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DECOMPOSITION OF COMPLETE GRAPHS INTO FACTORS WITH DIAMETER TWO

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1 Introduction

We shall consider undirected graphs without loops and multiple edges. The complete graph on n vertices will be denoted K_n; $d(G)$ means the diameter of G. Other notions from graph theory are used in the sence of [5].

Denote $f(k)$ the smallest such natural that the edge-set of $K_{f(k)}$ can be decomposed into k factors of diameter two. That $f(k)$ is finite for every $k \geqq 2$ was proved in [2]. In 1970 Sauer [7] proved that

$$f(k) \leqq 7k$$

and in 1974 Bosák [4] showed that

$$f(k) \leqq 6k.$$

On the other hand in 1980 Bollobás showed in [1] that for $k \geqq 6$ we have

$$f(k) \geqq 6k - 9$$

and later in [8] Znám improved that:

$$f(k) \geqq 6k - 7 \quad \text{for} \quad k \geqq 664.$$

Finally in [9] Znám showed that for $k \geqq 10^{17}$ we have

$$f(k) = 6k.$$

For small k the values of $f(k)$ were studied by Bollobás, Bosák and Nedela [6]. Presently the following bounds and exact values of $f(k)$ are known:

$$f(2) = 5$$

$$13 \geqq f(3) \geqq 12$$

$$24 \geqq f(4) \geqq 15$$

$$30 \geqq f(5) \geqq 22$$

$$36 \geqq f(6) \geqq 28$$

$$6k \geqq f(k) \geqq 6k - 9 \quad \text{for} \quad 7 \leqq k \leqq 663$$

$$6k \geqq f(k) \geqq 6k - 7 \quad \text{for} \quad k \geqq 664$$

$$6k = f(k) \quad \text{for} \quad k \geqq 10^{17}.$$

The cases $k = 4, 5, 6$ were studied in detail by Nedela. Our paper is devoted to the cases $k = 7$ and 8. We show that $f(7) \geqq 34$, which is an improvement of the known bound given above.

In what follows we shall use the following notations:

$V(G)$ — the vertex set of G,
$O(x)$ — the set of neighbours of the vertex x,
$S(x)$ — the sum of degrees of vertices from $O(x)$,
$\deg x$ — the degree of the vertex x,
$e(G)$ — the number of edges of G,
F_i — some factor from a decomposition of K_n into factors with diameter two.

The following assertions are proved in [3], [6] and [9].

P1. If K_n $(n > 1)$ can be decomposed into m factors of diameter 2, then for $N > n$ the complete graph K_N can also be decomposed into m factors of diameter two.

P2. For $k \geqq 3$ we have

$$3 \leqq \deg x \leqq n - 3k + 2,$$

in any factor of decomposition of K_n into k factors of diameter 2.

P3. In the decomposition of K_n into k factors of diameter 2 there exists a (minimal) factor F_0 with

$$e(F_0) \leqq \lfloor n(n-1)/2k \rfloor .$$

P4. Let $d(G) = 2$ and let for $x \in V(G)$ there exist r edges with both endpoints in $O(x)$. Then

$$S(x) \geqq n - 1 + 2r + \sum_{y \in V(G) - \{x\} - O(x)} (s_y - 1),$$

where s_y means the number of edges connecting y with vertices of $O(x)$.

In our consideration we shall use also the following Lemma by Bollobás [1]:

Lemma 1. Let $a > 0$ and let for all vertices of G the following inequality hold

$$(1) \qquad D(x) = \deg x + \sum_{\substack{y \in O(x) \\ \deg y > a}} (\deg y - a)/\deg y \geqq a.$$

Then $e(G) \geqq a/2\,|V(G)|$.

2 $f(7)$ is at least 34

According to P1 it is sufficient to prove that K_{33} cannot be decomposed into 7 factors of diameter two. In the proof we shall proveed indirectly. Suppose K_{33} is decomposed into 7 factors with diameter 2. Denote the factors $F_0, \ldots, F_6$, where F_0 has minimal number of edges.

Lemma 2. A vertex of degree 3 cannot be adjacent to a vertex of degree 3, 4, or 5 in any factor of the decomposition.

