It is easy to see that

$$S(u) = m + c_1 + 2c_2 + \ldots + mc_m + 2r \tag{1.2}$$

The graph $F$ has diameter 2. Thus $p_1 = n - m - 1$. Now we obtain the equality (1.1) from (1.2) by the substitution $p_i = \sum_{j=i}^{m} c_j$.

The following properties of decompositions were proved in [6].

**Proposition 2.** Let $F$ be a factor of a decomposition of $K_n$ into $k$ factors of diameter 2 and $u$ be its vertex. Then

$$3 \leq d(u) \leq n - 3k + 2 \qquad \text{for} \quad k \geq 3.$$

**Proposition 3.** Let $F$ be a factor with a minimum number of edges of a decomposition of $K_n$ into $k$ factors of diameter 2, then

$$e(F) \leq [n(n-1)/2k].$$

## 2 Case $k = 4$

**Theorem 2.** $f(4) \geq 17$.

**Proof.** It follows by Theorem 1 that it is sufficient to prove that $K_{16}$ cannot be decomposed into 4 factors with diameter 2. We prove it by contradiction. Let such a decomposition exist and $F$ be its minimal factor.
Propositions 1, 2 and 3 give us:

let $x$ be a vertex of $F$, then

$$S(x) \geq 15, \tag{2.1}$$

$$3 \leq d(x) \leq 6, \tag{2.2}$$

$$e(F) \leq 30. \tag{2.3}$$

First of all we will show that no two vertices of degree 3 are adjacent in $F$.

Assume, on the contrary, that $u$, $v$ are such vertices. If the vertices $u$, $v$ have a common vertex in their neighbourhoods, then it follows by Proposition 1 that $S(u) \geq 17$, which contradicts (2.2). Thus the vertices $u$, $v$ have no common neighbour. Denote the vertices adjacent to $u$ by $u_1$, $u_2$ and the vertices adjacent to $v$ by $v_1$, $v_2$. By (2.1) and (2.2) we get

$$d(u_1) = d(u_2) = d(v_1) = d(v_2) = 6 \tag{2.4}$$

Put $J = V - \{u, u_1, u_2, v, v_1, v_2\}$, where $V$ is the vertex set of $F$. The factor $F$ has diameter 2, thus each of the ten vertices of $J$ is adjacent to at least one pair $(u_i, v_j)$ $i, j \in \{1, 2\}$. No three vertices from the set $\{u_1, u_2, v_1, v_2\}$ can be adjacent to any vertex from $J$, otherwise we would get $S(u) > 15$ or $S(v) > 15$, which contradicts Proposition 1. As we can see by (2.3) and (2.4) all vertices of $J$ are of degree 3 and so $S(x) = 15$ for $x \in J$. By Proposition 1, no two vertices of $J$ have the same pair $(u_i, v_j)$ in their neighbourhoods. From the above mentioned facts it follows there is one-one correspondence between the sets $J$ and $I = \{(u_i, v_j);\ i, j = 1, 2\}$. But this is impossible since $|J| = 10$ and $|I| = 4$. Thus no two vertices of degree 3 are adjacent in $F$.

There is at least one vertex $u$ of degree 3 in $F$, otherwise $\sum_{u \in F} d(u) \geq 64 > 60$, a contradiction with (2.3). Let $m$ be the number of vertices of degree at least 4 in $F$. Then the following inequality holds:

$$60 \geq \sum_{v \in F} d(v) \geq S(u) + 3(16 - m) + 4(m - 3),$$

and consequently

$$9 \geq m. \tag{2.5}$$

From (2.5) it follows that there are at least 7 vertices of degree 3 in $F$. Assume that all these vertices are adjacent to at least two vertices in $\Gamma(u)$. Then Proposition 1 implies $S(u) \geq 21$, which contradicts (2.2). Thus there is a vertex $v$ of degree 3, which has only one common neighbour $w$ with the vertex $u$. It follows by (2.1) that $v$ is adjacent to at least one vertex different from $w$ of degree at least 5. Thus we obtain

$$9 > m. \tag{2.6}$$

According to (2.2) we have $d(w) \leq 6$ and thus there is a vertex of degree 3 which is not adjacent to $w$. Denote by $e_1$ the number of edges which are incident with the vertices of degree 3 in $F$ and by $e_2$ the number of other edges. Then

$$30 \geq e(F) = e_1 + e_2 \geq 3(16 - m) + 2(m - 6) + 3, \tag{2.7}$$

and consequently $m \geq 9$, a contradiction. The proof is complete.

## 3 On a method of Bollobás

In [1] Bollobás used the following lemma in the proof of the bound $f(k) \geq 6k - 9$ for $k \geq 6$.

