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TRANSITIVITY OF EXPANDING MAPS OF THE INTERVAL

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1 Introduction

We show that for any continuous piecewise expanding self-mapping f of a real compact interval I with a finite number of turning points there is an iterate f^n, $n \in N$, and an interval $J \subset I$ such that $f^n \colon J \to J$ is transitive in J. Consequently, f is chaotic in the sense of Li and Yorke.

2 Notations and notions

Throughout this paper f denotes a continuous self-mapping of a real compact interval I, f^n denotes the n-th iterate of f. A point $p \in I$ is periodic point of period n if n is the least integer with the property that $f^n(p) = p$. The basic notions used in this paper are the notions of expanding and transitive self-mapping.

Definition 1. A self-mapping $f \colon I \to I$ is piecewise expanding with expansion constant $\lambda > 1$ if f is piecewise monotonic and $|f(x) - f(y)| \geqslant \lambda |x - y|$ whenever both x and y belong to some interval on which f is monotonic.

Definition 2. A self-mapping $f \colon I \to I$ is transitive in I if there is $x \in I$ such that the set $\operatorname{Orb} x = \{x, f(x), \ldots, f^n(x), \ldots\}$ is dense in I.

Transitivity is characterized by the following well-known

Lemma 1. A self-mapping $f \colon I \to I$ is transitive in I if and only if, for any interval $K \subset I$, $I = \overline{\bigcup_{k \geqslant 0} f^k(K)}$.

For completeness, we present here a proof of this result.

Proof. Assume that $I = \overline{\bigcup_{k \geqslant 0} f^k(K)}$ for any interval $K \subset I$. Let $\{I_n\}_{n \in N}$ be a sequence of all subintervals of I with rational endpoints. Then there is a natural number $n(1)$ with the property that $f^{n(1)}(I_1) \cap I_2$ is an interval, hence there is a closed interval $J_1 \subset I_1$ with the property that $f^{n(1)}(J_1) \subset I_2$. Similarly, $f^{n(2)}(J_1) \cap I_3$ is an interval for some $n(2)$, hence $f^{n(2)}(J_2) \subset I_3$ for an appropriate closed interval $J_2 \subset J_1$. We obtain a sequence of closed intervals $J_1 \supset J_2 \supset J_3 \supset \ldots$ with the property that $f^{n(i)}(J_i) \subset I_{i+1}$, $i = 1, 2, \ldots$. For $x \in \bigcap_{i=1}^{\infty} J_i$, $f^{n(i)}(x) \in I_{i+1}$, hence Orb x is dense in I.

The converse implication is trivial.

3 Results

Our main result is the following.

Theorem. If $f\colon I \to I$ is piecewise expanding, continuous, with a finite number of turning points then there exists $n \in N$ and a closed interval $J \subset I$ such that $f^n\colon J \to J$ is transitive in J.

As a consequence of this theorem we obtain another proof of a result from [6], according to which any expanding self-map f with a finite number of turning points has a cycle of order $\neq 2^n$, $n = 1, 2, \ldots$, and consequently, f is chaotic in the sense of Li and Yorke (cf. also [1], [2]).

Recall that a continuous function $f\colon I \to I$ is chaotic (cf. [3]) if there is an uncountable set $S \subset I$ such that for any $x, y \in S$, $x \neq y$ and any periodic point p of f:

(1) $\limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0$

(2) $\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0$

(3) $\limsup_{n \to \infty} |f^n(x) - f^n(p)| > 0$.

Corollary. Let $f\colon I \to I$ be piecewise expanding with a finite number of turning points. Then f is chaotic in the sense of Li and Yorke.

Proof. Transitivity implies the existence of a cycle of order $\neq 2^n$, $n = 1, 2, \ldots$, (cf. [5]) and hence by [4] f is chaotic in the sense of Li and Yorke.

The following lemmas will be useful in the proof of our result. By an interval we always mean a non-trivial interval and $|\,.\,|$ denotes the length of the interval.

Lemma 2. Let $f\colon I \to I$ be piecewise expanding with a finite number of turning points and expansion constant $\lambda > 2$. Then the following holds.

(1) There exists $d(f) > 0$ such that for any interval $J \subset I$
   (i) $|J| \geqslant d(f)$ implies $|f(J)| \geqslant d(f)$
   (ii) $|J| < d(f)$ implies $|f(J)| > J$
   (iii) $f^k(J) \subset J$, $k \geqslant 1$ implies $|f^i(J)| \geqslant d(f)$, $i = 0, 1, \ldots, k$.