Proof. We shall prove it indirectly. Suppose that in F_i there is a vertex v_0 of degree 3 adjacent to some vertex of degree 5. Namely, let

$$O(v_0) = \{v_1, v_2, v_3\}, \quad \text{where} \quad 5 = \deg v_1 \leqq \deg v_2 \leqq \deg v_3.$$

Then owing to P2 and P4, v_2, v_3 are not adjacent in F_i and we have

$$\deg v_2 + \deg v_3 \geqq 27.$$

Hence due to P2: $\deg v_2 \geqq 13$, $\deg v_3 \geqq 14$. Thus in the remaining factors v_3 is of degree 3 and v_2 can be of degree 3 or 4. Suppose v_2 and v_3 are adjacent in F_j and

$$O(v_3) = \{v_2, v_4, v_5\}, \ \deg v_2 \leqq 4 \ \text{in} \ F_j.$$

Then

$$\deg v_4 + \deg v_5 \geqq 28$$

and hence, using P2

$$\deg v_4 = \deg v_5 = 14.$$

Therefore, in the remaining factors the vertices v_4 and v_5 are of degree 3. Then there exists a factor F_l in which v_4 and v_5 are adjacent and we have

$$O(v_4) = \{v_5, v_6, v_7\}, \ \deg v_5 = 3.$$

From this it follows (see P4) that

$$\deg v_7 + \deg v_6 \geqq 29 \quad \text{in } F_l$$

which is a contradiction with P2.

Lemma 3. For vertices of degree at least 4 the inequality (1) holds for $a = 4.55$ in any factor.

Proof. First of all consider the values of $(\deg x - a)/\deg x$ for different values of $\deg x$ (see *table 1*)

Table 1

$\deg x$	14	13	12	11	10	9	8	7	6
$\dfrac{\deg x - a}{\deg x}$	0.67	0.65	0.62	0.58	0.54	0.49	0.43	0.35	0.24

Let v_0 be of degree 4. If $O(v_0)$ contains some vertex of degree at least 11, then using the above list it can be easily checked, that $D(v_0) > 4.55$. If no vertex of degree at least 11 is contained in $O(v_0)$, then $O(v_0)$ contains at least two vertices of degree at least 8, and using the above list, we get $D(v_0) > 4.55$ again. The proof is finished.

Theorem 1. $f(7) \geqq 34$.

Proof. We shall proceed indirectly. Consider the minimal factor F_0. If all vertices are of degree at least 4, then due to Lemma 3, F_0 contains at least $\frac{4.55}{2} \cdot 33 = 75.075$ edges, a contradiction with P3 wich asserts that $e(F_0) \leqq 75$. The proof is finished in this case.

Hence in what follows we shall suppose that F_0 contains a vertex v_0 of degree 3. Suppose $O(v_0) = \{v_1, v_2, v_3\}$. We shall distinguish 3 cases and show that in all of them we have $D(v_0) > 4.55$.

a) There exists and edge between two vertices of $O(v_0)$. Since $d(F_0) = 2$, in this case (due to P4) we have $S(v_0) \geqq 32 + 2 = 34$. Using P2 we get that all vertices of F_0 are of degree at most 14. Thus none of v_1, v_2, v_3 is of degree smaller than 6. Hence if $S(v_0) = 34$ than for degrees of vertices v_i we have the following possibilities:

$$(6, 14, 14),\quad (7, 13, 14),\quad (8, 12, 14),\quad (8, 13, 13),\quad (9, 11, 14),$$

$$(9, 12, 13), (10, 10, 14), (10, 11, 13), (10, 12, 12), (11, 11, 12).$$

However, using the above list we can check in all cases the ineguality $D(v_0) > 4.55$. From this obviously the inequality follows for $S(v_0) > 34$, as well, because $D(x)$ is an increasing function. Now using Lemma 1 we have a contradiction with P3.

b) There is no edge between vertices of $O(v_0)$ but there is a vertex (different from v_0) belonging to the neighbourhood of two vertices v_i.

Without loss of generality it can be supposed $v_4 \in O(v_1) \cap O(v_2)$. From P4 and the fact that $d(F_0) = 2$ follows that in this case

$$S(v_0) \geqq 32 + 1 = 33.$$

According to Lemma 2 if one of vertices v_1, v_2, v_3 is of degree 14, then the remaining are of degree at most 11. Hence if $S(v_0) = 33$ we have only the following possibilities for degrees of v_1, v_2, v_3:

$$(7, 13, 13),\quad (8, 14, 11),\quad (8, 13, 12),\quad (9, 14, 10),\quad (9, 13, 11),$$
$$(9, 12, 12),\quad (10, 13, 10),\quad (10, 12, 11),\quad (11, 11, 11).$$

It can be easily checked that in all cases the inequality

$$D(v_0) > 4.55 \text{ holds.}$$

Futher argument as above.

c) There is no edge between v_1, v_2, v_3 and their neighbourhoods contain the only common element v_0. Then $S(v_0) \geqq 32$ and it is obviously sufficient to consider only the case $S(v_0) = 32$.

