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ON THE SUM OF INDEPENDENT RANDOM VARIABLES

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In his paper [1] D. Laugwitz has introduced an alternative approach to non-standard analysis using "Ω-calculus". Our aim is to show that, using "Ω-calculus", each random variable can be considered to be "discrete" (in a non-standard sense). The discrete probability distribution of the sum of two independent random variables is also presented in this paper.

Let T be a consistent theory, containing the theory of real analysis and L be its language. Add a new extra-logical constant Ω. A new theory $T(\Omega)$, with language $L \cup \{\Omega\}$, is obtained adding to T the following scheme of axioms

SA: Let $S(n)$ be a formula in the language L. Then the following formula is an axiom of the theory $T(\Omega)$:

$$(\exists k)(\forall n)\ (n \geqslant k \rightarrow S(n)) \rightarrow S(\Omega).$$

For the consistency and further properties of $T(\Omega)$ see [1].

Denote Z, N and R the sets of all integers, natural and real numbers, respectively and put $B(n, y) = \{x \in \langle -n;\ y);\ (\exists m \in Z)(x = m/n)\}$ and for any function $k\colon N \rightarrow N$

$$A(k(n)) = \{x \in \langle -n;\ n);\ (\exists m \in Z)(x = m/k(n))\}.$$

Theorem 1. Let G be a distribution function, y its continuity point and h any function with bounded variation such that $\int_{-\infty}^{\infty} h\, \mathrm{d}G$ exists and is finite. Denote $g_n(x) = G(x + 1/n) - G(x)$. Then there holds

a) for each $\varepsilon > 0$ $|G(y) - \sum_{x \in B(\Omega, y)} g_\Omega(x)| < \varepsilon$

b) there exists a function $k\colon N \rightarrow N$ such that for each $\varepsilon > 0$

$$\left|\int_{-\infty}^{\infty} h\, \mathrm{d}G - \sum_{x \in A(k(\Omega))} h(x)\, g_{k(\Omega)}(x)\right| < \varepsilon.$$

Proof. a) There holds

$$\sum_{x \in B(n,y)} g_n(x) = \sum_{x \in B(n,y)} (G(x + 1/n) - G(x)) = G(a) - G(-n),$$

where $a \in (y - 1/n; y + 1/n)$. Fix an $\varepsilon > 0$. Since G is a distribution function and y its continuity point, there exists an $m \in N$ such that for all $n \in N$, $n > m$

$$|G(y) - \sum_{x \in B(n,y)} g_n(x)| = |G(y) - G(a) + G(-n)| < \varepsilon.$$

Hence, since ε was arbitrary, SA implies Statement a).

b) Since $\int_{-\infty}^{\infty} h \, dG$ is finite, for each $i \in N$ there exists an $m \in N$ such that for all $n > m$

$$\left| \int_{-\infty}^{\infty} h \, dG - \int_{-n}^{n} h \, dG \right| < 1/i, \tag{1}$$

Fix for each i such an n. Then there exists a $k(n)$ such that

$$\left| \int_{-n}^{n} h \, dG - \sum_{x \in A(k(n))} h(x)(G(x + 1/k(n)) - G(x)) \right| =$$

$$= \left| \int_{-n}^{n} h \, dG - \sum_{x \in A(k(n))} h(x) g_{k(n)}(x) \right| < 1/i, \tag{2}$$

since $A(k(n))$ is a partition of $\langle -n; n \rangle$ having norm $1/k(n)$.

Take an $\varepsilon > 0$. (1), (2) and SA imply

$$\left| \int_{-\infty}^{\infty} h \, dG - \sum_{x \in A(k(\Omega))} h(x) g_{k(\Omega)}(x) \right| < \varepsilon.$$

Since ε was arbitrary, Statement b) follows.

Definition. Function g_Ω, mentioned in Theorem 1, will be called the discrete probability distribution.

Theorem 2. Let ξ and η be independent random variables with their distribution functions F and G, respectively. Denote H the distribution function of $\xi + \eta$ and $f_n(x) = F(x + 1/n) - F(x)$, $g_n(x) = G(x + 1/n) - G(x)$ and $h_n(x) = H(x + 1/n) - H(x)$. Then for each increasing sequence $(m_j)_{j=1}^{\infty}$ of natural numbers there exists a function $k: N \to N$ such that

$$|h_\Omega(y) - \sum_{x \in A(k(\Omega))} f_\Omega(y - x) g_{k(\Omega)}(x)| < 1/m_\Omega.$$

Proof. Obviously $H(y) = \int_{-\infty}^{\infty} F(y - x)\, dG(y)$, hence

$$h_j(y) = H(y + 1/j) - H(y) = \int_{-\infty}^{\infty} f_j(y - x)\, dG(x).$$

Fix an increasing sequence $(m_j)_{j=1}^{\infty}$. Since h_j are finite functions, for each $j \in N$ there exists an $k(j)$ such that

$$\left| h_j(y) - \int_{-k(j)}^{k(j)} f_j(y - x)\, dG(x) \right| < 1/2m_j.$$

Obviously there holds

$$\left| \int_{-k(j)}^{k(j)} f_j(y - x)\, dG(x) - \sum_{x \in A(k(j))} f_j(y - x)\, g_{k(j)}(x) \right| < 1/2m_j,$$

hence

$$|h_j(y) - \sum_{x \in A(k(j))} f_j(y - x)\, g_{k(j)}(x)| < 1/m_j$$

and SA implies the assertion of this Theorem.

REFERENCE

1. Laugwitz, D.: Ω-calculus as a generalization of field extension. — an alternative approach to nonstandard analysis. In Non-standard Analysis Recent Developements, pp. 120—133, Lecture Notes in Mathematics № 983, Springer Verlag, Berlin, Heidelberg, New York, Tokyo 1983.

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SÚHRN

O SÚČTE NEZÁVISLÝCH NÁHODNÝCH PREMENNÝCH

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Cieľom článku bolo ukázať, že použitím „Ω-kalkulu" môžeme každú náhodnú premennú považovať za diskrétnu (v neštandardnom zmysle). V článku sme odvodili aj „diskrétne rozdelenie" súčtu dvoch nezávislých náhodných premenných.

РЕЗЮМЕ

ОБ СУММЕ НЕЗАВИСИМЫХ СЛУЧАЙНЫХ ПЕРЕМЕННЫХ

МАРТИН КАЛИНА, Братислава

В статьи показано, что используя «Ω-калькул», каждую случайную переменную можно считать дискретной (в нестандартном смысле). В статьи дана тоже формула для дискретного распределения суммы двух независимых случайных переменных.