(2) The set $P(f) = \{k\colon$ there exists an interval $J \subset I$ with $f^k(J) \subset J$ and any two intervals $J, f(J), \ldots, f^{k-1}(J)$ have at most one point in common$\}$ is bounded.

Proof. (1) Let $I = [a, b]$ and let $a = a_0 < a_1 < \ldots < a_n = b$ be the turning points of f together with the endpoints of I. Put $d(f) = \min\{a_i - a_{i-1}, i = 1, 2, \ldots, n\}$.

(i) If $|J| \geqslant d(f)$ then J contains at least two points $a_i \neq a_{i-1}$, $i = 1, 2, \ldots, n$ and consequently $|f(J)| \geqslant |a_i - a_{i-1}| \geqslant d(f)$.

(ii) Assume that $J \subset I$, $|f(J)| \leqslant J$. Since $\lambda > 2$, J must contain at least two points $a_i \neq a_j$, $0 \leqslant i, j \leqslant n$, hence $|J| \geqslant d(f)$.

(iii) Let $J \subset I$, $f^k(J) \subset J$, $k \geqslant 1$. Since $f^k(J) \subset J$, by (ii) there is some i, $0 \leqslant i \leqslant k-1$, with $|f^i(J)| \geqslant d(f)$. But then by (i) $|f^{i+n}(J)| \geqslant d(f)$ for all n, hence $|J| \geqslant |f^k(J)| \geqslant d(f)$ and by (i) $|f^i(J)| \geqslant d(f)$, $i = 0, 1, \ldots, k$.

(2) Let $J \subset I, f^k(J) \subset J$ and let any two of the intervals $J, f(J), \ldots, f^{k-1}(J)$ have at most one point in common. Then by (iii) for any $0 \leqslant j \leqslant k$, $|f^j(J)| \geqslant d(f)$, hence $k \leqslant \dfrac{|I|}{d(f)}$.

For $f\colon I \to I$, $n(f)$ denotes $\max P(f)$, where $P(f)$ is the same as in Lemma 2.

Lemma 3. Let $f\colon I \to I$ be expanding with a finite number of turning points and expansion constant $\lambda > 2$. Then there is a minimal invariant interval $J \subset I$, i.e. an interval with the property that $f(J) = J$ and for any interval $K \subsetneq J$, $f(K) \not\subset K$.

Proof. We use Zorn's lemma. Let $\mathscr{S}$ be the system of all subintervals L of I with $f(L) \subset L$ partially ordered by the inclusion $\subset$. By Lemma 2, $|L| \geqslant d(f)$ for any $L \in \mathscr{S}$. Take a linearly ordered family $\{L_t\}_{t\in T}$ of members of $\mathscr{S}$. Denote $L = \bigcap_{t\in T} L_t$. Since

$$f(L) = f\left(\bigcap_{t\in T} L_t\right) \subset \bigcap_{t\in T} f(L_t) \subset \bigcap_{t\in T} L_t = L,$$

L is an interval from $\mathscr{S}$. Thus by Zorn's lemma, there is a minimal member J of $\mathscr{S}$.

We show that $f(J) = J$. Let $J = [a, b]$. First consider the case that $b \notin f(J)$. Put $H = \{x \in J\colon f(x) \geqslant x\}$. Clearly $H \neq \emptyset$. Let $x_1 = \max\{f(x), x \in H\}$. Then $f(x) \leqslant x_1$ and $f(x) \geqslant a$ for all $a \leqslant x \leqslant x_1$. Hence for $J_1 = [a, x_1]$ we have $f(J_1) \subset J_1$, which is impossible. In the case that $a \notin f(J)$ the argument is similar.

Lemma 4. Let $g = f^k$, $k \geqslant 1$, f be expanding with a finite number of turning points. Let J be a minimal invariant interval of $g = f^k$. Then there is $r \leqslant k$, $r \mid k$ with the property that any two of the intervals $J, f(J), \ldots, f^{r-1}(J)$ have at most one point in common and $f^r(J) = J$.

Proof. First consider the case that $J \cap f^i(J)$ is not an interval for all $0 < i \leqslant k-1$. Assume that $f^m(J) \cap f^n(J)$ is an interval for some $0 \leqslant m$, $n \leqslant k-1$, $m < n$. Then since f is expanding and

$$f^{k-m}(f^m(J) \cap f^n(J)) \subset f^k(J) \cap f^{k+n-m}(J) = J \cap f^{n-m}(J)$$

we obtain that $J \cap f^{n-m}(J)$ is an interval, which is impossible. Hence any two of the intervals $J, f(J), \ldots, f^{k-1}(J)$ have at most one point in common and the lemma holds for $r = k$.