**Lemma 4.** Suppose $a > 0$ and the graph $G$ is such that for every vertex $x \in G$

$$d(x) + \sum_{\substack{y \in \Gamma(x) \\ d(y) > a}} (d(y) - a)/d(y) \geq a. \tag{3.1}$$

Then

$$e(G) \geq (a/2)\,|V(G)|. \tag{3.2}$$

**Proof** [1]. If $x, y \in G$ and $d(y) \geq a$ let $W(x, y) = (d(y) - a)/d(y)$, if $x \in G$ and $d(x) < a$ put

$$d'(x) = d(x) + \sum_{\substack{y \in \Gamma(x) \\ d(y) > a}} W(x, y),$$

if $y \in G$ and $d(y) \geq a$ set

$$d'(y) = d(y) - \sum_{x \in \Gamma(y)} W(x, y) = a.$$

By the assumption we have $d'(x) \geq a$ for every vertex $x \in G$. Therefore

$$2e(G) = \sum_{x \in G} d(x) \geq \sum_{x \in G} d'(x) \tag{3.3}$$

$$\sum_{x \in G} d'(x) \geq a|V(G)|. \tag{3.4}$$

The original formulation in [1] of the present Lemma 4 claimed, moreover, that if there is a vertex for which the inequality (3.1) is sharp then so is (3.2). However this claim is false. In fact take as counterexample the star with $n \geq 4$ vertices and put $a = 2(n - 1)/n$. The following proposition solves this problem.

**Proposition 5.** Let $a$ be such that for every vertex $x$ of $G$ (3.1) holds. then $e(G) = (a/2)\,|V(G)|$ iff the following two conditions are fulfilled for every vertex $x$ of $G$

(i) If $d(x) < a$, then $d(x) + \sum_{\substack{y \in \Gamma(x) \\ d(y) > a}} (d(y) - a)/d(y) = a$,

(ii) if $d(x) > a$ and $y \in \Gamma(x)$, then $d(y) < a$.

**Proof.** Let $2e(G) = a|V(G)|$. Then the inequalities (3.1), (3.2) are transformed to equalities

$$\sum_{x \in G} d(x) = \sum_{x \in G} d'(x), \tag{3.5}$$

$$\sum_{x \in G} d'(x) = a|V(G)|. \tag{3.6}$$

We obtain the conditions (i) and (ii) from (3.6) and (3.5) respectively. on the other hand if the conditions (i) and (ii) are fulfilled, then so are (3.5), (3.6) and thus $e(G) = (a/2)|V(G)|$ holds.

If we want to apply Lemma 4 we must verify the validity of the assumption (3.1). We can use the following properties of (3.1).

Let $a > 0$ and a graph $G$ be fixed. Let $x$ be a vertex of degree $d$ in $G$. Denote by $y_1, y_2, \ldots, y_d$ the degrees of $d$ vertices adjacent with $x$. Suppose that the first $p$ of them are greater than $a$. In this case we shall say that $x$ has type $(y_1, y_2, \ldots, y_p)$. Consider the left hand side of (3.1) as a function

$$g(d, p, y_1, y_2, \ldots, y_p) = d + p - \sum_{i=1}^{p} a/y_i \tag{3.7}$$

with arguments $d, p, y_1, y_2, \ldots, y_p$ and a parameter $a$. From (3.7) it is easy to see that:

the value of $g$ is independent of the ordering of the arguments

$$y_1, y_2, \ldots, y_p, \tag{3.8}$$

$$g(d, p, y_1, y_2, \ldots, y_p) < g(d, p, y_1 + 1, y_2, \ldots, y_p) \tag{3.9}$$

if $y_1 \leq y_2$ and $y_1 - 1 > a$ then

$$g(d, p, y_1 - 1, y_2 + 1, y_3, \ldots, y_p) < g(d, p, y_1, y_2, \ldots, y_p). \tag{3.10}$$

Consider the function $g$ independently of the graph $G$. Let the numbers $n$, $k$, $a$, $d$ and $p$ be given. Let $A$ be set of $d$-tuples of integers $y_i$ ($i = 1, \ldots, d$) satisfying the following conditions:

$$\sum_{i=1}^{d} y_i \geq n - 1, \tag{3.11}$$

for every $i \in \{1, \ldots, d\}$

$$3 \leq y_i \leq n - 3k + 1, \tag{3.12}$$

for $\quad i = 1, \ldots, p \quad p \leq d \quad$ and $\quad y_i > a > 0.$ (3.13)

The conditions (3.11) and (3.12) correspond with the Proposition 1 and 2 respectively. It is clear, that if $A$ is non-empty we can find $(y_1, y_2, \ldots, y_d)$ in $A$ minimizing the function $g$. Since the value of the function $g$ depends directly only on the first $p$ integers of $y_1, y_2, \ldots, y_d$ we use the notation $(y_1, y_2, \ldots, y_p)$ to describe the $d$-tuple minimizing the value of $g$. We shall called such $p$-tuple the minimal type of neighbourhood. Obviously it is sufficient to verify (3.1) only for the minimal type of neighbourhoods.