For degrees of vertices v_1, v_2, v_3 we have the following possibilities (see Lemma 2):

$$(6, 13, 13),\quad (7, 11, 14),\quad (7, 12, 13),\quad (8, 10, 14),\quad (8, 11, 13),$$
$$(8, 12, 12),\quad (9, 9, 14),\quad (9, 10, 13),\quad (9, 11, 12),\quad (10, 10, 12),$$
$$(10, 11, 11).$$

Using similar considerations as above, we can show that $D(v_0) > 4.55$ holds for all cases with the only exeption: (6, 13, 13). Hence in what follows we shall deal with this case. Suppose $O(v_0) = \{v_1, v_2, v_3\}$ and $\deg v_1 = 6$ and put

$$A = O(v_1) - \{v_0\},\quad B = O(v_2) - \{v_0\},\quad C = O(v_3) - \{v_0\}.$$

If v_0 is the only vertex of degree 3, then due to P4 we have

$$2e(F_0) \geqq 3 + S(v_0) + 29.4 \geqq 151,$$

thus we have a contradiction with P3. Hence we can suppose that a futher vertex v_4 of degree 3 exists.

Now we shall show that in F_0 there exist at least 13 vertices of degree 3. If v_5 and v_6 are the neighbours of v_4 (different from v_1, v_2, v_3) then

$$\deg v_5 + \deg v_6 \geqq 19$$

(see P4).

Now, if the number of vertices of degree 3 in F_0 is less than 13, we have:

$$2e(F_0) \geqq S(v_0) + \deg v_5 + \deg v_6 + 12.3 + 16.4 \geqq 151$$

a contradiction with P3.

Note that if $v_4 \in A$, then $\deg v_5 + \deg v_6 \geqq 26$ and reasoning similarly as above we can show that F_0 contains at least 20 vertices of degree 3.

In that last case both B and C have to contain a vertex of degree 3. We shall show now that it is true also if A contains no vertex of degree 3.

Suppose no vertex of degree 3 exists in A. As we know that at least 13 vertices of degree 3 exist in F_0, only the following two cases can happen:

c1) All vertices of B are of degree 3. A vertex from C can be reached (by a path of lenght 2) from v_2 only trough vertices of B; thus every vertex of C is adjacent to some vertex of B and hence, due to L2, all vertices of C are of degree at least 6. Thus we have

$$2e(F_0) \geqq S(v_0) + 12.6 + 18.3 \geqq 158$$

— a contradiction with P3.

c2) Both B and C contain a vertex of degree 3. Then both B and C contain a vertex of degree at least 6. Now suppose that the number of vertices of degree 3 in $B \cup C$ is less then 14. Then we get:

$$2e(F_0) \geqq S(v_0) + S(v_1) + 2.6 + 13.3 + 9.4 \geqq 151$$

— a contradiction with P3.

Hence $B \cup C$ contains always at least 14 vertices of degree 3 and hence both B and C contain at least one vertex of degree 3. Take two vertices of degree 3: $v_7 \in B$, $v_8 \in C$. These two vertices can be connected by a way of length 2 only trough a vertex $v_9 \in A$. But then every vertex of degree 3 from B must be adjacent to v_9 and the same is true for vertices of degree 3 in C. Thus v_9 is of degree 15, what is a contradiction with P2.

Since we get a contradiction in all cases, F_0 cannot contain any vertex of degree 3 having neighbours of degree 6, 13 and 13, respectively. Thus for all vertices of F_0 (1) is fulfilled for $a = 4.55$. Hence F_0 has at least 76 edges, which contradicts P3.

The proof of the theorem is finished.

Remark. Using similar but rather more complicated considerations as above we proved recently

Theorem 2. $f(8) \geqq 40$.

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SÚHRN

ROZKLAD KOMPLETNÝCH GRAFOV NA FAKTORY PRIEMERU DVA

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Označme $f(k)$ také najmenšie prirodzené číslo, že hranová množina kompletného grafu $K_{f(k)}$ môže byť rozložená na k faktorov priemeru dva. Článok obsahuje vylepšenie tvrdenia Bollobása pre $k \geqq 6$: $f(k) \geqq 6k - 9$; kde v našom prípade $f(7) \geqq 34$.

РЕЗЮМЕ

РАЗЛОЖЕНИЕ ПОЛНЫХ ГРАФОВ НА ФАКТОРЫ ДИАМЕТРА ДВА

ИВЕТА МАРКОВА, Братислава

Пусть $f(k)$ такое наименьшее натуральное число, что множество ребер полного графа $K_{f(k)}$ может быть разложено на k факторов диаметра два. Статья содержит улучшения утверждения Боллобаса для $k \geqq 6$: $f(k) \geqq 6k - 9$; конкретно для случая $k = 7$ мы показываем, что $f(7) \geqq 34$.