Now consider the case that $J \cap f^i(J)$ is an interval for some $0 < i \leqslant k-1$. We can suppose that i is the minimal positive integer with this property. Since

$$f^k(J \cap f^i(J)) \subset J \cap f^i(J)$$

we have $J \cap f^i(J) = J = f^i(J)$. We show that $i|k$. Really, if $k = qi + j$, q, j positive integers, $j < i$, then we have

$$f^{i-j}(J) = f^{i-j}(f^k(J)) = f^{k+i-j}(J) = f^{qi+j+i-j}(J) = f^{(q+1)i}(J) = J$$

which is a contradiction with the minimality of i. Hence $f^i(J) = J$, $i|k$ and J, $f(J), \ldots, f^{i-1}(J)$ have at most one point in common. Thus the lemma is true for $r = i$.

Lemma 5. Let $f\colon I \to I$ be expanding with a finite number of turning points and expansion constant $\lambda > 2$. Then there is such m that $n(f^m) = 1$. (For the definition of $n(f)$ see Lemma 2.)

Proof. Put $m = n(f)!$. Let $J \subset I$, $f^{km}(J) \subset J$, $k \geqslant 1$ and any two of the intervals $J, f^m(J), \ldots, f^{(k-1)m}(J)$ have at most one point in common. By Lemma 3 there is a minimal invariant interval of $g = f^{km}$. Denote it by J_1. Then by Lemma 4 there is $r|km$ such that any two of the intervals $J_1, f(J_1), \ldots, f^{r-1}(J_1)$ have at most one point in common and $f^r(J_1) = J_1$. Clearly $r \leqslant n(f)$, hence $r|m$ and $f^m(J_1) = J_1$. Hence $k = 1$, q.e.d.

Proof of Theorem. We may assume that the expansion constant λ of f is greater than 2, since otherwise we replace f by an appropriate iterate. We show that $g = f^m$, where $m = n(f)!$, is transitive in its minimal invariant interval J. Now the condition from Lemma 1 will be used. Let $K \subset J$ be any interval. We show that $\overline{\bigcup_{n \geqslant 0} g^n(K)} = J$. Lemma 2 implies that $|g^n(K)| \geqslant \min\{d(g), |K|\}$ and it is easy to see that $\overline{\bigcup_{n \geqslant 0} g^n(K)} = K_1 \cup K_2 \cup \ldots \cup K_s$, where $K_1, K_2, \ldots, K_s$ are pairwise disjoint closed intervals. Since $g\left(\bigcup_{n \geqslant 0} g^n(K)\right) \subset \bigcup_{n \geqslant 0} g^n(K)$, there is natural number $r \leqslant s$ such that for some $i \in \{1, \ldots, s\}$ $g^r(K_i) \subset K_i$. Again by Lemma 2, $|g^r(K_i)| \geqslant d(g)$ and also $|g^{rn}(K_i)| \geqslant d(g)$ for all $n \in N$. Put $L = \bigcap_{n \geqslant 0} g^{rn}(K_i)$. Since $K_i \supset g^r(K_i) \supset \ldots \supset g^{rn}(K_i) \supset \ldots$, L is a closed interval, $|L| \geqslant d(g)$. Further $g^r(L) = L$ and $n(g) = 1$ (Lemma 5) implies $r = 1$. Thus L is an invariant interval of g, $L \subset J$, hence $L = J$. Since $L \subset \overline{\bigcup_{n \geqslant 0} g^n(K)}$ we obtain that $\overline{\bigcup_{n \geqslant 0} g^n(K)} = J$ and by Lemma 1 g is transitive in J.

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SÚHRN

TRANZITIVITA EXPANZÍVNYCH ZOBRAZENÍ INTERVALU

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Ukazuje sa, že pre spojité expanzívne zobrazenie kompaktného intervalu do seba, pozostávajúce z konečného počtu monotónnch častí existuje iterácia f^n, $n \in N$, a interval $J \subset I$ tak, že $f^N: J \to J$ je tranzitívne v J.

РЕЗЮМЕ

ТРАНСИТИВНОСТЬ РАСТЯГИВАЮЩИХ ОТОБРАЖЕНИИ ИНТЕРВАЛА

К. ЯНКОВА—М. ПОЛАКОВИЧ, Братислава

Показывается, что для любого непрерывного растягивающего отображения f компактного интервала I в себя, состоящего из конечного числа промежутков монотонности, существует итерация f^n и интервал $J \subset I$ так, что $f^n: J \to J$ транситивно в J.