## 4 Cases $k = 5$ and $k = 6$

**Theorem 3.** $f(5) \geq 22$

**Proof.** According to Theorem 1 it is sufficient to prove that $K_{21}$ cannot be decomposed into 5 factors with diameter 2. Assume, on the contrary, that there exists such a decomposition. Let $F$ be a factor with minimum number of edges and $u$ be its vertex. then Propositions 1, 2 and 3 give us:

$$S(u) \geq 20 \tag{4.1}$$

$$3 \leq d(u) \leq 8 \tag{4.2}$$

$$e(F) \leq 42 \tag{4.3}$$

Now we shall use Lemma 4. Put $a = 4$. The inequality (3.1) is trivial for the vertices with degrees $\geq 4$. If a vertex $x$ of $F$ has degree 3 we distinguish two cases according to as $p = 2$ or $p = 3$. If $p = 2$, then by (4.1) and (4.2) we get $x$ has a type (8,8). Then from (3.7) we obtain $g(3, 2, 8, 8) = 4 = a$.

Now let $p = 3$. Considering (4.1) and (4.2) and using (3.8), (3.9), (3.10) we get (5, 7, 8) is the minimal type of neighbourhood for which $g(3, 3, 5, 7, 8) =$ $= 299/70 > a$.

Therefore
$$e(F) \geq (a/2)\, V(F) = 42. \tag{4.4}$$

If there is equality in (4.4) then Proposition 5 gives us that all vertices of degree 3 have type (8,8). In such case by Proposition 5 (ii) we get that there are at least 15 vertices of degree 3 in $F$. By (4.1) and (4.2) these vertices cannot be adjacent. Then by trivial calculation we have $e(F) \geq 15 \,.\, 3 = 45 > 42$, a contradiction with (4.3).

**Theorem 4.** $f(6) \geq 28$.

**Proof.** It is sufficient to prove that $K_{27}$ cannot be decomposed into 6 factors of diameter 2. Assume, on the contrary, that there exists such a decomposition. Propositions 1, 2 and 3 give us that for minimal factor F and its vertex $u$ the following inequalities hold:

$$S(u) \geq 26 \tag{4.5}$$

$$3 \leq d(u) \leq 11 \tag{4.6}$$

$$e(F) \leq 58 \tag{4.7}$$

Assume $5 > a > 116/27$. As in the previous case we must verify (3.1). The assumption (3.1) holds trivially for vertices with degree $\geq 5$. using (4.5), (4.6), (3.8), (3.9), (3.10) we determine the minimal types of neighbourhoods for $d = 3$ and $d = 4$. We summarise the results in the following table:

If $220/51 > a > 116/27$ (note that such a exists), then (3.1) holds and it is sharp. in all cases except $d = 3$ and $p = 2$. Consequently, if there is no vertex with degree 3 and type (11, 11) in $F$, then by Lemma 4 and we have $e(F) > 58$,

| $d$ | $p$ | min. type |
|---|---|---|
| 4 | 2 | (7, 11) |
| 4 | 3 | (5, 6, 11) |
| 4 | 4 | (5, 5, 5, 11) |
| 3 | 2 | (11, 11) |
| 3 | 3 | (5, 10, 11) |

a contradiction with (4.7). The inequality (3.1) does not hold for $d = 3$ and $p = 2$, so we must use another method in this case.

Suppose that there exists a vertex $u$ of degree 3 and of type (11, 11) in $F$. Denote by $v$ a vertex of degree 4 adjacent to $u$. By (4.5) we have $S(u) + S(v) \geq 52$. Then from (4.7) it follows that there are at least 17 vertices with degree 3 in F. Therefore

$$m \leq 10, \tag{4.8}$$

where $m$ is the number of vertices with degree $\geq 4$ in $F$. As we can see from (4.5), no two vertices of degree 3 are adjacent in $F$. In the same way as in the proof of theorem 2 (see 2.7) we can obtain

$$e(F) \geq 3(27 - m) + 2(m - 6) + 3 = 72 - m.$$

Hence using (4.8) we get $e(F) \geq 62$, a contradiction with (4.7).

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*Author's address:*



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Roman Nedela
Katedra matematiky
Pedagogická fakulta
975 49 Banská Bystrica

## SÚHRN

## O JEDNOM PROBLÉME Z EXTREMÁLNEJ TEÓRIE GRAFOV

ROMAN NEDELA, Banská Bystrica

Nech $f(k)$ $(k \geq 3)$ je najmenšie prirodzené číslo $n$ také, že existuje rozklad kompletného grafu $K_n$ na $k$ faktorov priemeru 2. Je známe, že $f(k) \leq 6k$ pre $k \geq 3$. V článku je dokázané, že $f(4) \geq 17$, $f(5) \geq 22$ a $f(6) \geq 27$.

## РЕЗЮМЕ

## ОБ БОДНОЙ ПРОБЛЕМЕ В ЭКСТРЕМАЛЬНОЙ ТЕОРИИ ГРАФОВ

РОМАН НЕДЕЛА, Банска Быстрица

Пусть $f(k)$ $(k \in N,\ k \geq 3)$ наименьшее натуральное число $n$, при котором существует декомпозиция полного графа с $n$ вершинами на $k$ остовных подграфах диаметра 2. Известно, что $f(k) \leq 6k$ для всех $k \geq 3$. В статье приведены нижние оценки $f(4) \geq 17$, $f(5) \geq 22$ а $f(6) \geq 27$.